Freely Indecomposable Groups in Wild Topology

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Motivation and History – Non Free Splitting

- G. Higman in 1952: $\hat{F} := \varprojlim_{i \geq 1} F(x_1, \dots, x_i)$ does not split nontrivially as a free product.
- K. Eda 1998: Any homomorphism

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either factors through a canonical projection $p_n : \hat{G} \to G_n$ or the image under ϕ belongs, up to conjugation, to one of the free factors $\mathbb{Z}$.

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- W. Hojka 2014: Let A and B be groups not containing involutions. Then any homomorphism

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Where to go? A Generalization of Higman Completeness

Definition

Let G be a group and $\mathcal{U}$ a nested sequence of subgroups. Then G is $\mathcal{U}$-Higman complete provided for every choice of elements $f_i \in H_i$, every bivariate word $w_i \in F(x, y)$, there is a solution sequence $(h_i)_{i \geq 0}$ of the infinite system of equations

$$h_{i-1} = w_i(f_i, h_i) \quad i \geq 1$$

with $h_i \in H_i$ for all $i \geq 1$.

- If for all $i \geq 0$ one has $H_i = G$ and for $\mathcal{U} = (H_i)_{i \geq 0}$ the group G is $\mathcal{U}$-Higman complete then it is Higman complete. If it is in addition abelian then it is cotorsion.
- The inverse limit $\hat{G} = \varprojlim_{i \geq 1} G_i$ is $\mathcal{U}$-Higman complete for $H_i := \ker(\hat{G} \to G_i)$ and $\mathcal{U} := (H_i)_{i \geq 1}$.
- The topologist's product $G := \circledast_{i \geq 1} G_i$ is $\mathcal{U}$-Higman complete for $\mathcal{U}$ the sequence of subgroups $(\circledast_{j \geq i} G_j)_{i \geq 1}$.

Pertinent Examples

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Main Results

Theorem (A)

Let the free product $G = A * B$ be $\mathcal{U}$-Higman complete. Then there exists $H \in \mathcal{U}$ and $g \in G$ with

$$H^g \subseteq A \quad \text{or} \quad H^g \subseteq B.$$

Theorem (B)

Let G be $\mathcal{U}$-Higman complete and $\phi : G \to A * B$ be a homomorphism. Then one of the following holds:

(i) *There is $H \in \mathcal{U}$ contained in the kernel of ϕ.*

(ii) *Up to conjugacy the image of ϕ is a subgroup of either A or of B.*

Consequences / Remarks

1. Eda's 2011 result can be recovered without making use of the concept of σ-words.
2. The estimates of word length are less sensitive to the appearance of involutions.

Theorem

Every homomorphism

$$\phi : \hat{G} := \varprojlim_{i \geq 1} G_i \to \circledast_{i \geq 1} H_i$$

either factors through a canonical projection $p_n : \hat{G} \to G_n$ or the image under ϕ, up to conjugation, is contained in some free factor H_j.

This generalizes Eda's 1998 result.

Thank you for your Attention.