Families of labyrinth fractals and their magic

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FWF stand-alone project P27050-N26
and F5508-N26, part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications"

joint work with Bertran Steinsky and Gunther Leobacher

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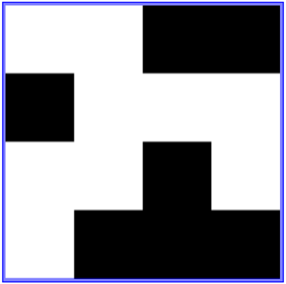
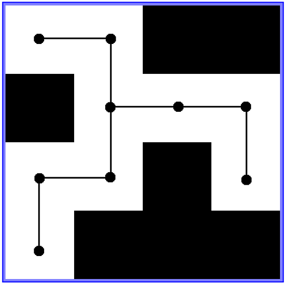
ÖMG-DMV, Salzburg, September 11-15, 2017

- LLC, B. Steinsky, **Paths of infinite length in** 4×4 **- labyrinth fractals**, *Geometriae Dedicata (2009)*
- LLC, B. Steinsky, **Curves of Infinite Length in Labyrinth Fractals**, *Proceedings of the Edinburgh Math. Soc. (2010)*
- LLC, B. Steinsky, **Mixed labyrinth fractals**, *J. Topology and its Applications* (2017)
- LLC, G. Leobacher, **A note on lengths of arcs in mixed labyrinth fractals**, *Monatshefte f. Mathematik* (2017)

(Labyrinth) patterns. The graph of a (labyrinth) pattern

A 4×4 (labyrinth) pattern and its graph

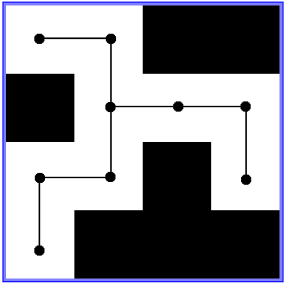




What is a labyrinth pattern?

Property 1 (The Tree Property)

The graph of the labyrinth pattern



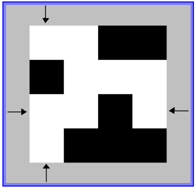
Property

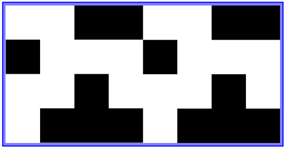
The graph of the labyrinth pattern is a tree. (the Tree Property)

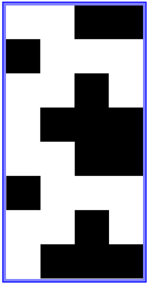
Property 2 (The Exits Property)

Property

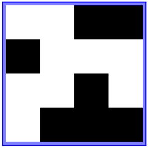
There is exactly one horizontal and exactly one vertical exit pair in the labyrinth pattern. (the Exits Property)





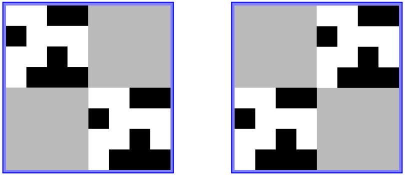


Property 3 (The Corner Property)



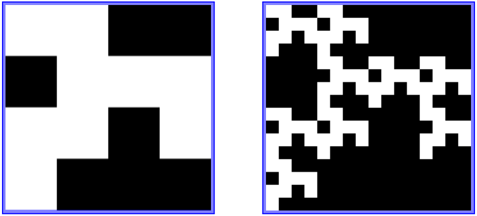
Property

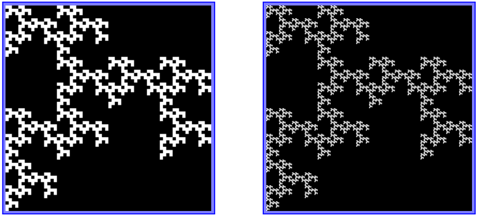
If there is a white square at a corner of the labyrinth pattern, then there is no white square at the diagonally opposite corner of the labyrinth pattern. (the Corner Property)



Construction of a labyrinth fractal

A 4×4-labyrinth pattern/set.



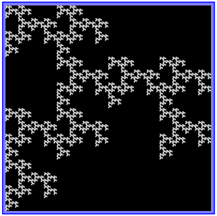


. . . labyrinth fractal

Dendrites

Theorem

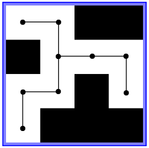
For **all** *$m \times m$ labyrinth patterns, the constructed self-similar fractal L is a dendrite.*



Dendrite

A *dendrite* is a connected and locally connected compact Hausdorff space that contains no simple closed curve.

A Fourth Property



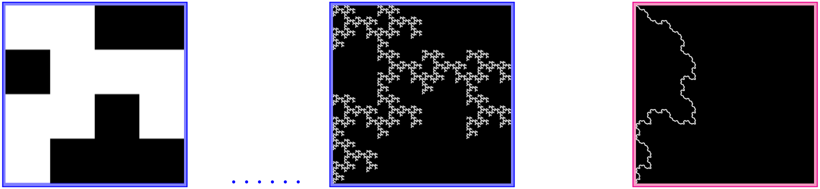
Horizontally Blocked

A labyrinth pattern is called *horizontally blocked* if the row (of squares) from the left exit to the right exit contains at least one black square.

Vertically Blocked

A labyrinth pattern is called *vertically blocked* if the column (of squares) from the top exit to the bottom exit contains at least one black square.

Self-similar labyrinth fractals. Main Result



Theorem

Let L_∞ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-labyrinth pattern.

(a) *Between any two points in L_∞ there is a unique arc a.*

(b) *The length of a is infinite and* $\dim_B(a) = \frac{\log r}{\log m}$.

(c) *The set of all points, at which no tangent to a exists, is dense in a.*

Mixed labyrinth fractals

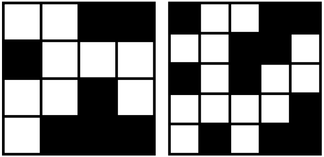


Figure: Two labyrinth patterns, $\mathcal{A}_1$ (a 4-pattern) and $\mathcal{A}_2$ (a 5-pattern)

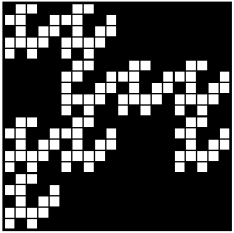


Figure: The mixed (labyrinth) set $\mathcal{W}_2$, constructed based on the above patterns $\mathcal{A}_1$ and $\mathcal{A}_2$, that can also be viewed as a 20-pattern

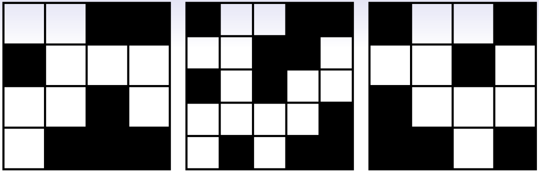


Figure: Labyrinth patterns: $\mathcal{A}_1$, $\mathcal{A}_2$ (as before), and $\mathcal{A}_3$ (4×4)

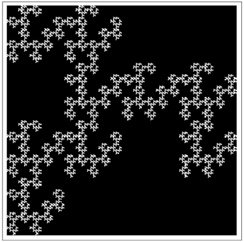


Figure: The mixed (labyrinth) set of level 4 defined by a sequence $\{\mathcal{A}_k\}_{k\geq 1}$ where the first three patterns are $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, respectively, shown above, and the fourth is $\mathcal{A}_1$

Topological properties of mixed labyrinth fractals

Lemma

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of non-empty patterns, $m_k \geq 3$, and $n \geq 1$. If $\mathcal{A}_1, \dots \mathcal{A}_n$ are labyrinth patterns, then $\mathcal{W}_n$ is an $m(n) \times m(n)$-labyrinth set (i.e., it has the Tree Property, Exits Property, Corner property), for all $n \geq 1$, where $m(n) = \prod_{k=1}^{n} m_k$.

We call the limit set $L_\infty = \bigcap_{n\geq 1} \bigcup_{W \in \mathcal{W}_n} W$ the mixed labyrinth fractal generated by $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

Theorem

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of labyrinth patterns, $m_k \geq 3$, for all $k \geq 1$. Then L_∞ is a dendrite.

The construction of the path between exits

Example: The path between the bottom exit and the right exit

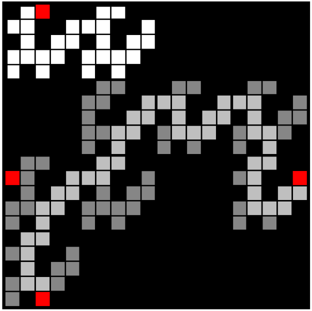


Figure: The set $\mathcal{W}_2$ constructed with the patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ shown before, and the path from the bottom exit to the right exit of $\mathcal{W}_2$ (in lighter gray).

One can check that $◲(2) = 48$.

The idea of the construction described in the following works for all mixed labyrinth fractals.

Paths in mixed labyrinth sets. Paths in patterns

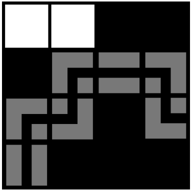


Figure: The path from the bottom exit to the right exit of $\mathcal{A}_1$

- first, we find the path between the bottom and the right exit of $\mathcal{W}_1$
- then we denote each white square in the path according to its neighbours within the path: there are 6 possible types of squares: ◫, ⊟, ◳, ◰, ◲, and ◱-*square*

Paths in mixed labyrinth sets. Paths in patterns

In order to obtain the ◲-path in $\mathcal{G}(\mathcal{W}_2)$, we replace each ◧-square of the path in $\mathcal{G}(\mathcal{W}_1)$ with the ◧-path in $\mathcal{G}(\mathcal{A}_2)$.
Analogously, we do this for the other marked white squares.

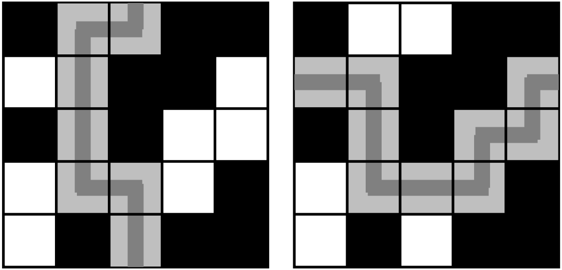
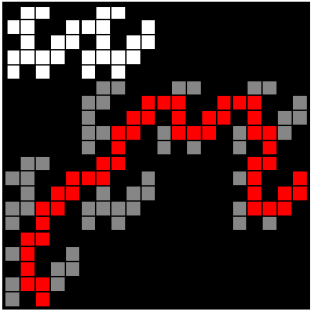


Figure: Paths from bottom to top and from left to right exit of $\mathcal{A}_2$

Paths in mixed labyrinth sets



In general, for any pair of exits and $n \geq 1$, we replace each marked white square in the path of $\mathcal{G}(\mathcal{W}_n)$ by its corresponding path in $\mathcal{G}(\mathcal{A}_{n+1})$ and obtain the path of $\mathcal{G}(\mathcal{W}_{n+1})$.

Paths in mixed labyrinth sets

Let $\{\mathcal{A}_k\}_{k\geq 1}$ be a sequence of labyrinth patterns, that defines the sequence $\{\mathcal{W}_n\}_{n\geq 1}$ of mixed labyrinth sets.

Proposition

There exist non-negative 6×6*-matrices* M_k, $k = 1, 2, \dots,$ *such that for all* $n \geq 1$, *and for* $M(n) = M_1 \cdot M_2 \cdot \dots \cdot M_n$, *the element in row* x *and column* y *of* $M(n)$ *is the number of* y*-squares in the* x*-path in* $\mathcal{G}(\mathcal{W}_n)$, *for* $x, y \in$ {┃, ━, ┗, ┏, ┓, ┛}.
Furthermore,

$$\begin{pmatrix} \text{┃}(n) \\ \text{━}(n) \\ \text{┗}(n) \\ \text{┏}(n) \\ \text{┓}(n) \\ \text{┛}(n) \end{pmatrix} = M(n) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Paths in mixed labyrinth sets

Sketch of the proof

For $k \geq 1$, we define the matrix M_k (the path matrix of $\mathcal{A}_k$):

- the columns of M_k from left to right and the rows of M_k from top to bottom correspond to , , , , , and , (ordered set of indices)
- the element in row x and column y of M_k is the number of y-squares in the x-path in $\mathcal{G}(\mathcal{A}_k)$.

One can easily check that the **matrix multiplication reflects the substitution of paths**.

(Proof by induction)

Remark. The proposition yields

- in the *self-similar* case $M(n) = M^n$
- in the *mixed* case $M(n+1) = M(n) \cdot M_{n+1}$

Mixed labyrinth fractals generated by special cross patterns

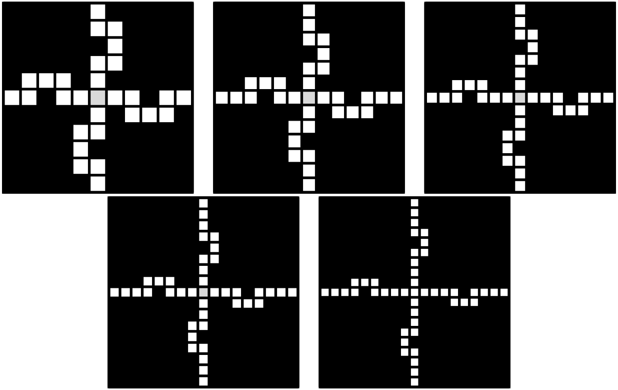


Figure: Example: the first five elements of a sequence of special cross patterns, where $m_k = 2k + 9$, and $a_k = k + 4$

Theorem

There **exist** *sequences* $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of both horizontally and vertically blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **finite**. *For almost all* points $x_0 \in a$ *(with respect to the length) there exists the tangent at* x_0 *to the arc* a.

Proposition

There exist sequences $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set* L_∞ *has the property that for any two points in* L_∞ *the length of the arc* $a \subset L_\infty$ *that connects them is* **infinite**.

Wild labyrinth patterns/Wild labyrinth fractals

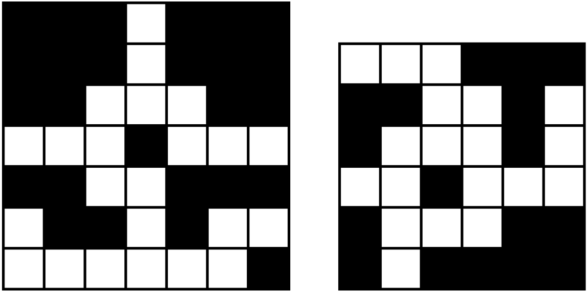
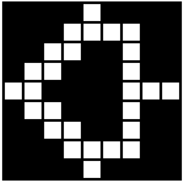


Figure: Examples: wild labyrinth patterns, both *vertically* and *horizontally blocked*

- tree $\longleftrightarrow$ connected graph
- uniqueness of v/h exit pair $\longleftrightarrow$ existence of v/h exit pair
- corner property

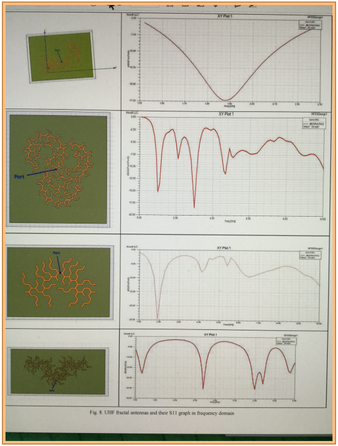
Paths in wild labyrinth fractals. Example

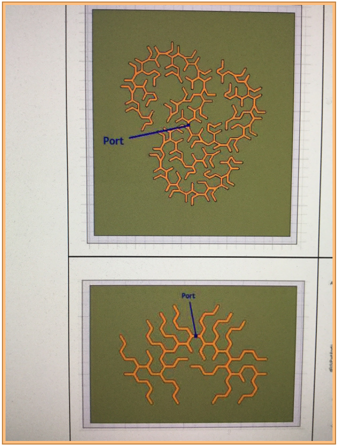
For wild labyrinth fractals the Lemma about the path construction does not hold in general: the squares in the shortest path from the top exit to the bottom exit in $\mathcal{G}(\mathcal{W}_2)$ do not lie whithin the shortest path from the top exit to the bottom exit in $\mathcal{G}(\mathcal{W}_1)$



- in $\mathcal{G}(\mathcal{W}_1)$: $\square_1^{left} = 15$, $\square_1^{right} = 13$, $\square_1 = 15$, $\square_1 = \square_1 = \square_1 = \square_1 = 9$
- the length of the "right" $\square$-path in $\mathcal{G}(\mathcal{W}_2)$ is $7\square_1 + 2\square_1 + \square_1 + \square_1 + \square_1 + \square_1 = 7 \cdot 13 + 2 \cdot 15 + 4 \cdot 9 = 157$
- the length of the "left" $\square$-path in $\mathcal{G}(\mathcal{W}_2)$ is $3\square_1 + 0 \cdot \square_1 + 3\square_1 + 3\square_1 + 3\square_1 + 3\square_1 = 3 \cdot 13 + 4 \cdot 3 \cdot 9 = 147$

A. A. Potapov, W. Zhang, CIE International Conference on Radar (October 2016) : prototypes of ultra-wide band radar antennas based on **labyrinth fractals**





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