Self-similarity in the entropy graph for a family of piecewise linear maps.

Henk Bruin (University of Vienna)

Carlo Carminati (University of Pisa)
Stefano Marmi (Scuola Normale di Pisa)
Alessandro Profeti (University of Pisa)

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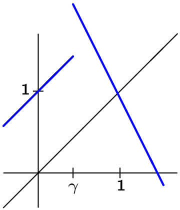
The family Q_γ

For fixed slope $s > 1$ and $\gamma \in [0, 1]$, take:

$$Q_\gamma(x) = \begin{cases} x + 1, & x \leq \gamma \\ 1 + s(1 - x), & x > \gamma \end{cases}$$



There is **matching** if there are $\kappa^\pm \geq 1$ such that

$$\begin{aligned} Q_\gamma^{\kappa^+}(\gamma^+) &= Q_\gamma^{\kappa^-}(\gamma^-), \\ (Q_\gamma^{\kappa^+})'(\gamma^+) &= (Q_\gamma^{\kappa^-})'(\gamma^-). \end{aligned}$$

The number

$$\Delta = \kappa^+ - \kappa^-$$

is the **matching index**.

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  (In this case $Q_\gamma^n(\gamma^+) = Q_\gamma^n(\gamma^-) = 1$ is fixed for all n large enough.)
- The non-matching set $\mathcal{E}$ has Hausdorff dimension

$$\dim_H(\mathcal{E}) = 1,$$

but $\dim_H(\mathcal{E} \setminus [0, \delta)) < 1$ for every $\delta > 0$.

The family Q_γ

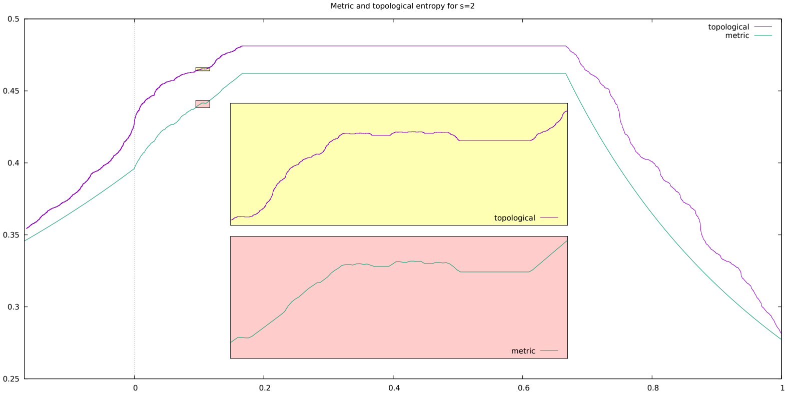


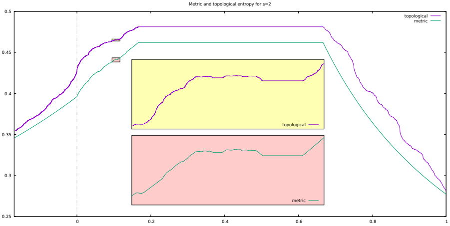
Figure: Topological & metric entropies of Q_γ for $s = 2$ as functions of γ.

The family Q_γ

Theorem: Topological and metric entropy

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta > 0, \end{cases}$$

as function of γ within matching intervals. (Recall $\Delta = \kappa^+ - \kappa^-$.)



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  - pseudo-centers
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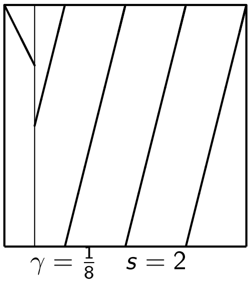
- Tuning windows and explanation of the self-similarity of the entropy graphs.

Why there is matching

Let $g(x) := s(1-x) \bmod 1$ and $R : (0,1) \to (0,1)$ be the first return of Q_γ to $[0,1)$.

Lemma:

$$R(x) = \begin{cases} g(x) & \text{if } x \in (0,\gamma) \\ g^2(x) & \text{if } x \in (\gamma, 1) \end{cases}$$



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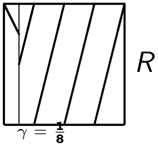

Lemma: For fixed $\gamma \in [0,1]$, the following conditions are equivalent:

(i) $g^k(\gamma) < \gamma$ for some $k \in \mathbb{N}$;

(ii) matching holds for γ.

In other words, the bifurcation set is

$$\mathcal{E} = \{\gamma \in [0,1] \ : \ g^k(\gamma) \geq \gamma \ \forall k \in \mathbb{N}\}.$$

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Definition

- For binary string u, let $\check{u}$ be the bitwise negation of u.
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- For $\xi \in \mathbb{Q}_{\mathrm{dyd}} \setminus \{1\}$ and let w be the shortest even binary expansion of ξ and v be the shortest odd binary expansion of $1 - \xi$.
- Define the interval $I_\xi := (\xi_L, \xi_R)$ containing ξ where,
- $\xi_L := .\overline{\check{v}v}, \quad \xi_R := .\overline{w}.$
- Also define the "degenerate" interval $I_1 := (2/3, +\infty)$.

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If $\xi = 1/2$ then $w = 10$, $v = 1$ and $\xi_L = .\overline{01}$, $\xi_R = .\overline{10}$.

(01) $w = u01 \Rightarrow \xi_L = .\overline{u001\check{u}110}$;

(11) $w = u11 \Rightarrow \xi_L = .\overline{u101\check{u}010}$;

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ξ			ξ_R			ξ_L		
$\frac{1}{2}$	$=$	$.10$	$\frac{2}{3}$	$=$	$.\overline{10}$	$\frac{1}{3}$	$=$	$.\overline{01}$
$\frac{1}{4}$	$=$	$.01$	$\frac{1}{3}$	$=$	$.\overline{01}$	$\frac{2}{9}$	$=$	$.\overline{001110}$
$\frac{7}{32}$	$=$	$.001110$	$\frac{2}{9}$	$=$	$.\overline{001110}$	$\frac{7}{33}$	$=$	$.\overline{0011011001}$
$\frac{3}{16}$	$=$	$.0011$	$\frac{1}{5}$	$=$	$.\overline{0011}$	$\frac{2}{11}$	$=$	$.\overline{0010111010}$
$\frac{9}{64}$	$=$	$.001001$	$\frac{1}{7}$	$=$	$.\overline{001}$	$\frac{4334}{16383}$	$=$	$.\overline{00100011101110}$
$\frac{1}{8}$	$=$	$.0010$	$\frac{2}{15}$	$=$	$.\overline{0010}$	$\frac{1}{9}$	$=$	$.\overline{000111}$

Pseudo-centers

Theorem:

- All matching intervals have the form I_ξ, where $\xi \in \mathbb{Q}_{\text{dyd}}$ are precisely the pseudo-centers of the components of $[0, \frac{2}{3}] \setminus \mathcal{E}$.
- The matching index is

$$\Delta(\xi) = \frac{3}{2}(|w|_0 - |w|_1),$$

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Remark: For general slope $s \geq 2$, the formulas are

$$\Delta(\xi) = \frac{s+1}{2}\sum_{a=0}^{s-1}(s-1-2a)|w|_a, \quad M = (\frac{s}{s+1} - \frac{1}{s}\,,\, \frac{s}{s+1})$$

Pseudo-centers (period doubling)

Pseudo-center $\xi = .w$ (even exp.) and $1 - \xi = .v$ (odd exp.).
The matching interval is $I_\xi = [\xi_L, \xi_R]$ for $\xi_L = .\overline{\check{v} v}$ and $\xi_R = .\overline{w}$.

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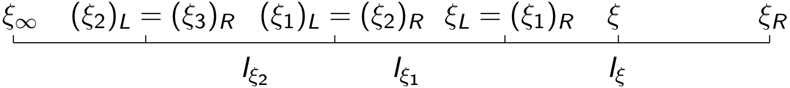
But ξ_L is also the right end-point of I_{ξ_1} for $\xi_1 = .\check{v}v$. We call this "period doubling". It repeats countably often, converging to ξ_∞.



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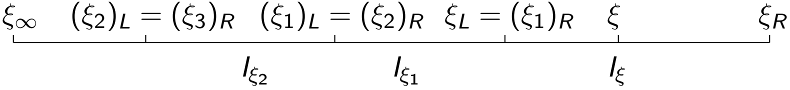

Lemma: The pseudo-center of the next period doubling can be obtained from the previous using the substitution:

$$\chi : \begin{array}{ll} w \mapsto \check{v}v & \check{w} \mapsto v\check{v} \\ v \mapsto vw & \check{v} \mapsto \check{v}\check{w} \end{array}.$$

Thus the limit ξ_∞ has s-adic expansion

$$\xi_\infty = .\check{v}\check{w}v\check{v}vw\check{v}\check{w}vw\check{v}v\check{v}\check{w}\dots$$

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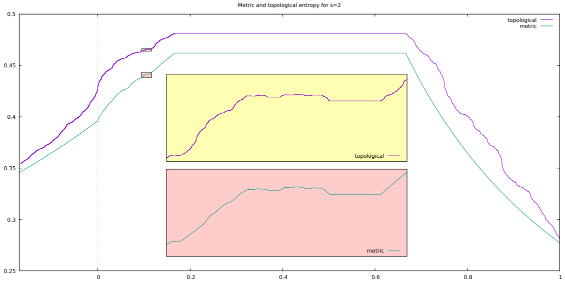

Remark: This proposition explains constant entropy on all matching intervals in $M = (\frac{1}{6}, \frac{2}{3})$. A no devil's staircase argument gives:

$$h_\mu(Q_\gamma) = \log(\frac{1+\sqrt{5}}{2}) \quad \text{and } h_{top}(Q_\gamma) = \frac{2}{3}\log 2,$$

for all $\gamma \in [\frac{1}{6}, \frac{2}{3}]$.

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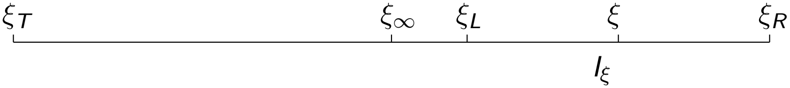

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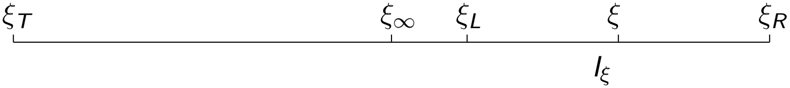
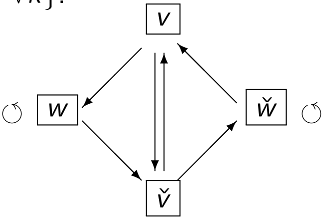


Theorem: Let $K(\xi_T) = \{x : g^k(x) \geq \xi_T\ \forall k\}$.
Then $x \in K(\xi_T) \cap T_\xi$ if and only if

$$x = .\sigma_1\sigma_2\sigma_3\sigma_4\ldots$$

for $\sigma_1 \in \{w, \check{v}\}$, $\sigma_j \in \{w, v, \check{w}, \check{v}\}$ describing a path in the diagram.



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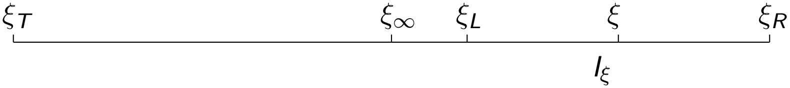

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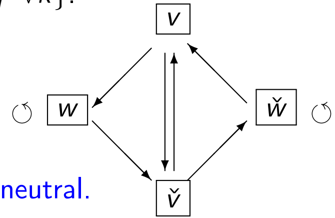
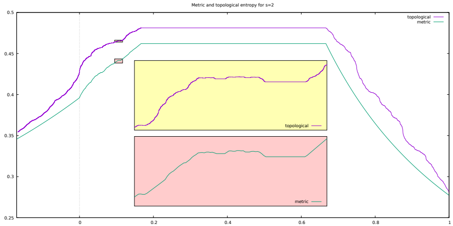
If $\Delta(\xi) = 0$, then all matching in T_ξ is neutral.



Shape of the entropy function



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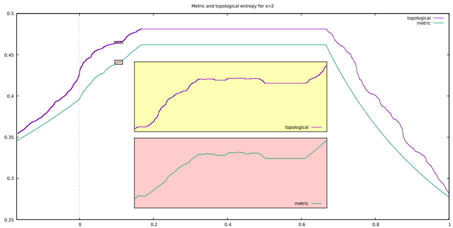

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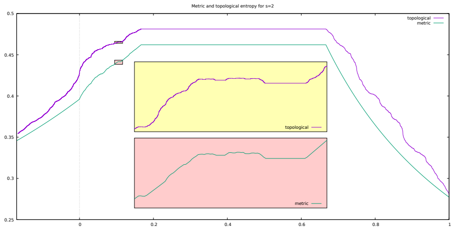

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Corollary: The shape of the entire entropy function (i.e., pattern of increase/decrease) is repeated in every tuning window T_ξ with $\Delta(\xi) > 0$, and reversed in every tuning window T_ξ with $\Delta(\xi) < 0$.