FUNDAMENTAL GROUP OF TILES ASSOCIATED TO QUADRATIC CANONICAL NUMBER SYSTEMS

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Abstract. If $\mathbf{A}$ is a 2×2 expanding matrix with integral coefficients, and $\mathcal{D} \subset \mathbb{Z}^2$ a complete set of coset representatives of $\mathbb{Z}^2/\mathbf{A}\mathbb{Z}^2$ with $|\det(\mathbf{A})|$ elements, then the set $\mathcal{T}$ defined by $\mathbf{A}\mathcal{T} = \mathcal{T} + \mathcal{D}$ is a self-affine plane tile of $\mathbb{R}^2$, provided that its two-dimensional Lebesgue measure is positive.

It was shown by Luo and Thuswaldner that the fundamental group of such a tile is either trivial or uncountable.

To a quadratic polynomial $x^2 + Ax + B$, $A, B \in \mathbb{Z}$ such that $B \geq 2$ and $-1 \leq A \leq B$, one can attach a tile $\mathcal{T}$. Akiyama and Thuswaldner proved the triviality of the fundamental group of this tile for $2A < B+3$, by showing that a tile of this class is homeomorphic to a closed disk. The case $2A \geq B+3$ is treated here by using the criterion given by Luo and Thuswaldner.

1. Introduction

This paper is devoted to the fundamental group of tiles related to quadratic canonical number systems.

To an iterated function system (IFS) $\{f_i\}_{i=1}^m$ of injective contractions on a complete metric space ($\mathbb{R}^n$ throughout this paper), there corresponds a unique nonempty compact set $\mathcal{T}$ with the *self-similarity* property $\mathcal{T} = \bigcup_{i=1}^m f_i(\mathcal{T})$ (see [8]). This set is called the *attractor* of the IFS.

We say that the IFS (or its attractor) satisfies the open set condition whenever there exists a bounded open set U with $\bigcup_{i=1}^m f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j$.

We are interested in *self-affine tiles*, attractors having nonempty interior, satisfying the open set condition and corresponding to IFS of the form

$$\{f_d(x) = \mathbf{A}^{-1}(x+d),\ x \in \mathbb{R}^n\}_{d \in \mathcal{D}},$$

where $\mathbf{A}$ is a real $n \times n$ matrix with eigenvalues greater than 1 and $\mathcal{D} \subset \mathbb{R}^n$ with $|\mathcal{D}| = |\det \mathbf{A}|$ is supposed to be an integer.

More precisely, we will be concerned with *integral self-affine tiles with standard digit set*: this means that A is an integer matrix and $\mathcal{D} \subset \mathbb{Z}^n$ is a complete set of coset representatives of $\mathbb{Z}^n/\mathbf{A}\mathbb{Z}^n$. Moreover the set $\mathcal{T} + \mathbb{Z}^n$ will be assumed to be a *tiling* of $\mathbb{R}^n$, *i.e.*,

$$\mathbb{R}^n = \mathcal{T} + \mathbb{Z}^n$$

and

$$(\mathrm{int}(\mathcal{T} + d_1)) \cap (\mathcal{T} + d_2) = \emptyset \text{ for } d_1 \neq d_2 \ (d_1, d_2 \in \mathbb{Z}^n).$$

We also say that $\mathcal{T}$ is a $\mathbb{Z}^n$*-tile* in $\mathbb{R}^n$.

There is a vast literature on topological properties of attractors and tiles. Hata showed in [7] that a connected attractor is even a locally connected continuum.

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Concerning plane attractors ($n = 2$), Luo, Rao and Tan obtained in [10] the disk-likeness of self-similar connected tiles with connected interior. Bandt and Wang characterized self-affine plane tiles that are homeomorphic to a disk by the number of their neighbors in the tiling (see [4]). The structure of the interior components of self-similar tiles with disconnected interior is studied by Ngai and Tang in [12].

In this paper we will deal with a class of self-affine tiles associated to *canonical number systems* (CNS). For $n \geq 1$, let $P = x^n + b_{n-1}x^{n-1} + ... + b_0 \ \in \mathbb{Z}[x]$, $\mathcal{N} = \{0, 1, ...|b_0| - 1\}$ and $\mathcal{R} = \mathbb{Z}[x] \ / \ P\,\mathbb{Z}[x]$. Let us denote the projection of x into $\mathcal{R}$ by $[x]$. Then the pair $(P, \mathcal{N})$ is called *CNS with digit set* $\mathcal{N}$ if each element γ of $\mathcal{R}$ can be written in the form

$$\gamma \ = \ a_0 \ + \ a_1[x] \ + \ \dots \ + a_{l(\gamma)}[x]^{l(\gamma)}$$

with $a_i \in \mathcal{N}$ and $l(\gamma) \in \mathbb{N}$. Note that if P is irreducible and α is a root of P, then $\mathcal{R}$ is isomorphic to $\mathbb{Z}[\alpha]$, thus $[x]$ can be replaced by α in the above expansion. Characterizations of CNS have been studied for example by Scheicher and Thuswaldner in [13], Akiyama and Rao in [1] and Brunotte in [5].
In the case of quadratic CNS, we write $P = x^2 + Ax + B \in \mathbb{Z}[x]$. Then it is known ([5], [9]) that

$$(P, \mathcal{N}) \ \text{ is a CNS} \quad \text{iff} \quad B \geq 2 \ \text{ and } \ -1 \leq A \leq B.$$

To each quadratic CNS, a tile $\mathcal{T}$ is attached in the following way: let

$$\mathbf{A} \ = \ \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\},$$

then the set defined by

$$\mathbf{A}\mathcal{T} \ = \ \bigcup_{d \in \mathcal{D}} (\mathcal{T} + d)$$

is a self-affine plane tile of $\mathbb{R}^2$ satisfying

$$\mathcal{T} \ = \ \left\{ \sum_{i \geq 1} \mathbf{A}^{-i} d_i, \ d_i \in \mathcal{D} \right\}.$$

We will use later the quantity

$$J \ = \ \max\left\{1, \left\lfloor \frac{B-1}{B-A+1} \right\rfloor \right\}.$$

Note that

$$J > 1 \quad \text{iff} \quad 2A \geq B + 3.$$

Some topological properties of quadratic CNS tiles have been studied by Akiyama and Thuswaldner in [3]: depending on J, the tile is either homeomorphic to a disk ($J = 1$) or it has a disconnected interior ($J > 1$). This means that the fundamental group $\pi_1(\mathcal{T})$ of $\mathcal{T}$ is trivial in the former case. Another result by Luo and Thuswaldner ([11]) states that the fundamental group of such tiles is either trivial or uncountable. An overview of these results can be found in [2]. We will use the criterion given in [11] to prove the uncountability of $\pi_1(\mathcal{T})$ in the latter case ($J > 1$).

2. Quadratic Canonical Number Systems: Graph of Neighbors and Set of Vertices

2.1. **Adding graph.** We define in $\mathbb{Z}^2$ the directed labelled graph $\mathcal{G}_1(\mathbb{Z}^2)$ as follows:

- each $s \in \mathbb{Z}^2$ is a state of $\mathcal{G}_1(\mathbb{Z}^2)$.
- for $s, s' \in \mathcal{G}_1(\mathbb{Z}^2)$ and $d, d' \in \mathcal{D}$, there exists an edge $s \xrightarrow{d|d'} s'$ from s to s' labelled by $d|d'$ if and only if $s + d \ = \ \mathbf{A}s' + d'$.

As $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^2/\mathbf{A}\mathbb{Z}^2$, s' and the *output digit* d' are uniquely determined by s and the *input digit* d, and this addition is well-defined for all $s \in \mathbb{Z}^2$ and all $d \in \mathcal{D}$. Thus $\mathcal{G}_1(\mathbb{Z}^2)$ is a so-called *adding machine*.
If $d_0, \ldots, d_n$ are *digits* (*i.e.*, elements of $\mathcal{D}$), then $w = (d_n \ldots d_0)$ is called a *string*. If n is the maximal i for which d_i is non zero, the *length* of the string, written $\mathcal{L}(w)$, is said to be equal to $n + 1$.
For $s \in \mathcal{G}_1(\mathbb{Z}^2)$, one can associate an *output string* $c = (d'_n \ldots d'_0)$ to an *input string* by "feeding" the graph with the input string from right to left as input digits, starting at the sate s and collecting the corresponding output digits (see also [11]).

2.2. **Graph of neighbors.** We define the *set of neighbors* $\mathcal{S}$ by

$$\mathcal{S} \ = \ \left\{ s \in \mathbb{Z}^2, s \neq 0, \mathcal{T} \cap (\mathcal{T} + s) \ \neq \emptyset \right\}.$$

The *graph of neighbors* $\mathcal{G}_1(\mathcal{S})$ is the restriction of $\mathcal{G}_1(\mathbb{Z}^2)$ to the subset $\mathcal{S}$ of $\mathbb{Z}^2$. It is shown that the graph $\mathcal{G}_1(\mathcal{S} \cup \{0\})$ is stable by addition of any digit to any state.

Remark 1. *This graph is called $\mathcal{G}_1(S)$ in* [3] *and $G^T(S)$ in* [11].

The graph of neighbors for quadratic CNS has been found in [3]: defining the points

$$P_n \ = \ \begin{pmatrix} n - (n-1)A \\ -(n-1) \end{pmatrix}, \ \ Q_n \ = \ \begin{pmatrix} -n + nA \\ n \end{pmatrix}, \ \ R \ = \ \begin{pmatrix} -A \\ -1 \end{pmatrix}, \ \ n \geq 1,$$

then the set of neighbors consists of the $2 + 4J$ elements

$$\pm P_1, \ldots, \pm P_J, \pm Q_1, \ldots, \pm Q_J, \pm R.$$

The states $\pm P_1, \pm Q_1, \pm R$ are said to be of *first level*, the states $\pm P_n, \pm Q_n$ of *level n* for $2 \leq n \leq J$. The edges are given in [3, p.1471] and are reproduced in Table 1. We obtain the graph of Figure 1, explicitely depicted there until $J = 2$, and where we also wrote the point $(0, 0)$ as an empty state and the corresponding edges. Moreover, if τ stands for the label $d|d'$, then $-\tau$ stands for $d'|d$. For the special case $J = 1$ we have the graph of Figure 2, which is a subgraph of the general graph.

An *infinite walk in the graph $\mathcal{G}_1(\mathcal{S})$ ending in a state s_0* is an infinite sequence of the form $s_0 \leftarrow s_1 \leftarrow s_2 \leftarrow \ldots$ with edges and states in $\mathcal{G}_1(\mathcal{S})$.
Considering the graph in Figure 1, we get the following result (see [3]):

Proposition 2. *Let W be an infinite walk ending in a state s_0, then one of the following possibilities occurs:*

1. *All states of W belong to level* 1.
2. *Going the walk W backwards from s_0, one comes to one of the cycles $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n$, for some n with $2 \leq n \leq J$.*

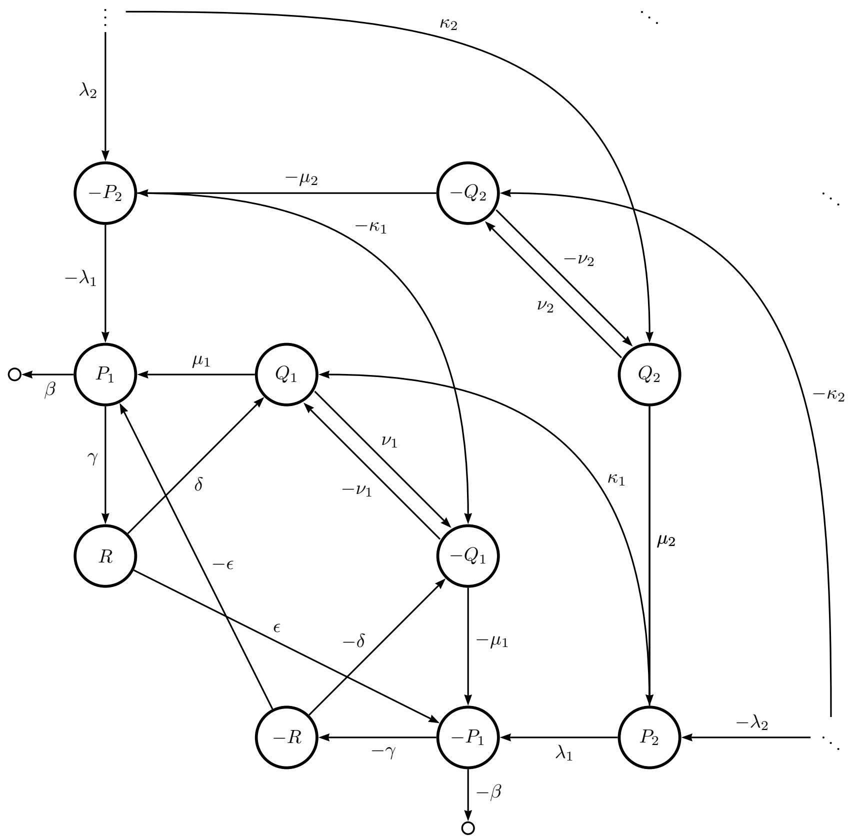

FIGURE 1. General neighbor graph $\mathcal{G}_1(\mathcal{S})$.

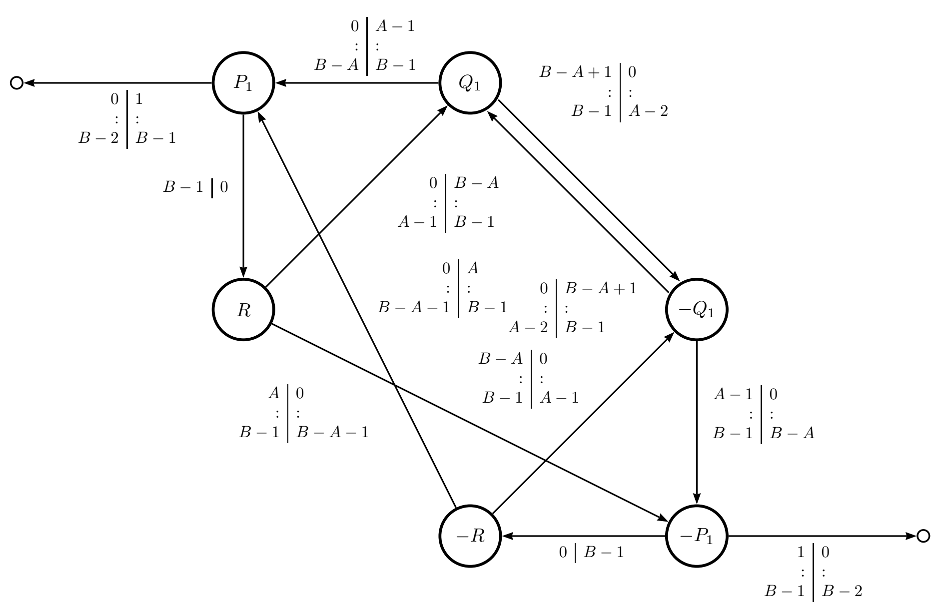

FIGURE 2. First level neighbor graph.

edge	labels		name
$0 \to 0$	0 $\vdots$ $B-1$	0 $\vdots$ $B-1$	
$P_1 \to 0$	0 $\vdots$ $B-2$	1 $\vdots$ $B-1$	β
$P_1 \to R$	$B-1$	0	γ
$R \to Q_1$	0 $\vdots$ $A-1$	$B-A$ $\vdots$ $B-1$	δ
$R \to -P_1$	A $\vdots$ $B-1$	0 $\vdots$ $B-A-1$	ϵ
$P_{n+1} \to Q_n$ $(1 \leq n < J)$	0 $\vdots$ $A-3-(n-1)(B-A+1)$	$1+n(B-A+1)$ $\vdots$ $B-1$	κ_n
$P_{n+1} \to -P_n$ $(1 \leq n < J)$	$A-2-(n-1)(B-A+1)$ $\vdots$ $B-1$	0 $\vdots$ $n(B-A+1)$	λ_n
$Q_n \to P_n$ $(1 \leq n \leq J)$	0 $\vdots$ $n(B-A+1)-1$	$A-1-(n-1)(B-A+1)$ $\vdots$ $B-1$	μ_n
$Q_n \to -Q_n$ $(1 \leq n \leq J)$	$n(B-A+1)$ $\vdots$ $B-1$	0 $\vdots$ $A-2-(n-1)(B-A+1)$	ν_n

TABLE 1. Edges of the general neighbor graph $\mathcal{G}_1(\mathcal{S})$.

2.3. **L-vertices of a tile.** An L-vertex of the tile $\mathcal{T}$ is a point of $\mathbb{R}^2$ where $\mathcal{T}$ coincides with exactly L translates of the shape $\mathcal{T}+s$, $s \in \mathcal{S}$: if $s_1, \dots, s_L$ are distinct points of $\mathcal{S}$, the set

$$V_L(s_1, \dots, s_L) \;=\; \left\{ x \in \mathbb{R}^2, x \in \mathcal{T} \cap \bigcap_{j=1}^{L} (\mathcal{T}+s_j) \right\}$$

leads to the definition of the set of L-vertices

$$V_L \;=\; \bigcup_{(s_1,\dots,s_L)\in \mathcal{S}^n} V_L(s_1, \dots, s_L).$$

One can obtain the L-vertices by finding infinite simple walks in the so-called L-fold power of $\mathcal{G}_1(\mathcal{S})$. This graph $\mathcal{G}_L(\mathcal{S})$ is constructed as follows:

- The states of $\mathcal{G}_L(\mathcal{S})$ are the $L-$subsets of $\mathcal{S}$.
- There exists an edge

$$\{s_{11}, \dots, s_{1L}\} \xrightarrow{d} \{s_{21}, \dots, s_{2L}\}$$

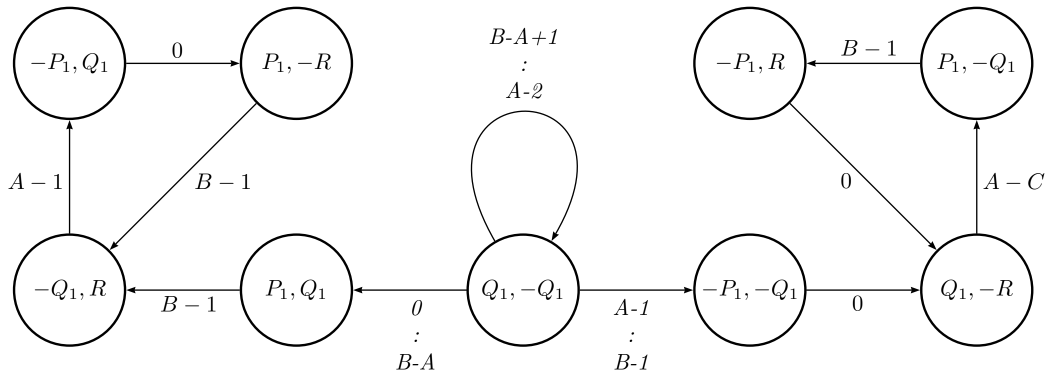

FIGURE 3. Graph $\mathcal{G}_2(\mathcal{S})$ (restriction to the states of level 1).

in $\mathcal{G}_L(\mathcal{S})$ if, after possible rearrangement of $s_{21}, \dots, s_{2L}$, there exist the edges

$$s_{1l} \xrightarrow{d|d_l} s_{2l} \quad (1 \le l \le L)$$

in $\mathcal{G}_1(\mathcal{S})$ for some $d_1, \dots, d_L \in \mathcal{D}$.

- The states that are not the endpoints of infinite walks are removed, together with the edges leading to them.

We have the following characterization of L-vertices:

Characterization 3. *The following assertions are equivalent.*

(1) *The point* $x = \sum_{j\ge1} \mathbf{A}^{-j} d_j$ *belongs to* $V_L(s_{01}, \dots, s_{0L})$
(2) *In* $\mathcal{G}_L(\mathcal{S})$*, there is an infinite walk*

$$\{s_{01}, \dots, s_{0L}\} \xleftarrow{d_1} \{s_{11}, \dots, s_{1L}\} \xleftarrow{d_2} \{s_{21}, \dots, s_{2L}\} \xleftarrow{d_3} \dots .$$

This follows from the fact that a point x belonging to $\mathcal{T} \cap (\mathcal{T} + s)$ admits the two representations $x = \sum_{j\ge1} \mathbf{A}^{-j} d_j = s + \sum_{j\ge1} \mathbf{A}^{-j} d'_j$ if and only if there is an infinite walk

$$s \xleftarrow{d_1|d'_1} s_1 \xleftarrow{d_2|d'_2} s_2 \xleftarrow{d_3|d'_3} \dots$$

in the graph $\mathcal{G}_1(\mathcal{S})$.

Remark 4.

- *For* $L = 2$ *we come to the subgraph of level* 1 *depicted in the figure above (Figure 3). Note that the edge from* $\{Q_1, -Q_1\}$ *to itself only exists for* $2A \ge B + 3$.
- *For* $L = 3$*, it is mentioned in* [3, p.1478] *that the subgraph of level* 1 *is empty. So there are no three infinite walks in level* 1 *of the graph* $\mathcal{G}_1(\mathcal{S})$ *with the same input digits that end in three different states of level* 1*. This can be checked here directly using the graphs of Figures 2 and 3.*

3. Fundamental Group of Quadratic CNS Tiles

3.1. **Criterion.** We recall the criterion for uncountability of the fundamental group of a tile given by Luo and Thuswaldner in [11]:

Criterion 5. *Let* $\mathcal{T}$ *be a connected* $\mathbb{Z}^2$*-tile in* $\mathbb{R}^2$*. Furthermore, suppose that there exist* $s_1, s_2 \in \mathcal{S}$ *such that:*

(1) $\#V_2(s_1, s_2) \ge 2$ *and* $V_2(s_1, s_2) \setminus V_3 \ne \emptyset$.

(2) *For each* $i \in \{0,1,2\}$, *there exists a string* w_i *with the property: using* w_i *as input string in* $\mathcal{G}_1(\mathcal{S} \cup \{0\})$ *starting at* $0, s_1, s_2$ *yields the output strings* c_0^i, c_1^i, c_2^i *satisfying:*

$$\max\left\{\mathcal{L}(c_i^i), \mathcal{L}(c_{i+1}^i)\right\} \; < \; \mathcal{L}(c_{i+2}^i) \quad (\textit{indices are written modulo } 3).$$

Then the fundamental group of $\mathcal{T}$ *is uncountable.*

Under the assumptions (1) and (2), the complement of the tile $\mathcal{T}$ in $\mathbb{R}^2$ is shown to be disconnected: two subpieces of $\mathcal{T}$ can be found whose union has a bounded complement component that also intersects the complement of $\mathcal{T}$. Thus the complement of this tile is disconnected, it even has infinitely many components. Therefore the tile $\mathcal{T}$ can not be locally simply connected, which is equivalent to the uncountability of its fundamental group by a result of Conner and Lamoreaux ([6]).

3.2. **Theorem.**

Theorem 6. *Let* $\mathcal{T}$ *be the quadratic CNS tile corresponding to the polynomial* $x^2 + Ax + B$. *Then the fundamental group of* $\mathcal{T}$ *is:*

- *trivial for* $2A < B + 3$,
- *uncountable for* $2A \geq B + 3$.

Proof. The first part has been proved in [3], we prove the second part by showing that both items of the above criterion are true. Let $s_1 = P_1$, $s_2 = -Q_1$.

(1) *Claim.* The point

$$x \; = \; \sum_{j \geq 1} \mathbf{A}^{-j} d_j$$

with

$$d_{1+3k} \; = \; \begin{pmatrix} B-A \\ 0 \end{pmatrix}, \quad d_{2+3k} \; = \; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad d_{3+3k} \; = \; \begin{pmatrix} B-1 \\ 0 \end{pmatrix}$$

belongs to $V_2(P_1, -Q_1) \setminus V_3$.
Indeed, looking at the first level subgraph of $\mathcal{G}_2(\mathcal{S})$ (Figure 3), the infinite cycle

$$\{P_1, -Q_1\} \xleftarrow{B-A} \{Q_1, -R\} \xleftarrow{0} \{-P_1, R\} \xleftarrow{B-1} \{P_1, -Q_1\} \xleftarrow{B-A} \; \ldots$$

provides a point of $V_2(P_1, -Q_1)$ because of Characterization 3.
Then, as seen in the second item of Remark 4, an infinite walk in $\mathcal{G}_1(\mathcal{S})$ with the same input digits as the cycle above and ending in $P \notin \{P_1, -Q_1\}$ could not have all states in level 1. Note that the levels would grow up going this infinite walk in $\mathcal{G}_1(\mathcal{S})$. Thus, one should come to a cycle in level $n \geq 2$ (see Proposition 2): $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n$; this would imply the existence of the edge $-Q_n \xleftarrow{B-A} Q_n$ in the walk, which is not true (according to Table 1 page 5). This proves the claim.
The point
$y = \mathbf{A}^{-1} \begin{pmatrix} B-A \\ 0 \end{pmatrix} + \mathbf{A}^{-3} \begin{pmatrix} A-1 \\ 0 \end{pmatrix} + \sum_{j \geq 4} \mathbf{A}^{-j} \begin{pmatrix} A-2 \\ 0 \end{pmatrix}$ is distinct from x and also belongs to $V_2(P_1, -Q_1)$ (this set is even easily seen to contain infinitely many elements, using Characterization 3 and the graph of Figure 3).
Thus the first item of the criterion is proved.

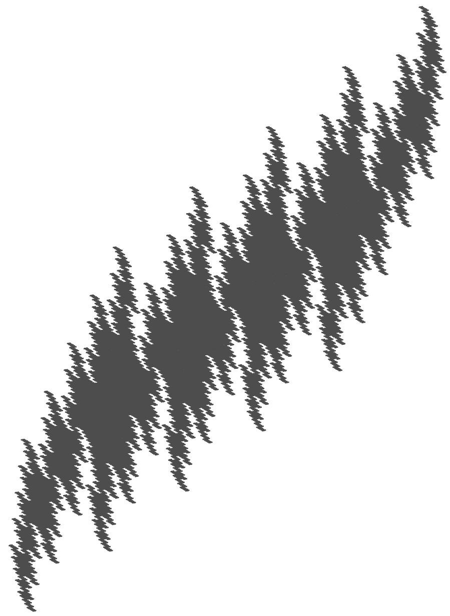

FIGURE 4. A CNS tile with uncountable fundamental group (limit case, with $A = 4, B = 5$).

(2) The second part is obtained by looking at the graph in Figure 2. With the input strings

$$\begin{aligned} w_0 &= (0000), \\ w_1 &= (01(A-1)(B-1)0(B-1)), \\ w_2 &= (0000(B-1)(B-1)), \end{aligned}$$

one gets

$$\begin{array}{rcccccl} \max\left\{\mathcal{L}(c_0^0), \mathcal{L}(c_1^0)\right\} & = & 1 & < & \mathcal{L}(c_2^0) & = & 3, \\ \max\left\{\mathcal{L}(c_1^1), \mathcal{L}(c_2^1)\right\} & = & 3 & < & \mathcal{L}(c_0^1) & = & 5, \\ \max\left\{\mathcal{L}(c_2^2), \mathcal{L}(c_0^2)\right\} & = & 2 & < & \mathcal{L}(c_1^2) & = & 5. \end{array}$$

Thus the second item of the criterion is fulfilled and Theorem 6 is proved. □

An example of CNS tile with uncountable fundamental group can be seen on Figure 4.

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