A characterization of cotorsion-free groups in terms of homomorphisms from fundamental groups of Peano continua

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joint work with
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- $\pi_1(X, x)$ is a topological group
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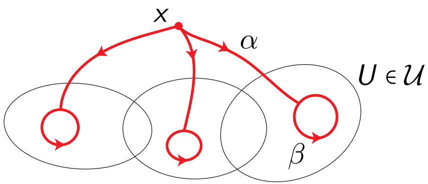
$\pi(\mathcal{U},x) \leqslant \pi_1(X,x)$ is generated by all elements $[\alpha\beta\alpha^-]$ with $\alpha : ([0,1],0) \to (X,x)$, $\beta : ([0,1],\{0,1\}) \to (U,\alpha(1))$, $U \in \mathcal{U}$.



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- (a) $\pi^s(X,x) = 1 \Leftrightarrow \pi_1(X,x)$ is Hausdorff
- (b) $\pi^s(X,x)$ is open $\Leftrightarrow X$ has a universal covering space
- (c) $\{1\}$ is open $\Leftrightarrow \pi_1(X,x)$ is discrete
  $\Leftrightarrow X$ is semilocally simply connected
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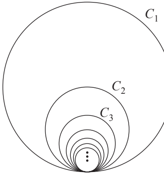

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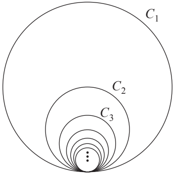

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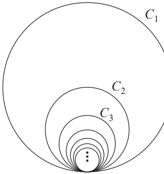

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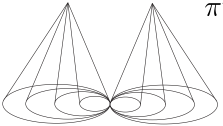

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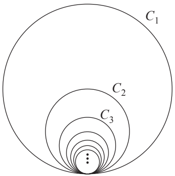

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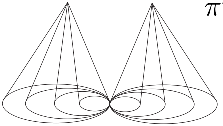

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- $\pi_1(\mathbb{G},*)$ is **not** Hausdorff (indiscrete)
- $\mathbb{G}$ has a universal covering space $(id : \mathbb{G} \to \mathbb{G})$
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Theorem [F-Zastrow 2007]

There exists a **generalized covering** $p : \widetilde{X} \to X$ w.r.t. $\pi^s(X, x)$:

(1) $\widetilde{X}$ is path connected (pc) and locally path connected (lpc).

(2) $p_\# : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$ is a monomorphism onto $\pi^s(X, x)$.

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$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{x}) \\ & \overset{\exists!\, \widetilde{f}}{\nearrow} & \downarrow p \\ (Y^{pc,lpc}, y) & \xrightarrow{\forall f} & (X, x) \end{array} \iff f_\# \pi_1(Y, y) \leqslant p_\# \pi_1(\widetilde{X}, \widetilde{x})$$

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If $\pi^s(X,x)=1$, then $p:\widetilde{X}\to X$ is a **generalized universal covering**.

Examples with $\pi^s(X,x)=1$ include:

- 1-dimensional spaces [Eda-Kawamura 1998]
- subsets of surfaces [F-Zastrow 2005]
- certain "trees of manifolds" [F-Guilbault 2005]

Definition

An abelian group A is called **slender** if for every homomorphism $h : \mathbb{Z}^{\mathbb{N}} \to A$, $\exists\ n \in \mathbb{N}\ \forall c_n, c_{n+1}, \dots \in \mathbb{Z}$: $h(0, \dots, 0, c_n, c_{n+1}, \dots) = 0$.

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Examples: Free groups are n-slender [Higman, Griffiths 1952-56]. Certain HNN extensions of n-slender groups are n-slender, including the Baumslag-Solitar groups [Nakamura 2015].

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Theorems [Eda 1992, 2005]

(1) An abelian group A is n-slender $\Leftrightarrow$ A is slender.

(2) A group G is n-slender $\Leftrightarrow$ for every Peano continuum X and every homomorphism $h:\pi_1(X,x)\to G$, $\exists\,\mathcal{U}:h(\pi(\mathcal{U},x))=1$.

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Examples of residually n-slender groups include π_1 of

- 1-dimensional spaces
- Planar sets
- Pontryagin surface $\prod_2$
- Pontryagin sphere $\varprojlim \left(T^2 \leftarrow T^2 \# T^2 \leftarrow T^2 \# T^2 \# T^2 \leftarrow \cdots \right)$

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A is slender $\Leftrightarrow$ is cotorsion-free and $\mathbb{Z}^{\mathbb{N}} \nleqslant A$.

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"A cotorsion-free $\Rightarrow$ A homom-T_2 rel. every Peano continuum X"

Suppose $h : \pi_1(X, x) \to A$ with $0 \neq a \in \bigcap_{\mathcal{U}} h(\pi(\mathcal{U}, x))$.

Find $\widehat{\mathbb{Z}} = \varprojlim (\mathbb{Z}/2!\mathbb{Z} \leftarrow \mathbb{Z}/3!\mathbb{Z} \leftarrow \mathbb{Z}/4!\mathbb{Z} \leftarrow \cdots) \xrightarrow{\phi} A$ with $a \in \phi(\widehat{\mathbb{Z}})$.

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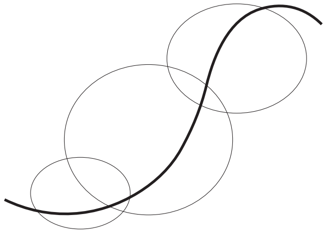

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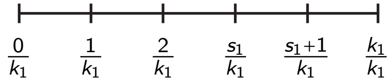

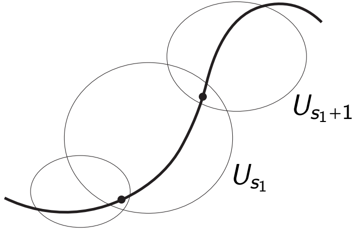

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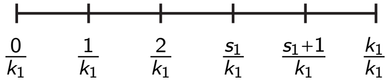

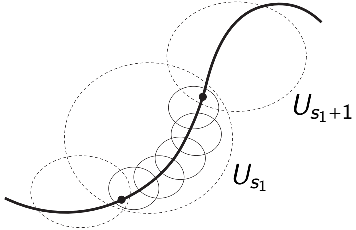

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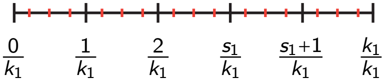

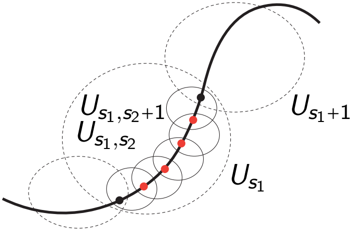

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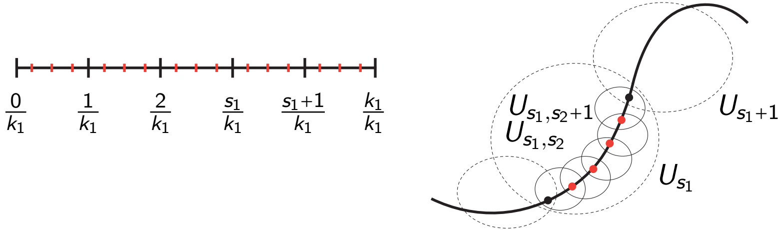

Get a sequence $\mathcal{U}_n = \{U_{s_1,s_2,\dots,s_n} \mid 0 \leqslant s_i < k_i\}$ of open covers of X and subdivision points $a_{s_1,s_2,\dots,s_n} = \sum_{i=1}^{n} \frac{s_i}{\prod_{j=1}^{i} k_j} \in [0,1]$ such that

(1) $U_{s_1,s_2,\dots,s_{n-1},s_n} \subseteq U_{s_1,s_2,\dots,s_{n-1}}$

(2) $\forall\ U \in \mathcal{U}_n$: U is path connected and $\operatorname{diam}(U) < 1/n$

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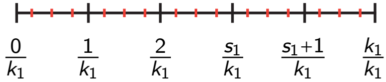

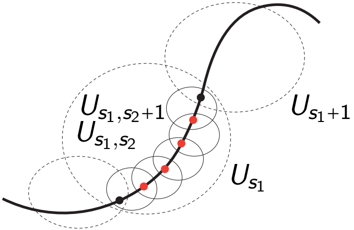

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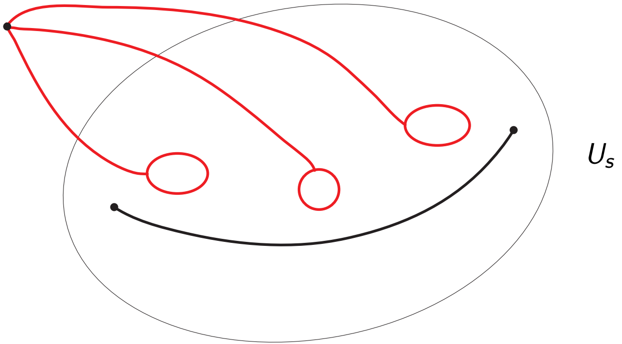
Since $u_n a \in h(\pi(\mathcal{U}_n, x))$ we have $u_n a = h(\prod_{s \in S_n} \prod_{i=1}^{r_s} [\alpha_{s,i} \beta_{s,i} \alpha_{s,i}^{-}]) \in A$ with $\alpha_{s,i} : ([0,1], 0) \to (X, x)$ and $\beta_{s,i} : ([0,1], \{0,1\}) \to U_s$.



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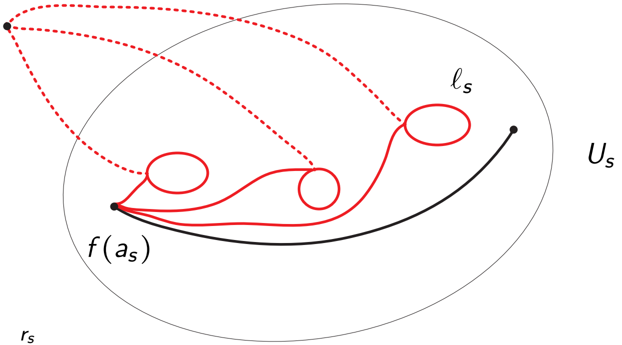

Put $\ell_s = \prod_{i=1}^{r_s} \gamma_{s,i} \beta_{s,i} \gamma_{s,i}^{-}$ with $\gamma_{s,i} : ([0,1], 0) \to (U_s, f(a_s))$.

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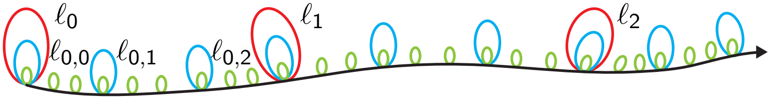
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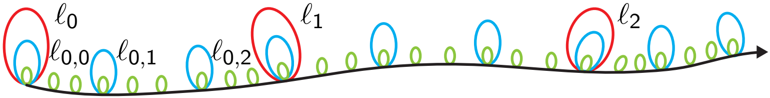
ℓ_1
ℓ_2



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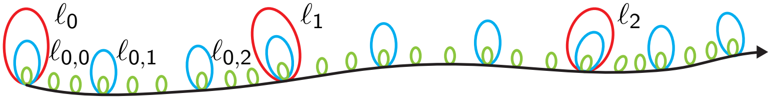

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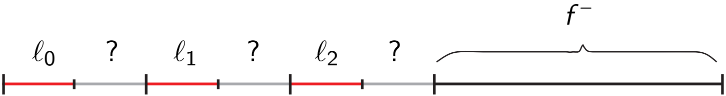

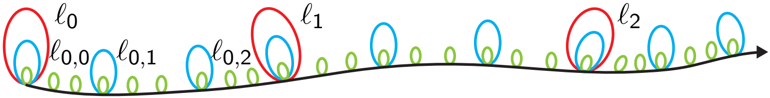

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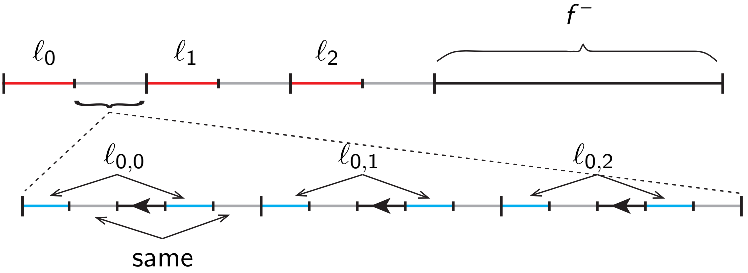

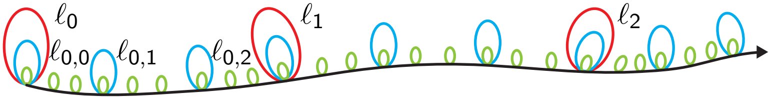

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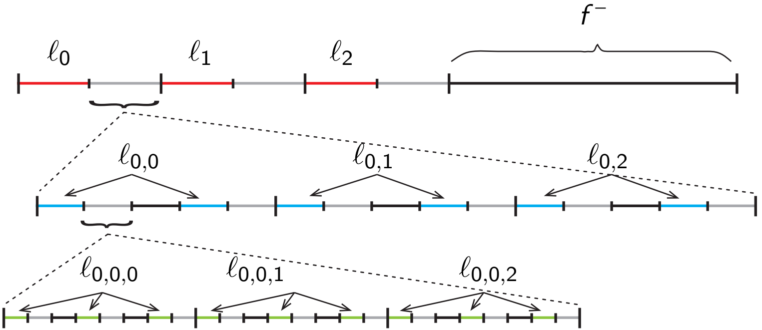

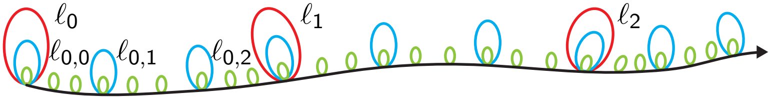

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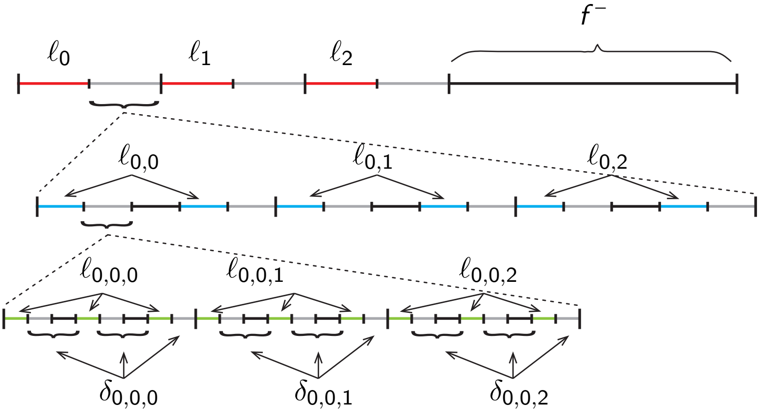

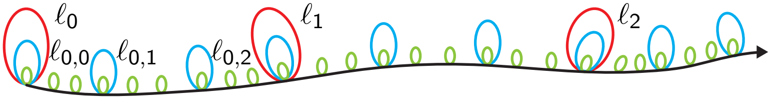

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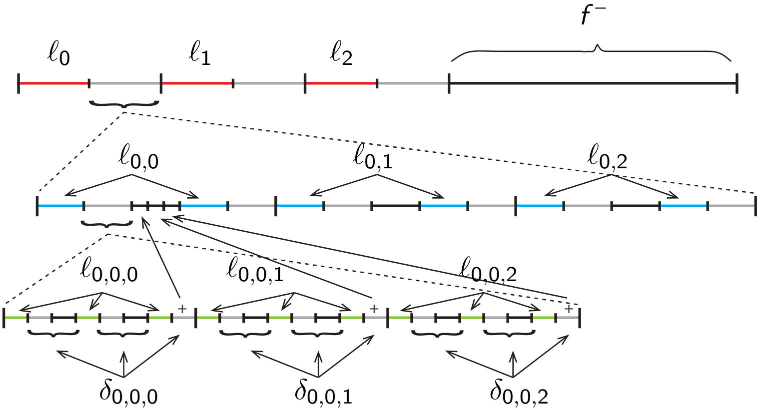

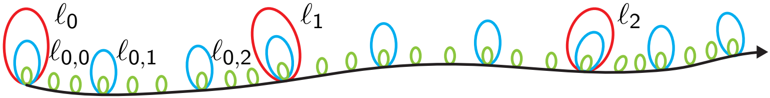

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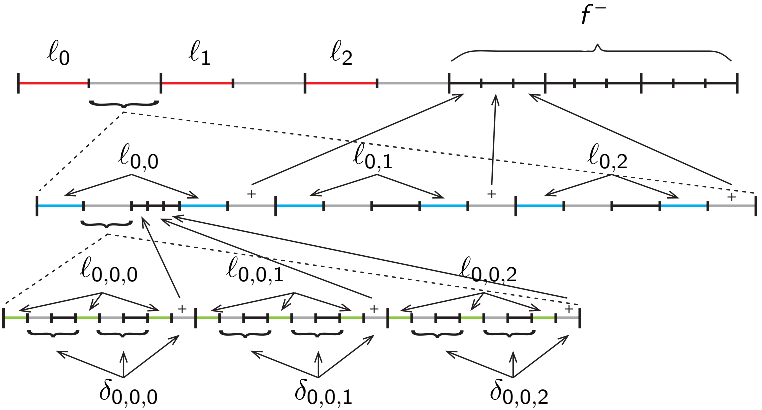

Rearrange the paths in $H_1(X)$ so that

$$[\ell] = \sum_{i=1}^{n-1} i! \sum_{s \in S_i} [\ell_s] + n! \left(\sum_{s \in S_n} [\ell_s] + [\delta_s] \right).$$

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Hence,

$$n! \mid h([\ell]) - \sum_{i=1}^{n-1} i! u_i a \qquad \text{in } A \text{ for all } n \geqslant 2.$$

□

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So, $h = \phi \circ \mu : \pi_1(\mathbb{H}, *) \to A$ with $\bigcap_{\mathcal{U}} h(\pi(\mathcal{U}, x)) = \mathbb{J}_p \neq \{0\}$. □

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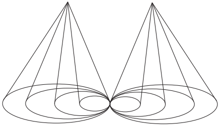

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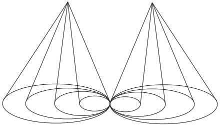

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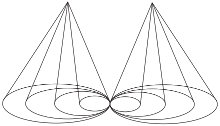

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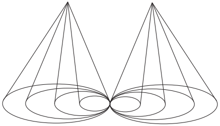

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$$H_1(\mathbb{G}) = \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} = \left(\bigoplus_{2^{\aleph_0}} \mathbb{Q} \right) \oplus \left(\prod_{p \in \mathbb{P}} A_p \right)$$

where $A_p = \prod_{\aleph_0} \mathbb{J}_p =$ **p-adic completion of** $\bigoplus_{2^{\aleph_0}} \mathbb{J}_p$

Reference:

Cotorsion-free groups from a topological viewpoint, K. Eda and H. Fischer, Topology and its Applications 214 (2016) 21–34.