ARNOUX-RAUZY INTERVAL EXCHANGES

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ARNOUX-RAUZY ON 3 LETTERS

Symbolic systems on $\{a, b, c\}$ defined by

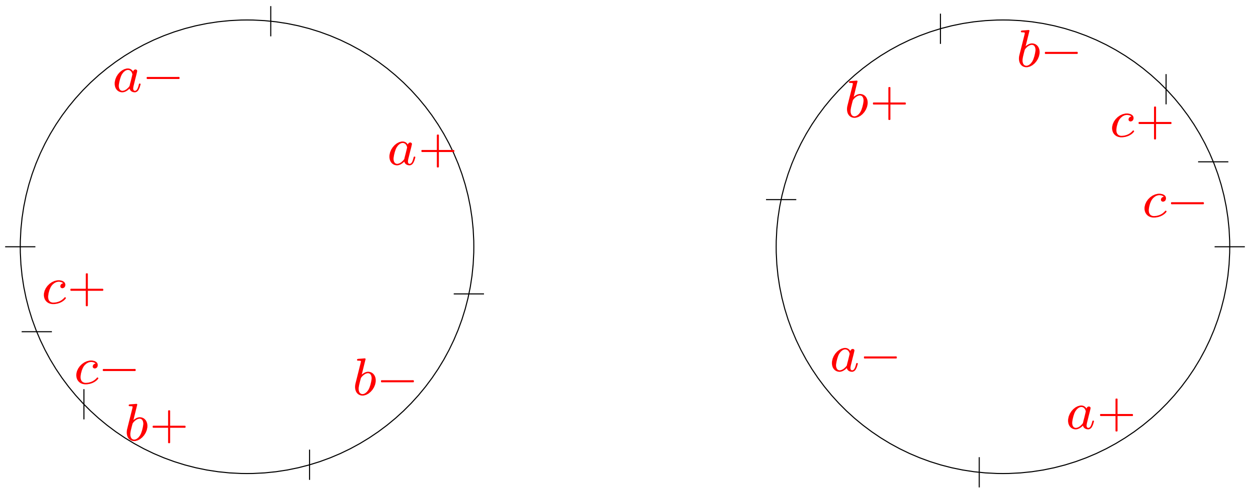
- rule I,
  - $A_{k+1} = A_k B_k$,
  - $B_{k+1} = A_k C_k$,
  - $C_{k+1} = A_k$;
- rule II,
  - $A_{k+1} = A_k B_k$,
  - $B_{k+1} = A_k$,
  - $C_{k+1} = A_k C_k$;
- rule III,
  - $A_{k+1} = A_k$,
  - $B_{k+1} = A_k B_k$,
  - $C_{k+1} = A_k C_k$.

For an AR3 system, infinitely many rules I $\rightarrow$ minimality and unique ergodicity, invariant measure μ.

Natural coding of a rotation of the 2-torus ? True for almost all AR3 (Berthé Steiner Thuswaldner), not for all (Cassaigne Ferenczi Messaoudi).

ARNOUX-RAUZY ON 6 LETTERS

Interval exchange on a circle



Three parameters = lengths a_0, b_0, c_0. Their possible values form the Rauzy gasket.

NOVIKOV'S CONJECTURE

An interval exchange with many intervals and few parameters should not be minimal in general.

The AR6 family gives minimal examples thus is worth studying.

Indeed the AR6 family is small : the Rauzy gasket has not full Hausdorff dimension (Avila Hubert Skripchenko).

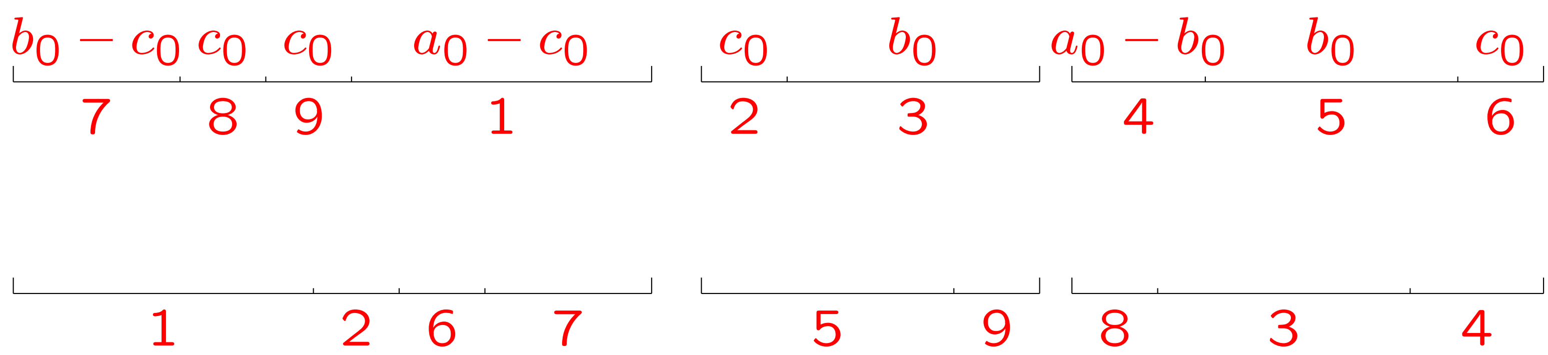
Isomorphism with AR3 ? At least semi-conjugacy : $a+ \to a$, $a- \to a$, ... gives a surjective coding.

For AR6, no set of rules generating the language : we need more letters.

ARNOUX-RAUZY ON 9 LETTERS

Interval exchange on the interval, defined in ABB for Tribonacci.

Up to permutation of the large intervals and symmetry :



$AR9 \to AR6 \to AR3$

$1, 2 \to a-$, $3, 4 \to a+$, $5 \to b-$, $6, 7 \to b+$, $8 \to c-$, $9 \to c+$.

$\phi(1) = \phi(2) = \phi(3) = \phi(4) = a$, $\phi(5) = \phi(6) = \phi(7) = b$, $\phi(8) = \phi(9) = c$.

Infinitely many rules I $\to$ minimality for AR6 and AR9, but not always unique ergodicity (Dynnikov Skripchenko).

MULTIPLICATIVE RULES FOR AR3

Partial quotients k_n

— rule Im

— $A_{m_{n+1}} = A_{m_n}^{k_{n+1}} B_{m_n}$,

— $B_{m_{n+1}} = A_{m_n}^{k_{n+1}} C_{m_n}$,

— $C_{m_{n+1}} = A_{m_n}$;

— rule IIm

— $A_{m_{n+1}} = A_{m_n}^{k_{n+1}} B_{m_n}$,

— $B_{m_{n+1}} = A_{m_n}$,

— $C_{m_{n+1}} = A_{m_n}^{k_{n+1}} C_{m_n}$.

Tribonacci : all $k_n = 1$, all rules Im.

MULTIPLICATIVE RULES FOR AR9

— RULE Im. $1_{m_{n+1}} = 3_{m_n} 4_{m_n}^{k_{n+1}-1} 5_{m_n}$,

— $2_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 5_{m_n}$,

— $3_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 6_{m_n}$,

— $4_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 7_{m_n}$,

— $5_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 8_{m_n}$,

— $6_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 9_{m_n}$,

— $7_{m_{n+1}} = 2_{m_n} 1_{m_n}^{k_{n+1}-1} 9_{m_n}$,

— $8_{m_{n+1}} = 2_{m_n}$, $\quad 9_{m_{n+1}} = 3_{m_n}$;

— RULE IIm. $1_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 7_{m_n}$,

— $2_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 6_{m_n}$,

— $3_{m_{n+1}} = 4_{m_n}^{k_{n+1}} 5_{m_n}$,

— $4_{m_{n+1}} = 3_{m_n} 4_{m_n}^{k_{n+1}-1} 5_{m_n}$,

— $5_{m_{n+1}} = 3_{m_n}$, $\quad 6_{m_{n+1}} = 2_{m_n}$, $\quad 7_{m_{n+1}} = 1_{m_n}$,

— $8_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 9_{m_n}$,

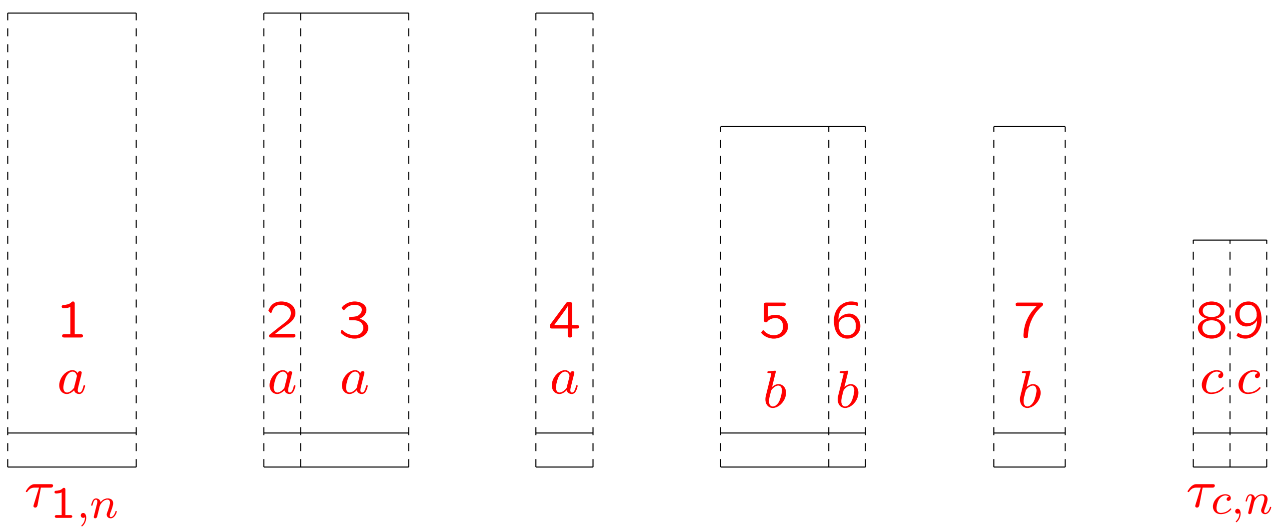
— $9_{m_{n+1}} = 1_{m_n}^{k_{n+1}} 8_{m_n}$.

ROKHLIN TOWERS

A Rokhlin tower τ_n = disjoint sets F_n, TF_n, ..., $T^{h_n-1}F_n$.

At each stage n, three towers $\tau_{a,n}$, $\tau_{b,n}$, $\tau_{c,n}$ for AR3, nine towers, $\tau_{1,n}$ to $\tau_{9,n}$, for AR9.

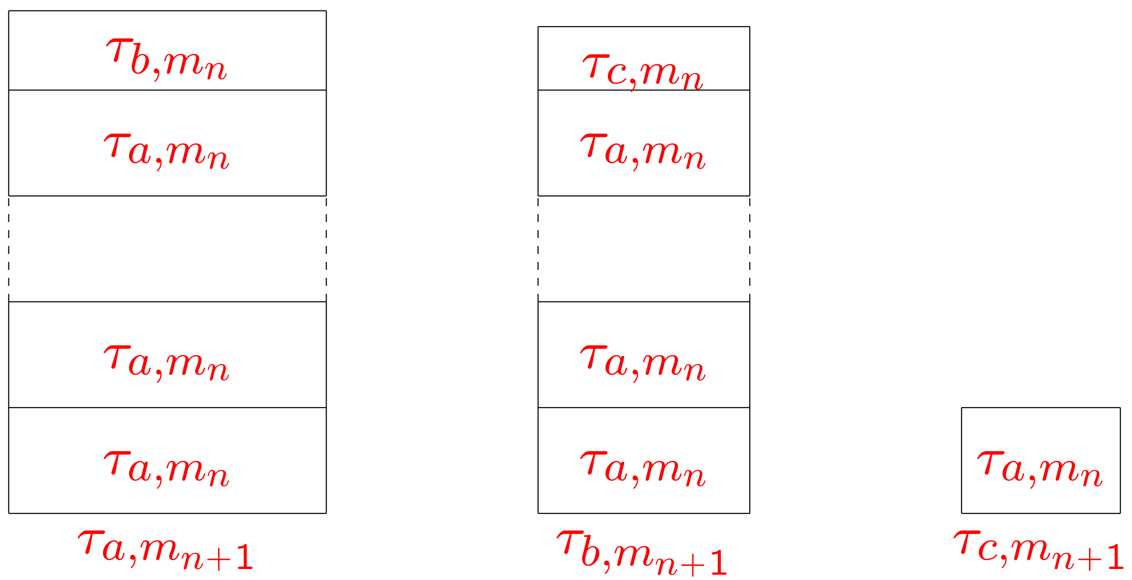
Lemma 1. *Towers 1 to 9 are made with intervals. Towers 2 and 3, 5 and 6, 8 and 9, are contiguous at each stage.*



CUTTING AND STACKING

Example : rule Im at multiplicative stages.

Cut the m_n-towers into columns and stack the columns to get the m_{n+1}-towers.



TOWARDS ISOMORHISM

$AR9 \to AR3$: $E_i = i$ pre-images by ϕ.

Proposition 1. *Every x in an AR3 is in E_1, E_2, or E_3. E_3 is countable. If $\mu(E_1) < 1$, the AR9 is a two-point extension of its AR3 factor.*

Lemma 2. *If y is in tower c infinitely many times, then y is in E_1.*
The condition is sufficient but not necessary.

Proposition 2. *Let*

— $\xi_n = \frac{1}{k_{n+2}}$ *if the $n+1$-th multiplicative rule is of type Im and $k_{n+1} \geqslant 2$,*

— $\xi_n = \frac{1}{3^l k_{n+2} \dots k_{n+l+1}}$ *if the $n+1$-th multiplicative rule is of type Im with $k_{n+1} = 1$ or of type IIm, and the next multiplicative rule of type Im is the $n+l$-th, $l \geqslant 2$.*

Suppose $\sum \xi_n = +\infty$. Then $\underline{\mu(E_1) = 1}$ for the unique invariant measure μ.

RESULTS ON ISOMORPHISM

Theorem 1. *Under the hypothesis of Proposition 2, an AR9 system is uniquely ergodic and measure-theoretically isomorphic to its AR3 factor.*

Theorem 2. *The hypothesis of Proposition 2 is satisfied by almost all AR systems.*

Theorem 3. *If* $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$, *an AR9 or AR6 systems is not uniquely ergodic. The AR9 or AR6 system has two ergodic invariant measures ; it is measure-theoretically isomorphic to its AR3 factor if and only if it is equipped with an ergodic measure.*

WEAK MIXING

Weak mixing $=$ no non-constant eigenfunction $f \circ T = \lambda f$.

There exists weak mixing among AR6 or AR9 equipped with ergodic measures : sufficient conditions $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$ or a weaker one deduced from CFM.

But we do not know any example with unique ergodicity and weak mixing.

RIGID LR IET

Rgidity = there exists a sequence $q_n \to \infty$ such that for any measurable set $\mu(T^{q_n} A \Delta A) \to 0$.

Linear recurrence of the coding = in the language of trajectories, every word of length L occurs in every word of length KL.

Find rigid linearly recurrent interval exchanges : an interval exchange with a circular permutation is a rotation, thus is rigid, even if it is LR. Otherwise ?

The Arnoux-Yoccoz interval exchange (= Tribonacci AR6) is LR (self-induced) and rigid.