CRYSTALLOGRAPHIC NUMBER SYSTEMS

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Dedicated to Shigeki Akiyama on the occasion of his 50th birthday.

Abstract. We introduce the notion of crystallographic number systems, generalizing matrix number systems. Let Γ be a group of isometries of $\mathbb{R}^d$, g an expanding affine mapping of $\mathbb{R}^d$ with $g\circ\Gamma\circ g^{-1}\subset\Gamma$ and $\mathcal{D}\subset\Gamma$. We say that $(\Gamma,g,\mathcal{D})$ is a Γ-number system if every isometry $\gamma\in\Gamma$ has a unique expansion

$$\gamma = g^n\delta_n g^{-n}\ g^{n-1}\delta_{n-1}g^{-(n-1)}\ \dots\ g\delta_1 g^{-1}\ \delta_0,$$

for some $n\in\mathbb{N}$ and $\delta_0,\dots,\delta_n\in\mathcal{D}$. A tile can be attached to a Γ-number system. We show fundamental topological properties of this tile: they admit the fixed point of g as interior point and tesselate the space by the whole group Γ. Moreover, we give several examples, among them a class of $p2$-number systems, where $p2$ is the crystallographic group generated by the π-rotation and two independent translations.

1. Introduction

Let $b\geq 2$ be an integral base. It is a well-known fact that every positive integer n has a unique expansion $n = d_0 + d_1 b + \cdots + d_m b^m$ for some $m\geq 0$ and $d_0,\dots,d_m\in\{0,\dots,b-1\}$. Such numeration systems gave rise to several generalizations in the last forty years. Rather than the set of integers, authors considered successively Gaussian integers [13], more generally quadratic fields [11, 12] and finally rings of polynomials [14, 21] as representation spaces. Algorithms were developped to decide whether a given system is a number system or not [5, 24]. However, it is usually a hard task to determine whole classes of number systems (see [1, 2] for recent criteria in this direction).

All these number systems have well-known connections to fractals [8, 10]. Important developments to investigate these connections were carried out by Gröchenig and Haas as well as Lagarias and Wang [9, 15]. They considered

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matrix systems $(M, \mathcal{N})$ and the corresponding self-similar sets T defined by the equation $MT = T + \mathcal{N}$. Here, the $d \times d$ matrix M is the base and $\mathcal{N} \subset \mathbb{R}^d$ a finite digit set. Fundamental topological results on self-similar tiles were proved in [9, 15].

In the present paper we generalize matrix systems. Let us first give some definitions. A *crystallographic group* Γ is a discrete cocompact subgroup of the group of isometries of $\mathbb{R}^d$. By a theorem of Bieberbach [6], it can be seen as a group of isometries containing a maximal abelian subgroup Λ isomorphic to $\mathbb{Z}^d$ such that the *point group* Γ/Λ is finite.

Let Γ be a crystallographic group and $g(x) = \mathbf{A}x + b$ an expanding affine mapping of $\mathbb{R}^d$ that conjugates Γ into itself ($g \circ \Gamma \circ g^{-1} \subset \Gamma$). Furthermore, let $\mathcal{D} \subset \Gamma$ be a complete set of right coset representatives of $\Gamma/g \circ \Gamma \circ g^{-1}$. Gelbrich [7] shows in 1994 that the unique non-empty compact set T satisfying

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T)$$

tiles the Euclidean space by some subset $\mathcal{J}$ of Γ. T is called a *crystallographic replication tile*. In the case of matrix systems, *i.e.*, if Γ is isomorphic to $\mathbb{Z}^d$, $\mathcal{J}$ is isomorphic to a sublattice of $\mathbb{Z}^d$ [15]. In the general case of a crystallographic group, nothing is known on the algebraic structure of $\mathcal{J}$. Gelbrich conjectured it to be a subgroup of Γ.

We will generate classes of crystallographic replication tiles for which $\mathcal{J} = \Gamma$. To this matter, we introduce the notion of crystallographic number system, analogously to the lattice case. $(\Gamma, g, \mathcal{D})$ will be called a Γ-number system if every isometry $\gamma \in \Gamma$ has a unique expansion

$$\gamma = g^n \delta_n g^{-n} \; g^{n-1} \delta_{n-1} g^{-(n-1)} \; \dots \; g \delta_1 g^{-1} \; \delta_0,$$

for some $n \in \mathbb{N}$ and $\delta_0, \dots, \delta_n \in \mathcal{D}$. This property has topological consequences for the associated tile T. We will assume that the identity mapping id belongs to $\mathcal{D}$. Let $\mathrm{Fix}(g)$ denote the fixed point of the expansion g. Then we show that the data $(\Gamma, g, \mathcal{D})$ gives rise to a crystallographic number system if and only if $\mathrm{Fix}(g)$ is an exclusive inner point of T. In this case, $\mathcal{J} = \Gamma$, that is, T tiles the space by the whole group. If Γ is a lattice, g is assumed to be linear, thus the fixed point is 0 and these results are well-known [10].

We give several examples of crystallographic number systems. By our results, it follows that the tiling set of the corresponding tiles is the whole crystallographic group, and their Lebesgue measure is $1/|\Gamma/\Lambda|$. As counterexamples, we will see that the well-known Levy dragon and Heighway dragon do not have the number system property. Restricting ourselves to the case of $p2$, the group generated by two independent translations and a

π-rotation, we will obtain a class of $p2$ number systems. Topological properties of the corresponding class of crystallographic replication tiles will be investigated in a forthcoming paper. More precisely, we wish to characterize the tiles among this class that are homeomorphic to a closed disk.

2. Topology and tiling criterion

Crystallographic replication tiles, or *crystiles* for short, generalize self-affine tiles : the digit set is not restricted to translations. They were introduced by Gelbrich in 1994 in [7], and further studied in [18, 19, 20]. In [7] the following result can be found.

Theorem 2.1 (*cf.* [7])**.** *Let Γ be a crystallographic group, $g : \mathbb{R}^d \to \mathbb{R}^d$ an affine exapanding mapping such that $g\Gamma g^{-1} \subset \Gamma$ and $\mathcal{D} \subset \Gamma$ a complete set of right coset representatives of $g\Gamma g^{-1}$ in Γ. Then there is a unique non-empty compact set T satisfying*

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T), \tag{2.1}$$

and there is a subset $\Gamma_0 \subset \Gamma$ such that $\{\gamma(T); \gamma \in \Gamma_0\}$ is a tiling of $\mathbb{R}^d$.

Remark 2.2.

1. Iterating (2.1), one can write explicitly
$$T = \left\{ \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a); (\delta_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}} \right\},$$
where a can be any point of $\mathbb{R}^d$. Given an infinite sequence of digits $(\delta_n)_n$, the limit point $\lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a)$ does not depend on the choice of a. Therefore we will frequently write the sequence of digits to denote the corresponding point of T:
$$\lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a) =: .(\delta_1, \delta_2, \dots, \delta_n, \dots)_g \tag{2.2}$$
for $(\delta_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$. Note that a given point may have several such addresses. For the approximations, we will often choose $a = \mathrm{Fix}(g)$, the fixed point of g.
2. Without loss of generality, $\mathrm{id} \in \mathcal{D}$. Indeed, if $\mathcal{D} = \{\delta_1, \dots, \delta_k\}$ and T is the solution of $g(T) = \bigcup_{i=1}^k \delta_i(T)$, then let $g' := \delta_1^{-1} g$ and $\mathcal{D}' = \{\mathrm{id}, \delta_1^{-1}\delta_2, \dots, \delta_1^{-1}\delta_k\}$. Then T is also the solution of $g'(T) = \bigcup_{\delta\in\mathcal{D}'} \delta(T)$.
3. As for lattice self-affine tiles (Γ isomorphic to $\mathbb{Z}^d$), every crystile T fulfills the following properties.
   (*i*) $\bigcup_{\gamma\in\Gamma} \gamma(T) = \mathbb{R}^d$.
   (*ii*) $\mathrm{int}(T) \neq \emptyset$.
   (*iii*) $T = \overline{\mathrm{int}(T)}$.

(*iv*) $\lambda_d(\partial T) = 0$, where λ_d is the d-dimensional Lebesgue measure.
Properties (*i*) − (*iii*) were proved by Gelbrich [7], whereas (*iv*) can be shown as in the lattice case [15].

From now on Γ is a crystallographic group, $g(x) = \mathbf{A}x + b$ an expanding affine mapping of $\mathbb{R}^d$ that conjugates Γ into itself ($g \circ \Gamma \circ g^{-1} \subset \Gamma$) and $\mathcal{D} \subset \Gamma$ is a complete set of right coset representatives of $\Gamma / g\Gamma g^{-1}$ with $\mathrm{id} \in \mathcal{D}$.

We will use the following notations. For $\gamma \in \Gamma$, we write M_γ its linear part and t_γ its translation part. Thus for all $x \in \mathbb{R}^n$, $\gamma(x) = M_\gamma x + t_\gamma$. We denote by $\mathbb{P}$ the set of linear parts of elements of Γ. Since the point group is finite, $\mathbb{P}$ is also finite. Moreover, $\mathbf{A}\mathbb{P} = \mathbb{P}\mathbf{A}$, because $g\Gamma g^{-1} \subset \Gamma$.

Proposition 2.3. *Let T be the crystallographic replication tile satisfying $g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T)$. Then $\Gamma(T) := \{\gamma(T); \gamma \in \Gamma\}$ is a multiple tiling of the space,* i.e.*, there is a p such that almost all points of the space are covered by p sets of $\Gamma(T)$.*

Proof. The proof runs as in the lattice case (see also [16]). By Remark 2.2.3.(*i*), $\bigcup_{\gamma \in \Gamma} \gamma(T) = \mathbb{R}^d$. Suppose that there are two integers $m_1 < m_2$ and two sets M_1, M_2 such that

- each $x \in M_1$ is covered m_1 times.
- each $x \in M_2$ is covered at least m_2 times.
- $\lambda_d(M_1) > 0, \;\; \lambda_d(M_2) > 0$.

Now by Remark 2.2.3.(*iv*) the boundaries have Lebesgue measure zero : $\lambda_d\left(\bigcup_{\gamma \in \Gamma} \partial\gamma(T)\right) = 0$. Therefore, for each $i = 1, 2$, there is an $x_i \in M_i \setminus \bigcup_{\gamma \in \Gamma} \partial\gamma(T)$. Consequently, one can find $\epsilon > 0$ such that the balls of radius ϵ around x_i are contained in M_i. In particular,

$$\bigcup_{\gamma \in \Gamma} \gamma(B_\epsilon(x_1)) = \bigcup_{\gamma \in \Gamma} B_\epsilon(\gamma(x_1)) \subset M_1$$

and M_1 is relatively dense.

On the other hand, the ball $B_\epsilon(x_2)$ is contained in M_2. We show inductively that the sets $g^n(B_\epsilon(x_2))$ remain in M_2 for all $n \geq 0$. Let $x \in B_\epsilon(x_2)$. Then there are $\gamma_1, \ldots, \gamma_{m_2}$ all distinct such that $x \in B_\epsilon(x_2) \subset \bigcap_{j=1}^{m_2} \gamma_j(T)$. Hence

$$g(x) \in \bigcap_{j=1}^{m_2} g\gamma_j(T) = \bigcap_{j=1}^{m_2} g\gamma_j g^{-1}\delta_{i_j}(T),$$

where $\delta_{i_j} \in \mathcal{D}$. Here, all of the $g\gamma_j g^{-1}\delta_{i_j}$ are distinct. Indeed, if $j \neq j'$, then $g\gamma_j g^{-1}\delta_{i_j} \neq g\gamma_{j'} g^{-1}\delta_{i_{j'}}$, as $\mathcal{D}$ is a right residue system modulo $g\Gamma g^{-1}$ and $\gamma_j \neq \gamma_{j'}$. Thus $g(x) \in M_2$, and this holds for all $x \in B_\epsilon(x_2)$. Inductively,

$g^n(B_\epsilon(x_2)) \subset M_2$ for all n. This means that M_2 contains arbitrarily large sets, hence it must intersect M_1, a contradiction. $\square$

We will see that if $(\Gamma, g, \mathcal{D})$ is a so-called crystallographic number system, then the value of p in Proposition 2.3 is 1.

We define a dynamical system on Γ. Since $\mathcal{D}$ is a complete right residue system of $\Gamma/g\Gamma g^{-1}$, each $\gamma \in \Gamma$ has a unique decomposition $\gamma = g\gamma' g^{-1}\delta$ with $\delta \in \mathcal{D}$ and $\gamma' \in \Gamma$. We set

$$\begin{array}{rccc} \phi: & \Gamma & \to & \Gamma \\ & \gamma & \mapsto & \gamma'. \end{array}$$

Therefore every $\gamma \in \Gamma$ expands in $\gamma = g\phi(\gamma)g^{-1}\delta$ for some $\delta \in \mathcal{D}$. We denote by $\mathcal{P}$ the set of periodic points of the dynamical system (Γ, ϕ):

$$\mathcal{P} = \{\gamma \in \Gamma \,;\, \exists\, n \geq 1, \phi^n(\gamma) = \gamma\}.$$

Lemma 2.4. *For each $\gamma \in \Gamma$, there is an $n \in \mathbb{N}$ such that $\phi^n(\gamma) \in \mathcal{P}$. Moreover, $\mathcal{P}$ is finite.*

Proof. We will show that for an element $\gamma \in \Gamma$ the translation parts of the iterates $\phi^n(\gamma)$ end up in a uniformly bounded region of the lattice Λ. Thus there are finitely many possible translation parts, and combining with the finitely many possible linear parts in the crystallographic group, we see that the iterates $\phi^n(\gamma)$ eventually remain inside a finite set of isometries, proving the assertions of the lemma.

The mapping g^{-1} need not be a uniform contraction for the Euclidean norm $||\cdot||$. Thus we will make use of the norm introduced by Lind in [17]. Let

$$\max\{1/\lambda; \lambda \in \mathrm{sp}(\mathbf{A})\} < \rho < 1$$

be larger than all eigenvalues of $\mathbf{A}^{-1}$. Then g^{-1} is a uniform contraction with respect to the norm

$$||x||' := \sum_{k\geq 0} \rho^k ||\mathbf{A}^{-k}x||.$$

Moreover, for any linear part $O \in \mathbb{P}$,

$$\begin{array}{rcl} ||Ox||' & = & \sum_{k\geq 0} \rho^k ||\mathbf{A}^{-k}Ox|| \\ & = & \sum_{k\geq 0} \rho^k ||O_k\mathbf{A}^{-k}x|| \\ & = & \sum_{k\geq 0} \rho^k ||\mathbf{A}^{-k}x|| \\ & = & ||x||' \end{array}$$

where O_k is such that $\mathbf{A}^{-k}O = O_k\mathbf{A}^{-k}$. Such an O_k exists because $\mathbf{A}\mathbb{P} = \mathbb{P}\mathbf{A}$.

From $\phi(\gamma) = g^{-1}\gamma\delta^{-1}g$ we deduce that

$$t_{\phi(\gamma)} = \mathbf{A}^{-1}\left(M_\gamma M_{\delta^{-1}} b \ + \ M_\gamma t_{\delta^{-1}} \ + \ t_\gamma \ - \ b\right).$$

Let $C_1 := \max\{2||b||' + ||t_{\delta^{-1}}||'; \delta \in \mathcal{D}\}$. Then

$$||t_{\phi(\gamma)}||' \ \leq \ \rho(C_1 + ||t_\gamma||').$$

By iteration,

$$||t_{\phi^n(\gamma)}||' \ \leq \ \rho C_1 \sum_{k=0}^{n-1} \rho^k \ + \ \rho^n ||t_\gamma||'.$$

Let $C_2 := \rho C_1 \sum_{k=0}^{\infty} \rho^k$. Then for each $\gamma \in \Gamma$, there is a n_γ such that

$$n \geq n_\gamma \Rightarrow ||t_{\phi^n(\gamma)}||' \leq 2C_2.$$

Therefore, the translation parts of the iterates $\phi^n(\gamma)$ end up in a finite subset of Λ. Combining with the finitely many possible linear parts, we obtain that the iterates themselves end up in a finite set $\Gamma_1 \subset \Gamma$. Therefore, $\phi^n(\gamma) = \phi^m(\gamma)$ for some $m > n \geq n_\gamma$, meaning that $\phi^n(\gamma)$ eventually belongs to $\mathcal{P}$. Finally, we see that $\mathcal{P} \subset \Gamma_1$, thus $\mathcal{P}$ is finite. □

It follows from this lemma that each $\gamma \in \Gamma$ has an expansion

$$\begin{aligned} \gamma &= g^n\gamma_n \, g^{-1}\delta_{n-1} \, g^{-1}\delta_{n-2} \, \ldots \, g^{-1}\delta_0 \\ &= \underbrace{g^n\gamma_n g^{-n}}_{\in\Gamma} \, \underbrace{g^{n-1}\delta_{n-1}g^{-(n-1)}}_{\in\Gamma} \, \ldots \, \underbrace{g\delta_1 g^{-1}}_{\in\Gamma} \, \delta_0, \end{aligned}$$

where $\gamma_n = \phi^n(\gamma) \in \mathcal{P}$ for some $n \in \mathbb{N}$.

Definition 2.5. We call $(\Gamma, g, \mathcal{D})$ a *crystallographic number system* (or *crystem* for short) if the set $\mathcal{P}$ of periodic points of ϕ is trivial, that is, $\mathcal{P} = \{id\}$.

By definition, $(\Gamma, g, \mathcal{D})$ is a crystem *if and only if* every element $\gamma \in \Gamma$ has a unique finite expansion

$$\gamma = g^n\delta_n g^{-n} \, g^{n-1}\delta_{n-1} \, g^{-(n-1)} \ \ldots \ g\delta_1 g^{-1} \, \delta_0 \tag{2.3}$$

for some $n \geq 0$, $\delta_j \in \mathcal{D}$ and $\delta_n \neq \mathrm{id}$ as soon as $\gamma \neq \mathrm{id}$. We call $n =: l(\gamma)$ the *length of* γ and simply write

$$\gamma = (\delta_n, \ldots, \delta_0)_g. \tag{2.4}$$

This notation can be combined with the notation (2.2) to write points of a tile $\gamma(T)$:

$$\gamma(z) = (\delta_n, \ldots, \delta_0)_g.(\delta_1', \delta_2', \ldots)_g$$

for $\gamma \in \Gamma$ and $z = \lim_{n\to\infty} g^{-1}\delta_1' \ldots g^{-1}\delta_n'(a) \in T$.

The following lemma shows that g acts as a shift on the set of strings

$$\left\{(\delta_n, \ldots, \delta_0)_g.(\delta_1', \delta_2', \ldots)_g; (\delta_j)_{0\leq j\leq n} \in \mathcal{D}^{n+1}, (\delta_j')_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}\right\}.$$

Lemma 2.6. *Let* $\delta_0, \ldots, \delta_n, \delta'_1, \delta'_2, \ldots \in \mathcal{D}$. *Then*

$$g(\delta_n, \ldots, \delta_0)_g.(\delta'_1, \delta'_2, \ldots)_g = (\delta_n, \ldots, \delta_0, \delta'_1)_g.(\delta'_2, \delta'_3, \ldots)_g. \tag{2.5}$$

Proof. By definition,

$$\begin{aligned}
&g(\delta_n, \ldots, \delta_0)_g.(\delta'_1, \delta'_2, \ldots)_g\\
&= \lim_{k\to\infty} g g^n \delta_n g^{-n}\; g^{n-1}\delta_{n-1}\; g^{-(n-1)}\; \ldots\; g\delta_1 g^{-1}\; \delta_0 g^{-1}\delta'_1 \ldots g^{-1}\delta'_k(a)\\
&= \lim_{k\to\infty} g^{n+1}\delta_n g^{-(n+1)}\; g^n\delta_{n-1}\; g^{-(n)}\; \ldots\; g^2\delta_1 g^{-2}\; g\delta_0 g^{-1}\delta'_1 g^{-1}\delta'_2 \ldots g^{-1}\delta'_k(a)\\
&= (\delta_n, \ldots, \delta_0, \delta'_1)_g.(\delta'_2, \delta'_3, \ldots)_g.
\end{aligned}$$

□

Theorem 2.7. *Let* $(\Gamma, g, \mathcal{D})$ *be a crystallographic number system and* T *be the associated crystallographic reptile. Then* $\{\gamma(T); \gamma \in \Gamma\}$ *is a tiling of* $\mathbb{R}^d$.

Proof. We already know from Remark 2.2.3 that $\{\gamma(T); \gamma \in \Gamma\}$ is a covering of $\mathbb{R}^d$. Suppose that $(\Gamma, g, \mathcal{D})$ is a crystem but that two tiles overlap. Without loss of generality, this means that

$$\operatorname{int}(T) \cap \gamma(\operatorname{int}(T)) \neq \emptyset$$

for some $\gamma \neq \mathrm{id}$. In particular, one can find a point with two addresses:

$$\lim_{n\to\infty} g^{-1}\delta_1 \ldots g^{-1}\delta_n(\mathrm{Fix}(g)) = \gamma \lim_{n\to\infty} g^{-1}\delta'_1 \ldots g^{-1}\delta'_n(\mathrm{Fix}(g))$$

for some sequences $(\delta_n), (\delta'_n)$ of digits. This can be written in the short form

$$.(\delta_1, \delta_2, \ldots)_g = (\epsilon_m, \ldots, \epsilon_0)_g.(\delta'_1, \delta'_2, \ldots)_g,$$

where $(\epsilon_m, \ldots, \epsilon_0)_g$ is the unique representation of γ. In particular, $\epsilon_m \neq \mathrm{id}$. By Remark 2.2.1., it is not restrictive to require that $(\delta_n)_{n\in\mathcal{N}}$ is eventually id:

$$.(\delta_1, \ldots, \delta_p, \mathrm{id}, \ldots, \mathrm{id}, \ldots)_g = (\epsilon_m, \ldots, \epsilon_0)_g.(\delta'_1, \delta'_2, \ldots)_g.$$

Taking the image of the above equality by g^p and using Lemma 2.6 we have

$$\underbrace{(\delta_1, \ldots, \delta_p)_g}_{=:\,\gamma_1}.(\mathrm{id}, \mathrm{id}, \ldots)_g = \underbrace{(\epsilon_m, \ldots, \epsilon_0, \delta'_1, \ldots, \delta'_p)_g}_{=:\,\gamma_2}.(\delta'_{p+1}, \ldots)_g.$$

Note that $\gamma_1 \neq \gamma_2$, since these elements have different lengths. Let $(\epsilon'_q, \ldots, \epsilon'_0)_g$ be the representation of $\gamma_1^{-1}\gamma_2$. Then $\epsilon'_q \neq \mathrm{id}$ and

$$\mathrm{Fix}(g) = (\epsilon'_q, \ldots, \epsilon'_0)_g.(\delta'_{p+1}, \ldots)_g$$

Now, applying $g^k, k = 0, 1, 2, \ldots$ to this equality, we obtain that the point $\mathrm{Fix}(g)$ belongs to the tiles $\alpha_0(T), \alpha_1(T), \alpha_2(T), \ldots$, where the elements $\alpha_k \in \Gamma$ are all distinct, since they have strictly increasing length $q + k$. However, by compactness of the tiles and discreteness of Γ, the covering $\{\gamma(T); \gamma \in \Gamma\}$ is locally finite, thus only finitely many tiles can meet at $\mathrm{Fix}(g)$, which is a contradiction. □

Definition 2.8. Let $\{\gamma(T); \gamma \in \mathcal{J}\}$ ($\mathcal{J} \subset \Gamma$) be a covering of the space. Then $x \in \gamma_0(T)$ ($\gamma_0 \in \mathcal{J}$) is *exclusive inner point* of $\gamma_0(T)$ with respect to $\mathcal{J}$ if it belongs only to $\gamma_0(T)$, that is,

$$x \in \gamma_0(T) \setminus \bigcup_{\gamma \in \mathcal{J}, \gamma \neq \gamma_0} \gamma(T).$$

Theorem 2.9. *Let T be a crystallographic replication tile with respect to the data $(\Gamma, g, \mathcal{D})$. Then $(\Gamma, g, \mathcal{D})$ is a crystallographic number system if and only if $\mathrm{Fix}(g)$ is an exclusive inner point of T with respect to the group Γ.*

Proof. Assuming that $(\Gamma, g, \mathcal{D})$ is a crystallographic number system, suppose that $\mathrm{Fix}(g)$ is not an inner point of T, that is,

$$\mathrm{Fix}(g) = (\epsilon_m \ldots, \epsilon_0)_g.(\delta_1, \delta_2, \ldots)_g.$$

with $\epsilon_m \neq \mathrm{id}$. As in the proof of Theorem 2.7, applying successively g^k to this equality for $k = 0, 1, 2, \ldots$, it follows that $\mathrm{Fix}(g)$ belongs to infinitely many tiles $\gamma(T)$, in contradiction with the local finiteness of the tiling $\{\gamma(T); \gamma \in \Gamma\}$. Therefore, $\mathrm{Fix}(g)$ belongs to no other tile and is an exclusive inner point of T.

Suppose now that $(\Gamma, g, \mathcal{D})$ is not a crystallographic number system, that is, there is a periodic point $\gamma \neq \mathrm{id}$. Then there is a $m \in \mathbb{N}$ and digits $\epsilon_0, \ldots, \epsilon_m$ with ϵ_m such that

$$\gamma = g^{m+1} \gamma g^{-1} \epsilon_m \ldots g^{-1} \epsilon_0.$$

Iterating this p times leads to

$$\gamma = g^{p(m+1)} \gamma (g^{-1} \epsilon_m \ldots g^{-1} \epsilon_0)^p,$$

thus for all p, at the point $a \in \mathbb{R}^d$,

$$g^{-p(m+1)} \gamma(a) = \gamma (g^{-1} \epsilon_m \ldots g^{-1} \epsilon_0)^p(a).$$

At the limit, we obtain a point in two distinct tiles

$$\lim_{p \to \infty} g^{-p(m+1)}(\gamma(a)) = \mathrm{Fix}(g) = \gamma \lim_{p \to \infty} (g^{-1} \epsilon_m \ldots g^{-1} \epsilon_0)^p(a) \quad \in T \cap \gamma(T),$$

and $\mathrm{Fix}(g)$ is not an exclusive interior point of T. □

3. Number systems and counting automata

We introduce the counting automaton as a tool to check whether a given $(\Gamma, g, \mathcal{D})$ gives rise to a crystallographic number system. This automaton informs us about the residue class and the carry of the composition $\delta\gamma$ for each $\gamma \in \Gamma$ and each $\delta \in \mathcal{D}$. This is similar to the lattice case [23, 24].

Definition 3.1. Let $(\Gamma, g, \mathcal{D})$ be the data for a crystallographic reptile. The associated counting automaton $\mathcal{A}$ has:

- states : the elements $\gamma \in \Gamma$;
- edges : $\gamma \xrightarrow{\delta|\delta'} \gamma'$ iff $\delta\gamma = g\gamma' g^{-1}\delta'$, where $\gamma, \gamma' \in \Gamma$ and $\delta, \delta' \in \mathcal{D}$.

Remark 3.2.

1. In the edges above, the pair (γ', δ') is uniquely defined by the pair (γ, δ), because $\mathcal{D}$ is a complete residue system of $g\Gamma g^{-1}$ in Γ.
2. By definition, the following edges always belong to $\mathcal{A}$:
$$\gamma \xrightarrow{\mathrm{id}|\delta'} \phi(\gamma),$$
with $\gamma = g\phi(\gamma)g^{-1}\delta'$.

For a set $S \subset \Gamma$ we define $\mathcal{A}(S)$ as the subautomaton of $\mathcal{A}$ generated by S. This is the smallest subautomaton of $\mathcal{A}$ whose set of states S' contains S and which is stable by left composition by any digit δ: for all $\gamma \in S'$ and $\delta \in \mathcal{D}$, the edge $\gamma \xrightarrow{\delta|\delta'} \gamma'$ remains in $\mathcal{A}(S)$.

Proposition 3.3. *Let* $(\Gamma, g, \mathcal{D})$ *be the data for a crystallographic reptile and* $S = S^{-1} \subset \Gamma$ *generating the whole group* Γ*, that is,* $\langle S \rangle = \Gamma$*. Then* $(\Gamma, g, \mathcal{D})$ *is a crystallographic number system if and only if for every state* γ *of* $\mathcal{A}(S)$ *there is the walk*

$$\gamma \xrightarrow{id|\delta_0} \gamma_1 \xrightarrow{id|\delta_1} \dots \xrightarrow{id|\delta_m} id \tag{3.1}$$

in the counting automaton for some $\delta_0, \dots, \delta_m \in \mathcal{D}$.

Proof. By definition of the edges, $(\Gamma, g, \mathcal{D})$ is a crystem if and only if a walk (3.1) exists for every $\gamma \in \Gamma$. Suppose that this property is satisfied by the set S' of states of $\mathcal{A}(S)$. Let γ, γ' be two elements of S'. Then γ has a finite expansion

$$\gamma = (\delta_{m-1}, \dots, \delta_0)_g,$$

and

$$\gamma\gamma' = (\delta'_p, \dots, \delta'_m, \delta'_{m-1}, \dots, \delta'_0)_g,$$

where the digits δ'_k $(0 \le k \le p)$ are defined via the walk

$$\gamma' \xrightarrow{\delta_0|\delta'_0} \gamma_1 \xrightarrow{\delta_1|\delta'_1} \dots \xrightarrow{\delta_{m-1}|\delta'_{m-1}} \gamma_m \xrightarrow{\mathrm{id}|\delta'_m} \dots \xrightarrow{\mathrm{id}|\delta'_p} \gamma_{p+1} = \mathrm{id}$$

in $\mathcal{A}(S)$. Thus $\gamma\gamma'$ also has a finite expansion. Since $S \subset S'$ generates Γ, we can infer that every element of Γ has a finite expansion, that is, $(\Gamma, g, \mathcal{D})$ is a crystallographic number system. □

4. Examples

In the first part of this section we treat six examples encountered in the literature and decide whether they correspond to crystallographic number systems or not. To this matter, we construct a finite automaton $\mathcal{A}(S)$ for some generator S of the group Γ in question and use Proposition 3.3. In the second part of the section we give a whole class of crystallographic number systems. The topological study of the corresponding tiles will be part of a forthcoming work.

Example 1. This example corresponds to Gelbrich's picture [7, Fig. 6 (i)]. Γ is the crystallographic group $p2$ in $\mathbb{R}^2$, *i.e.*,

$$\Gamma = \{a^p b^q c^r \ : \ p, q \in \mathbb{Z}\ , \ r \in \{0,1\}\}$$

where the isometries a, b are the canonical translations and c is the rotation by π around the origin:

$$a(x,y) = (x+1, y), \ \ b(x,y) = (x, y+1), \ \ c(x,y) = (-x,-y).$$

We consider

$$g(x,y) = (y, -3x - \frac{1}{2}), \ \ \mathcal{D} = \{\mathrm{id}, b, c\}.$$

Let $S = \{a^{-1}c, b^{\pm 1}, c\}$. Then the counting subautomaton $\mathcal{A}(S)$ is depicted on Figure 1. For simplicity, we wrote only the first digit on the label of each edge. Then one can see on this automaton that there is a walk

$$\gamma \xrightarrow{\mathrm{id}} \gamma_1 \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \mathrm{id} \tag{4.1}$$

starting from every state γ of this automaton. By Proposition 3.3, $(\Gamma, g, \mathcal{D})$ is a crystallographic number system.

Example 2. This example corresponds to Gelbrich's picture [7, Fig. 6 (b)]. The crystallographic group is again $p2$, the mapping and the digits are the following:

$$g(x,y) = (-y, 3x+1), \ \ \mathcal{D} = \{\mathrm{id}, b, c\}.$$

The digit tiles and a counting subautomaton are represented on Figure 2. Similarly as for Example 1, one can show that $(\Gamma, g, \mathcal{D})$ is a crystallographic number system.

Example 3. For this example, we consider the planar crystallographic group $p3$, generated by the translations

$$a(x,y) = (x+1, y), \ \ b(x,y) = (x + 1/2, y + \sqrt{3}/2)$$

and the $2\pi/3$-rotation d around the origin. Choosing

$$g(x,y) = \sqrt{3}(y, -x), \quad \mathcal{D} = \{\mathrm{id}, ad^2, bd^2\},$$

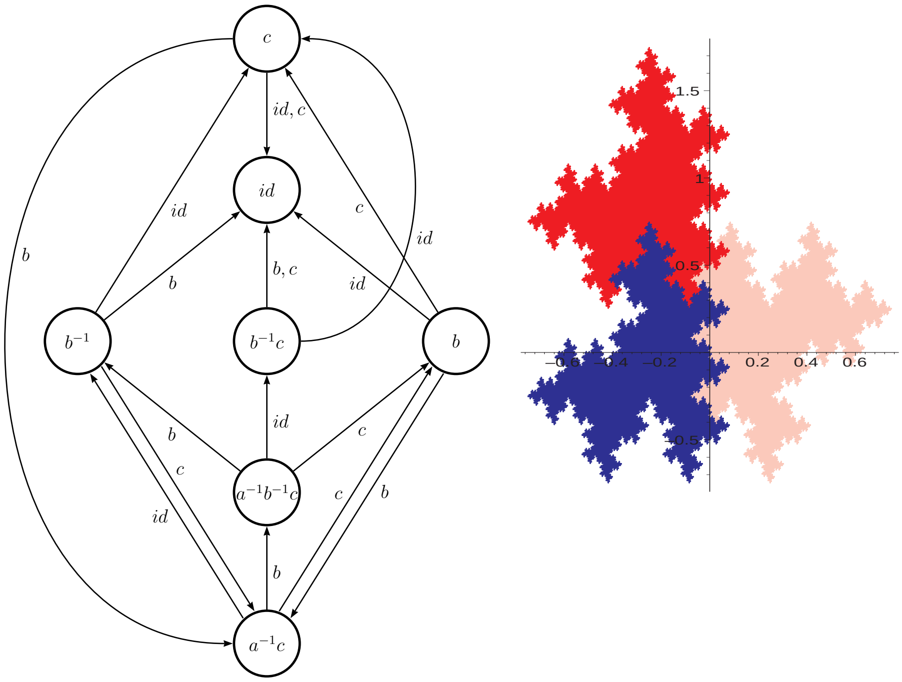

FIGURE 1. Example 1. Digit tiles $T, b(T), c(T)$ and counting subautomaton.

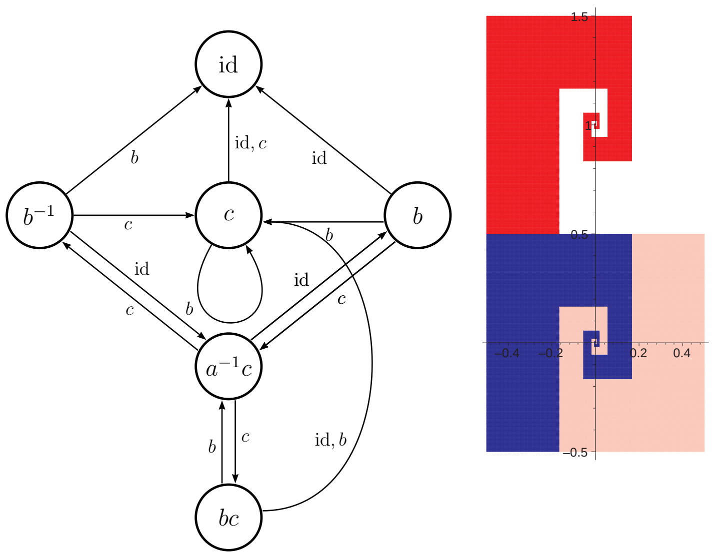

FIGURE 2. Example 2. Digit tiles $T, b(T), c(T)$ and counting subautomaton.

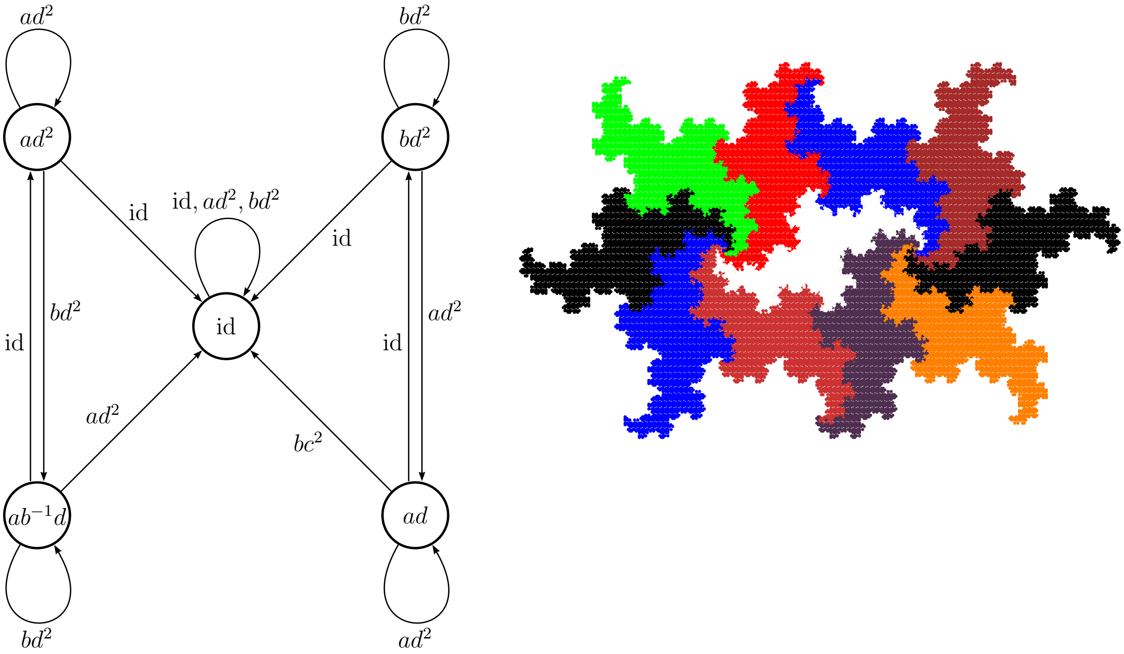

FIGURE 3. Example 3. Tiling by the terdragon and counting subautomaton.

we obtain the terdragon of Figure 3 (see also [4, Fig. 15],[7, Fig. 9]). Using the counting subautomaton depicted in this figure and Proposition 3.3, we can prove that $(\Gamma, g, \mathcal{D})$ is a crystallographic number system.

Remark 4.1. It follows from Theorem 2.7 that each tile T of these examples tiles the plane by the whole crystallographic group Γ. In particular, their Lebesgue measure is respectively $1/2, 1/2$ and $\sqrt{3}/8$.

We now give counterexamples.

Example 4. The crystallographic group is $p2$ (see Example 1), the example is taken from [7, Fig. 8 (c)]

$$g(x, y) = (-y, -3x - y), \quad \mathcal{D} = \{\mathrm{id}, b, a^{-1}c\}.$$

One reads off from the counting subautomaton that c is a periodic point of the dynamical system $(p2, \phi)$, since $\phi(c) = c$.

Example 5. The Heighway dragon is constructed as follows. Let $p4$ be the planar crystallographic group generated by the canonical translations a, b (see Example 1) and the rotation e by $\pi/2$ around the origin. The

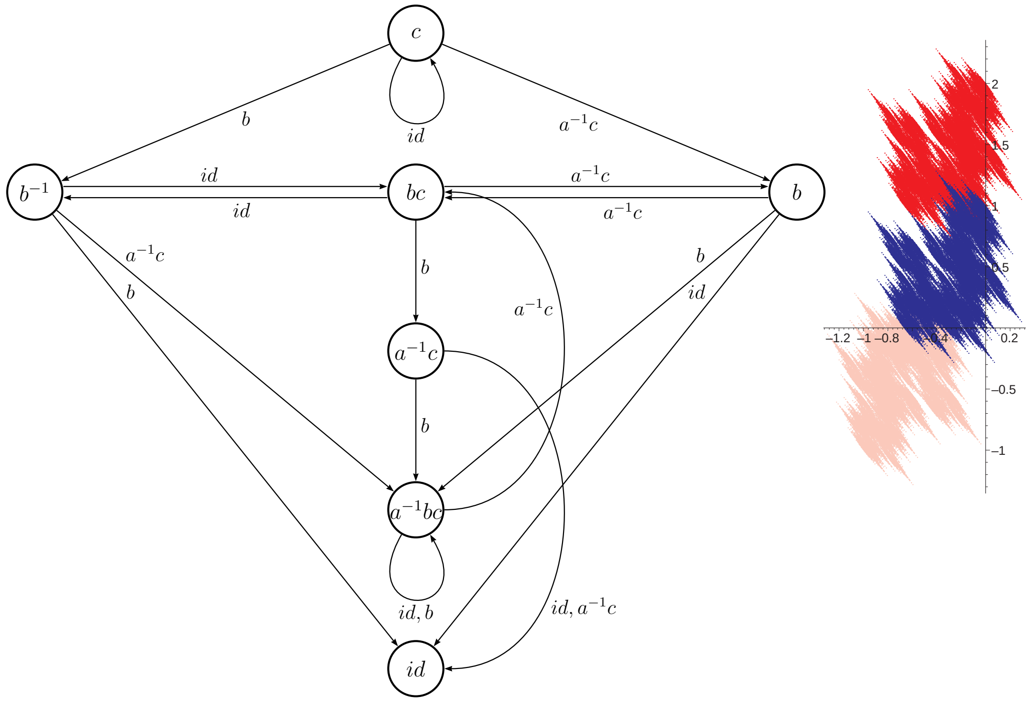

FIGURE 4. Example 4. Digit tiles $T, b(T), a^{-1}c(T)$ and counting subautomaton.

expansion and the digit set are

$$g(x, y) = (x + y, y - x), \quad \mathcal{D} = \{\mathrm{id}, \delta = abe\}.$$

A counting subautomaton is depicted in Figure 5. We see that e is a periodic element. More precisely, $\phi(e) = e$ and e is a fixed point of the dynamical system $(p4, \phi)$.

Example 6. The Levy dragon is also a $p4$ example that does not satisfy the number system property. The data for the Levy dragon are

$$g(x, y) = (x + y, y - x), \quad \mathcal{D} = \{\mathrm{id}, \delta = be^3\}.$$

It is easily computed that $e = geg^{-1}$, thus $\phi(e) = e$ and $(p4, g, \mathcal{D})$ is not a crystallographic number system. The counting subautomaton has more than twenty states and is not depicted here.

Remark 4.2. The tiles in the last three examples seem to induce tilings by the whole associated group Γ. This is already known for the Levy dragon (see [22]).

The rest of this section is devoted to a class of examples. We consider the planar crystallographic group $p2$, generated by the π-rotation c around

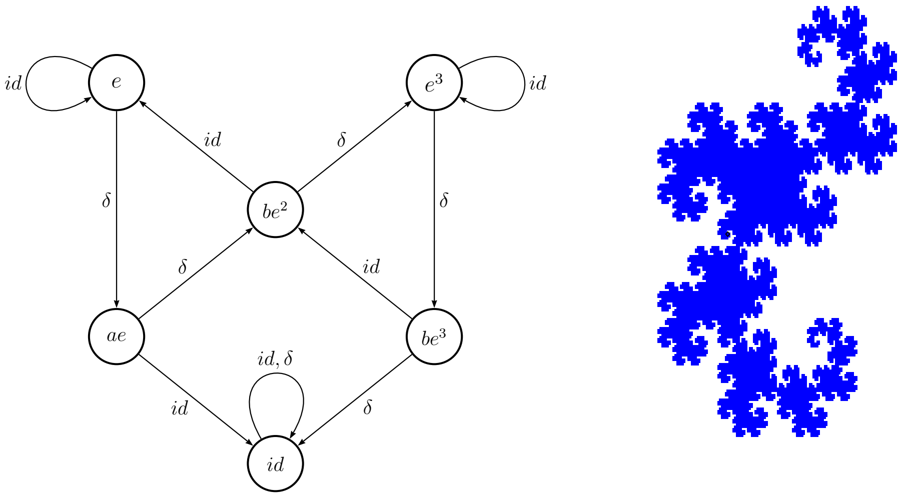

FIGURE 5. Heighway dragon and counting subautomaton.

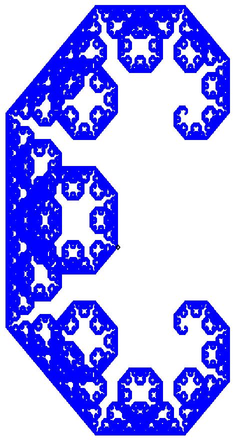

FIGURE 6. Levy dragon.

the origin and two translations a, b along the the lattice $\mathbb{Z}^2$:

$$a(x, y) = (x + 1, y) \quad b(x, y) = (x, y + 1) \quad c(x, y) = (-x, -y)$$

and $\Gamma = \{a^p b^q c^r; p, q \in \mathbb{Z}, r \in \{0, 1\}\}$. We choose

$$g(x, y) = \underbrace{\begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix}}_{M \in \mathbb{Z}^2 \times \mathbb{Z}^2} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}, \tag{4.2}$$

where $B := |\det(M)|$, and

$$\begin{array}{lcll} \mathcal{D} & = & \{\mathrm{id}, a, \ldots, a^{B-2}, c\} & \text{if } B \geq 3 \\ \mathcal{D} & = & \{\mathrm{id}, c\} & \text{if } B = 2. \end{array} \tag{4.3}$$

Then $\mathcal{D}$ is a complete right residue system of $\Gamma/g\Gamma g^{-1}$ if and only if $\epsilon = \pm 1$.

We will characterize the number systems among this class as follows.

Theorem 4.3. *Let $(g, \mathcal{D})$ as in (4.2)-(4.3). Then $(p2, g, \mathcal{D})$ is a crystallographic number system if and only if*

$$-1 \leq -\mathrm{Tr}(M) \leq \det(M) \geq 2.$$

In fact, the crystallographic data $(p2, g, \mathcal{D})$ is very close to the lattice data $(\mathbb{Z}^2, M, \mathcal{N})$, where

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\}. \tag{4.4}$$

Note that in the lattice case, we consider the above translation vectors rather than the translation mappings $\mathrm{id}, a, \ldots, a^{B-1}$. $(\mathbb{Z}^2, M, \mathcal{N})$ is a so-called *canonical number system* if and only if every integer vector t has a unique representation

$$t = d_0 + Md_1 + \ldots + M^l d_l$$

with $l \in \mathbb{N}$ and digits $d_i \in \mathcal{N}$. For this lattice data, the following was proved in [25].

Proposition 4.4 (*cf.* [25]). *Let $M := \begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix} \in \mathbb{Z}^2 \times \mathbb{Z}^2$ and*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\},$$

where $B := |\det(M)|$. Then $\mathcal{N}$ is a complete residue system of $\mathbb{Z}^2/M\mathbb{Z}^2$ if and only if $\epsilon = \pm 1$. In this case, $(\mathbb{Z}^2, M, \mathcal{N})$ is a canonical number system if and only if

$$-1 \leq -\mathrm{Tr}(M) \leq \det(M) \geq 2.$$

Proof of Theorem 4.3. We first show that a translation γ of $p2$ has a finite expansion in $(\Gamma, g, \mathcal{D})$ if and only if the corresponding translation vector t_γ has a finite expansion in $(\mathbb{Z}^2, M, \mathcal{N})$. To this matter, we give the exact correspondence between the two expansions. Let

$$\gamma = a^p b^q = g^m \delta_m g^{-m} \ldots g \delta_1 g^{-1} \delta_0.$$

Then c appears an even number of times, because γ is a translation. Let us consider the special case where c appears exactly twice in the following way : there are $1 < i_1 < i_2 + 1 = m$ with $\delta_{i_1+1} = \delta_{i_2+1} = c$ and the other digits are translations a^{k_i}. Note that

$$g^p c g^{-p}(x) = -x + M^{p-1}\begin{pmatrix} B-1 \\ 0 \end{pmatrix} + \ldots + M\begin{pmatrix} B-1 \\ 0 \end{pmatrix} + \begin{pmatrix} B-1 \\ 0 \end{pmatrix},$$

and

$$g^p a^k g^{-p}(x) = x + M^p\begin{pmatrix} k \\ 0 \end{pmatrix}.$$

Therefore, by a straightforward computation,

$$g^{i_1}\delta_{i_1}g^{-i_1}\ldots g\delta_1 g^{-1}\delta_0(x) = x + M^{i_1}\begin{pmatrix} k_{i_1} \\ 0 \end{pmatrix} + \ldots + M\begin{pmatrix} k_1 \\ 0 \end{pmatrix} + \begin{pmatrix} k_0 \\ 0 \end{pmatrix}.$$

After the first occurrence of c, we have

$$g^{i_1+1}cg^{-i_1-1}g^{i_1}\delta_{i_1}g^{-i_1}\ldots g\delta_1 g^{-1}\delta_0(x)$$

$$= -x + M^{i_1}\begin{pmatrix} B-1-k_{i_1} \\ 0 \end{pmatrix} + \ldots + M\begin{pmatrix} B-1-k_1 \\ 0 \end{pmatrix} + \begin{pmatrix} B-1-k_0 \\ 0 \end{pmatrix}.$$

Now, from $i_1 + 2$ to i_2, again translations are considered:

$$g^{i_2}\delta_{i_2}g^{-i_2}\ldots\delta_0(x)$$

$$= -x + M^{i_2}\begin{pmatrix} k_{i_2} \\ 0 \end{pmatrix} + \ldots + M^{i_1+2}\begin{pmatrix} k_{i_1+2} \\ 0 \end{pmatrix}$$

$$+M^{i_1}\begin{pmatrix} B-1-k_{i_1} \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} B-1-k_0 \\ 0 \end{pmatrix}$$

and finally after the second occurrence of c

$$g^m c g^{-m}\ldots\delta_0(x) = x + t_\gamma$$

$$= x + M^{m-1}\begin{pmatrix} B-1-k_{m-1} \\ 0 \end{pmatrix} + \ldots + M^{i_1+2}\begin{pmatrix} B-1-k_{i_1+2} \\ 0 \end{pmatrix}$$

$$+M^{i_1+1}\begin{pmatrix} B-1 \\ 0 \end{pmatrix} + M^{i_1}\begin{pmatrix} k_{i_1} \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} k_0 \\ 0 \end{pmatrix}.$$

Since all $\begin{pmatrix} B-1-d \\ 0 \end{pmatrix}$ above belong to $\mathcal{N}$, the last lines contains the finite expansion of t_γ we are looking for.

For any γ translation of $p2$, the correspondence between crystallographic and lattice expansions can be established in a similar way. For simplicity, we depict this correspondence as an automaton on Figure 7. Let $(\delta_m, \ldots, \delta_0)_g$ represent the translation γ. Then one obtains the string representing t_γ by feeding the automaton from the state P with the string of digits $(\delta_m, \ldots, \delta_0)_g$

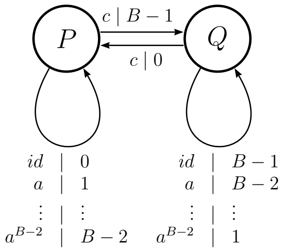

FIGURE 7. From crystallographic data to lattice data.

from right to left. The corresponding digits $(d_m, \dots, d_0)_M$ are read off from the edges.

In the same way, if t is an integer vector having a finite representation $(d_m, \dots, d_0)_M$, the automaton of Figure 7 gives the representation $(\delta_m, \dots, \delta_0)_g$ of the corresponding translation $\gamma(x) = x + t$ in $p2$. Hence a translation γ has a finite representation in $(p2, g, \mathcal{D})$ if and only if its translation vector t_γ has a finite representation in $(\mathbb{Z}^2, M, \mathcal{N})$.

Therefore, by Proposition 4.4, a necessary condition for $(p2, g, \mathcal{D})$ to be a crystallographic number system is that

$$-1 \leq -\mathrm{Tr}(M) \leq \det(M) \geq 2.$$

Now we check that if this condition is fulfilled, then also the rotations $a^p b^q c$ $(p, q \in \mathbb{Z})$ have a finite representation in $(p2, g, \mathcal{D})$. This follows from the counting action of c. We depict in Figure 8 part of the counting automaton involving the state c. Every translation $a^p b^q$ has a representation $(\delta_m, \dots, \delta_0)_g$, and the composition by c to obtain the representation of $a^p b^q c$ is given by the automaton of Figure 8. The output string has the form $(\delta'_m, \dots, \delta'_0)_g$: in particular, it remains finite.

$\square$

Eventually, we give a correspondence between the crystallographic tiles and the associated lattice tiles of the above class. More precisely, let $(g, \mathcal{D})$ as in (4.2)-(4.3) and $\mathcal{N}$ as in (4.4). Furthermore, let T be the solution of

$$g(T) = \mathcal{D}(T)$$

and T^l be the solution of

$$MT^l = T^l + \mathcal{N}.$$

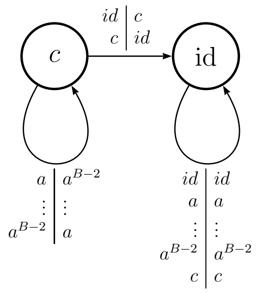

FIGURE 8. Counting automaton restricted to $\{\mathrm{id}, c\}$.

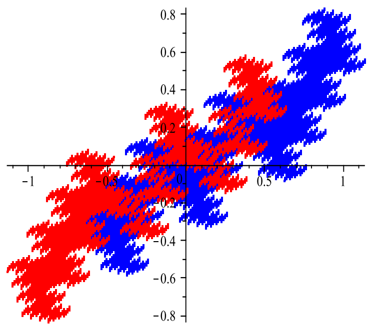

FIGURE 9. $T \cup (-T)$ for $\mathrm{Tr}(M) = -3,\ \det(M) = 4$.

Then, by unicity of the solutions of the above equations, it is easy to see that

$$T^l = T \cup (-T) \quad + (M - I_2)^{-1} \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}.$$

Here, I_2 is the 2×2 identity matrix. Therefore, T^l is a translate of $T \cup (-T)$. The topological study for T^l was investigated in [3]. However T and T^l may have a very different topological behaviour, as shown in Figure 9. We postpone this study to a forthcoming paper.

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