Geometry of compact lifting spaces

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An inverse limit of a sequence of covering projections over a given space X is not, in general, a covering projection over X but is still a lifting projection, i.e., a Hurewicz fibration with unique path lifting property. However, there are many other natural examples so it is hard to expect a classification theory of lifting projections similar to that of covering projections. In this talk we will present a characterization of a class of lifting projections that is of particular interest, namely those that arise as inverse limits of finite coverings (resp. finite regular coverings).

This is joint work with Gregory Conner and Wolfgang Hefort.

$$\begin{array}{ccccccccc}
\widetilde{X}_1 & \longleftarrow & \widetilde{X}_2 & \longleftarrow & \widetilde{X}_3 & \longleftarrow & \cdots & \longleftarrow & \widetilde{X}_\infty = \varprojlim \widetilde{X}_i \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & & & \downarrow{\scriptstyle p_\infty} \\
X & = & X & = & X & = & \cdots & = & X
\end{array}$$

Examples: dyadic solenoid over S^1

$$p\colon \mathbb{R}^\infty \to (S^1)^\infty$$

$\left.\begin{array}{l} p_\infty \text{ is a Hurewicz fibration} \\ \quad\ \text{has unique path-lifting property} \end{array}\right\}$ lifting projection

(lifting projection $\Leftrightarrow$ Hurewicz fibration with totally path-disconnected fibre)

fibres are inverse limits of discrete spaces

X_∞ may have many leaves

How far is a lifting projection from a covering projection?

$$\left.\begin{array}{l} p\colon E \to B \text{ lifting projection} \\ E \text{ locally } 0-\text{connected} \\ B \text{ semilocally } 1-\text{connected} \end{array}\right\} \Rightarrow p \text{ is a covering projection (Spanier)}$$

Is every lifting projection inverse limit of covering projections?

NO. Irrational slope line, compactified spiral, Morse-Thue dynamical system,...

profinite lifting projection = inverse limit of finite, regular covering projections

fibre is inverse limit of finite groups, i.e., a profinite group

all leaves are dense

deck transformations act transitively on the fibre (***regular*** lifting projection)

Theorem 1 $p\colon E \to X$ *regular lifting projection with dense leaf and compact fibre* F

Then:

◇ *all leaves are dense,*

◇ $A(p)$ *acts freely and transitively on fibres,*

◇ $\Theta\colon A(p) \to F, \quad \varphi \mapsto \varphi(\widetilde{x}_0)$ *is a bijection and endows* $A(p)$ *with profinite group structure.*

Sketch of the proof:

◇ multiplication is separately continuous
◇ by compactness and theorem of Ellis $A(p)$ is compact and totally disconnected

$p\colon E \to X$ lifting projection with fibre F

$\pi_1(X)$ acts on F in two ways: by liftings of paths and by deck transformations

liftings: defined on F, for any path from origin

$$\widetilde{x} \to \widetilde{x}\alpha \quad \text{right action}$$

deck transformations: for regular liftings, defined on E

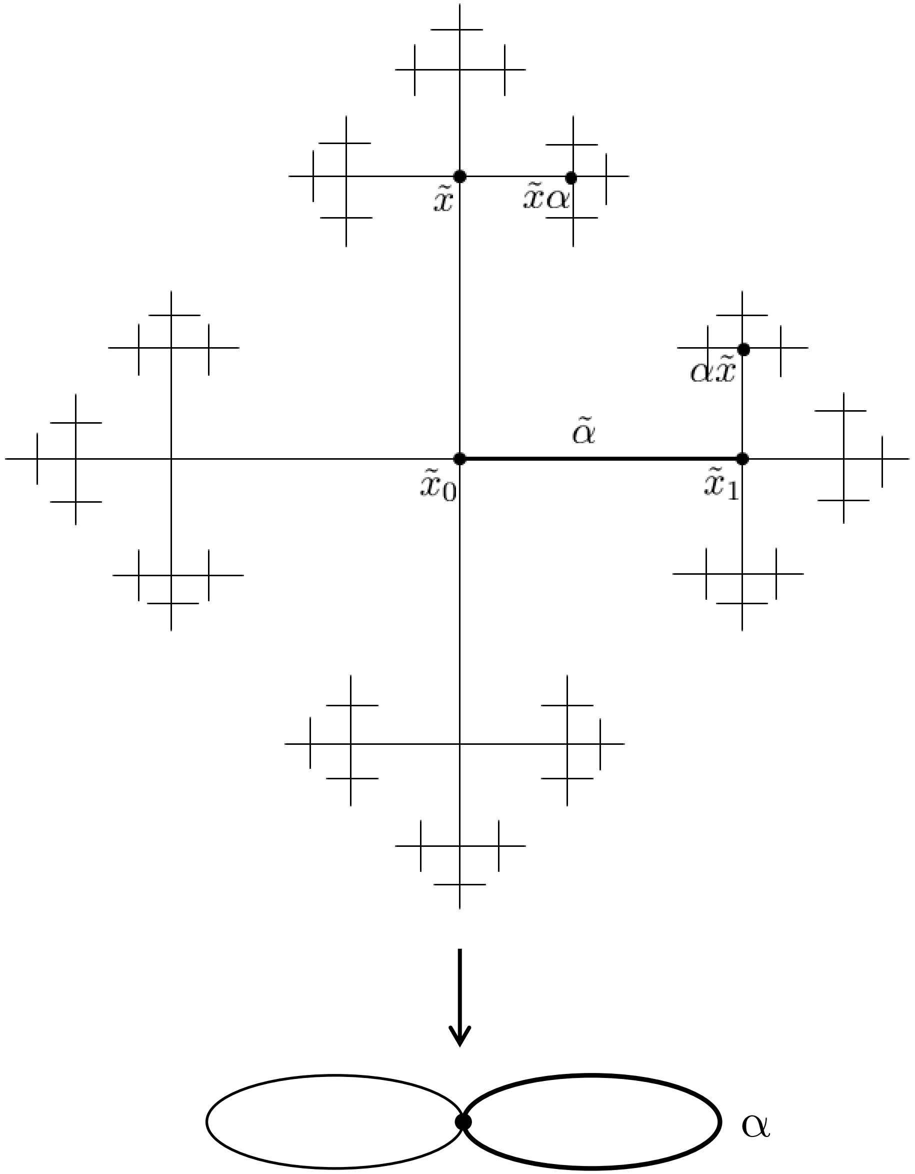
$$\widetilde{x} \to \alpha\widetilde{x} \quad \text{left action}$$

actions coincide on center of $\pi_1(X)$

$p\colon E \to X$ is π_1-*profinite* if F is compact and

$$\pi_1(X) \to F, \quad \alpha \mapsto x_0\alpha$$

is continuous with respect to profinite topology on $\pi_1(X)$



X connected, locally 0-connected, $\pi_1(X)$ residually finite

$\widehat{p}\colon \widehat{X} \to X$ inverse limit of all regular, finite coverings of X

leaf L of $\widehat{X}$ is 1-connected, $\pi_1(X)$ acts on L by deck transformations (left action)

$p\colon E \to X$ lifting projection with fibre F

equivalence relation on $F \times L$: $(u\gamma, \widehat{y}) \sim (u, \gamma\widehat{y})$ for $\gamma \in \pi_1(X, x_0)$

Borel construction $F \times_{\pi_1(X)} L := F \times L/\sim$

Define $\quad \Phi\colon F \times L \to E \quad$ as $\quad \Phi(u, \widehat{x}) := u\alpha,$

where $\alpha\colon (I, 0) \to (X, x_0)$ such that $\widehat{x}_0\alpha = \widehat{x}$

Theorem 2 *X 0-connected, locally categorical, $\pi_1(X)$ residually finite*

$p\colon E \to X$ regular, π_1-profinite, dense leaf $\Rightarrow$ $\overline{\Phi}\colon F \times_{\pi_1(X)} L \to E$ is a homeomorphism

Sketch of the proof:

- consider restriction to fibres and reduce to Theorem 1
- for general case use locally triviality

X 0-connected, locally categorical,
$\pi_1(X)$ residually finite,
$p\colon E \to X$ lifting projection

Theorem 3 *p is profinite $\Leftrightarrow$ p is regular, π_1-profinite, with dense leaves*

Sketch of the proof:

- ◇ F is inverse limits of quotients of $\pi_1(X)$
- ◇ $E \longleftarrow F \times_{\pi_1(X)} L \longrightarrow \varprojlim \left[(\pi_1(X)/G_i) \times_{\pi_1(X)} L\right]$ are fibre-preserving homeomorphisms