(NON-) RIGIDITY OF INTERVAL EXCHANGES

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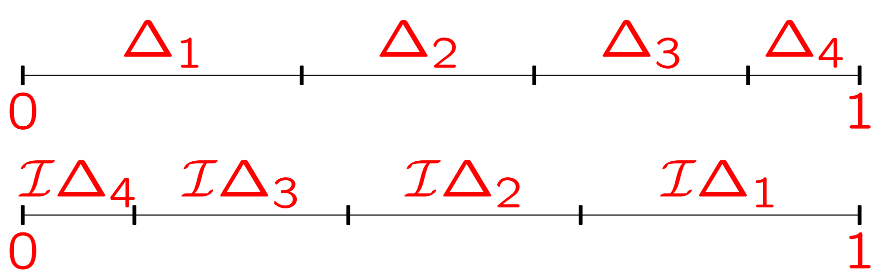
INTERVAL EXCHANGES

A k-interval exchange or k-iet $\mathcal{I}$ with probability vector $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, and permutation π is defined by

$$\mathcal{I}x = x + \sum_{\pi^{-1}(j)<\pi^{-1}(i)} \alpha_j - \sum_{j<i} \alpha_j.$$

when x is in the interval

$$\Delta_i = \left[\sum_{j<i} \alpha_j, \sum_{j \leqslant i} \alpha_j \right[.$$



THE QUESTION OF RIGIDITY

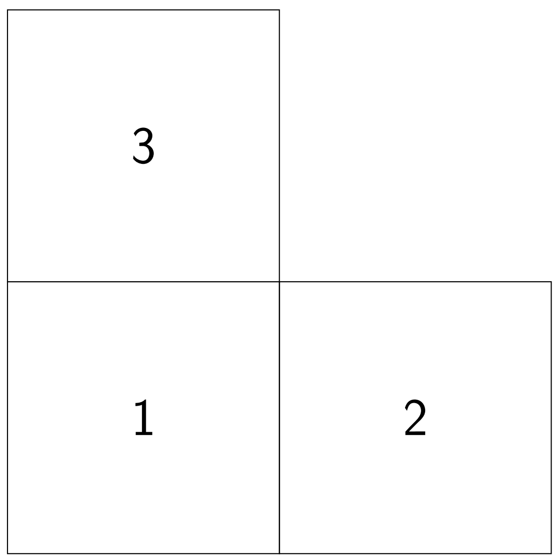
Veech (1982) : almost all iet are rigid $=$ there exists a sequence $q_n \to \infty$ such that for any measurable set

$$\mu(T^{q_n} A \Delta A) \to 0.$$

Examples of non-rigid iet were known only for 3 intervals. Until Robertson (2017) and F-H.

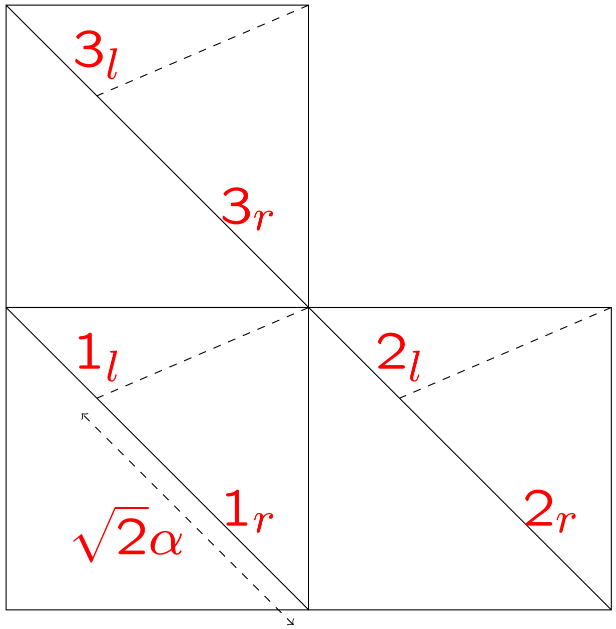
SQUARE TILED SURFACES

It is generated by d squares, whose sides are glued according to two permutations σ along the vertical and τ along the horizontal.
In this example $\tau(1,2,3) = (2,1,3)$ and $\sigma(1,2,3) = (3,2,1)$.



SQUARE-TILED IET

We take the directional flow of angle θ on a square tiled surface and its first return map on the union of negative diagonals. Let $\alpha = \frac{1}{1+\tan\theta}$.

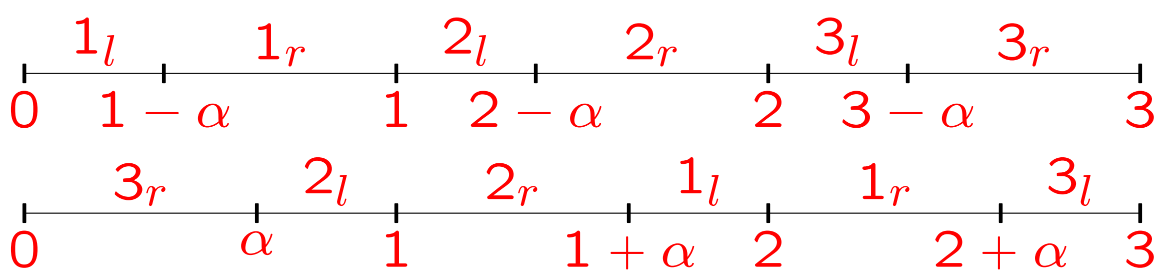


OUR FIRST FAMILY

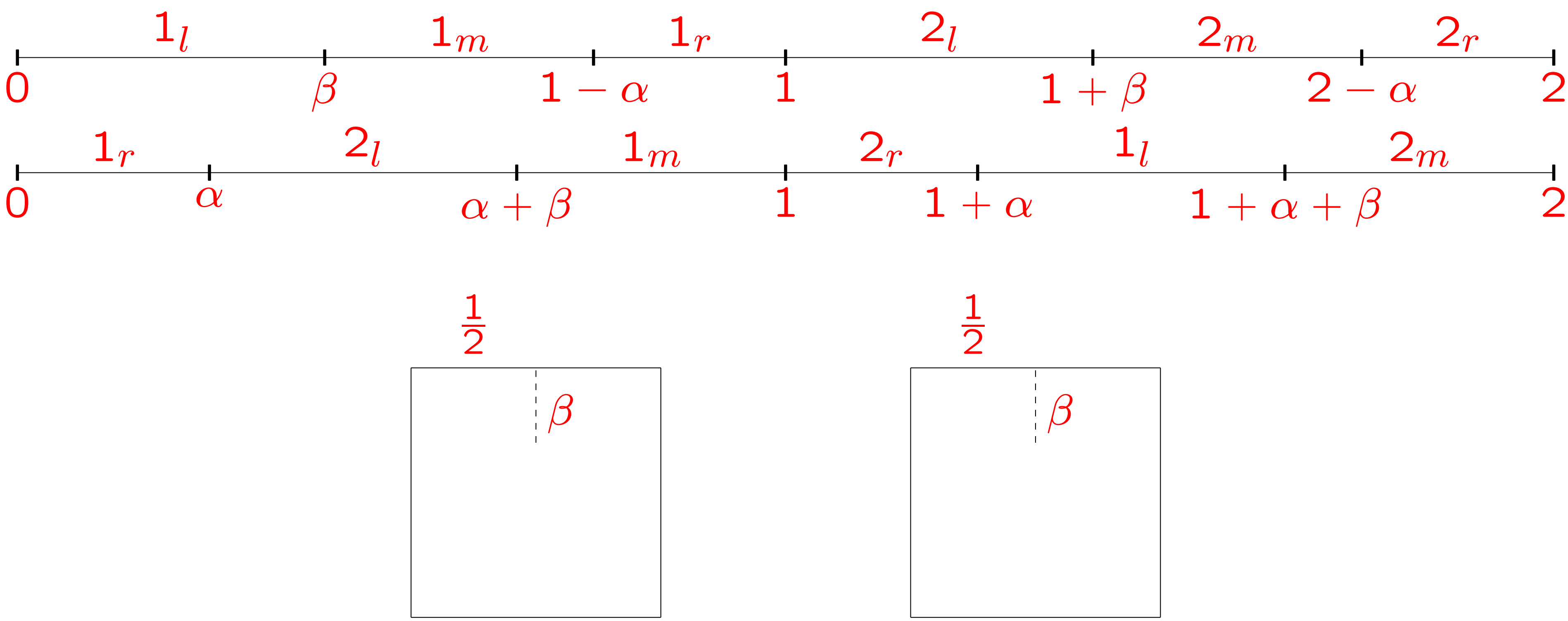
let T where the departure intervals are

$[i, i+1-\alpha[$, denoted by i_l, sent to $[j+\alpha, j+1[$, $j = \tau i$,

$[i+1-\alpha, i+1[$, denoted by i_r, sent to $[j, j+\alpha[$, $j = \sigma i$.



SECOND TYPE OF EXAMPLE : VEECH 1969



Glue two tori by the dotted edge. Take the directional flow of angle θ, going from one torus to the other when crossing this line, and its first return map on the union of verticals.

GRAND UNIFICATION

We take α irrational, β_i not in $\mathbb{Z}(\alpha)$, $0 = \beta_0 < \beta_1 < ...\beta_i < 1-\alpha < .\beta_{i+1} < ...\beta_r < \beta_{r+1} = 1$, σ_0, ..., σ_i, τ_i, σ_{i+1}, ..., σ_r permutations of $\{1, ...d\}$.

$Rx = x + \alpha$ modulo 1.

$T(x,s) = (x, \sigma_j s)$ if $\beta_j \leqslant x < \beta_{j+1}$, $j \neq i$,
$T(x,s) = (x, \sigma_i s)$ if $\beta_i \leqslant x < 1-\alpha$,
$T(x,s) = (x, \tau_i s)$ if $1-\alpha \leqslant x < \beta_{i+1}$.

If no strict subset of $\{1 \ldots d\}$ is invariant by all the σ_i and τ_i, the iet is minimal.

Theorem 1. *If $\sigma_l \neq \sigma_{l+1}$, $0 \leqslant l \leqslant i-1$ and $i+1 \leqslant l \leqslant r-1$, $\tau_i \neq \sigma_{i+1}$, and $\sigma_i\sigma_r \neq \tau_i\sigma_0$, then T is rigid (for any ergodic invariant measure) if α has unbounded partial quotients, T is uniquely ergodic and non-rigid if the coding of R by the partition determined by $\beta_1, ..., \beta_i, 1-\alpha, \beta_{i+1}, ..., \beta_r$ is linearly recurrent.*

SYMBOLIC SYSTEMS

Symbolic system = the shift on infinite sequences on a finite alphabet.

Trajectories = $y_n = j_l$ if $T^n y$ falls into the j-th interval in the l-th copy of $[0, 1[$.

A trajectory of T gives a trajectory of $R : u \to \phi(u)$ by $j_l \to j$, for all j, l.

Linear recurrence of the coding = in the language of trajectories of R, every word of length n occurs in every word of length Kn.

THE GREY ZONE

(work in progress)

Case where α has bounded partial quotients but the coding of R is NOT linearly recurrent.

This happens when there are β_i, under conditions on Ostrowski approximations of the β_i by α.

WE DON'T KNOW if T is rigid.

But we have examples where T is non-rigid and not linearly recurrent.

HOW TO PROVE RIGIDITY

When α has unbounded partial quotients,

— the trajectories of R are mainly made of words repeated many times, $A_n^{q_n}$,
— the trajectories of T are mainly made of cycles repeated many times $(A_{n,1}...A_{n,k})^{q'_n}$.

Thus rigidity.

$\bar{d}$ - SEPARATION

For two words of equal length $w = w_1 \dots w_q$ and $w' = w'_1 \dots w'_q$, Hamming distance $=$ $\bar{d}(w, w') = \frac{1}{q}\#\{i; w_i \neq w'_i\}$.

For a uniquely ergodic symbolic system, rigidity implies that for any infinite sequence $x_0 x_1 x_2 \dots$ in the system, for a given ϵ, $n > n_0$, $N > n_1$, $\bar{d}(x_0 \dots x_N, x_{q_n} \dots x_{q_n+N}) < \epsilon$.

$\bar{d}$-separated $=$ there exists C such that for any two words w and w' of length q produced by the system, if $\bar{d}(w, w') < C$, then $\{1, \dots q\} = I \cup J \cup K$, intervals in increasing order, $w_J = w'_J$, $\bar{d}(w_I, w'_I) = \bar{d}(w_K, w'_K) = 1$ except for empty I or K.

$\bar{d}$-separation was introduced by del Junco (1977) for the Thue - Morse sequence.

$\bar{d}$-separation and aperiodicity imply non-rigidity ; for primitive substitutions of constant length, $\bar{d}$-separation is equivalent to non-rigidity (Lemanczyk - Mentzen, 1988).

AVERAGE $\bar{d}$ - SEPARATION

Average $\bar{d}$-separated $=$ there exists C such that for d pairs of words of length q, if

— $\sum_{i=1}^{d} \bar{d}(v_i, v'_i) < C$,

— $\phi(v_i)$ is the same word u for all i,

— $\phi(v'_i)$ is the same word u' for all i,

— $v_i \neq v_j$ for $i \neq j$,

then $\{1, \ldots q\} = I \cup J \cup K$, intervals in increasing order

— $v_{i,J} = v'_{i,J}$ for all i,

— $\sum_{i=1}^{d} \bar{d}(v_{i,I}, v'_{i,I}) \geqslant 1$ if I is nonempty,

— $\sum_{i=1}^{d} \bar{d}(v_{i,K}, v'_{i,K}) \geqslant 1$ if K is nonempty,

HOW TO PROVE NON - RIGIDITY

Our iet is $\bar{d}$-separated if and only if for all t, $\sigma_l(t) \neq \sigma_{l+1}(t)$, $0 \leqslant l \leqslant i-1$ and $i+1 \leqslant l \leqslant r-1$, $\tau_i(t) \neq \sigma_{i+1}(t)$, and $\sigma_i\sigma_r(t) \neq \tau_i\sigma_0(t)$, In general, our iet is average $\bar{d}$-separated.

Average $\bar{d}$-separation implies non-rigidity, which proves the hard part of Theorem 1.

Counter-example Square-tiled iet with 4 squares and α golden ratio : non-rigidity is not equivalent to $\bar{d}$-separation for primitive substitutions.

CODINGS OF ROTATIONS

The bispecial words $w_{n,i}$ and their return words $M_{n,i}$, $P_{n,i}$ are built from the Euclid continued fraction expansion of α or its generalizations by Ferenczi - Holton - Zamboni.

For n large enough, $w_{n,i}$ has exactly two extensions of length $|w_{n,i}| + (|P_{n,i}| \wedge |M_{n,i}|)$, and these are of the form $w_{n,i}V'_{n,i}V_{n,i}$ and $w_{n,i}V''_{n,i}V_{n,i}$ for the same word $V_{n,i}$ and two words $V'_{n,i}$ and $V''_{n,i}$ of (common) length 1 or 2.

If the coding is linearly recurrent, there exists a constant K_1 such that for all n
$|P_{n,i}| \wedge |M_{n,i}| > K_1|w_{n,i}|$.