Return words in subshifts arising from non-primitive substitutions

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Definition 1 (Y, 2007)

$\sigma : A \to A^+$ is almost primitive $\overset{\text{def}}{\Leftrightarrow}$ $\exists p \in \mathbb{N}$ and $\exists a \in A$ s.t.

1. $\sigma(a) = a^p$;
2. $\forall n \in \mathbb{N}$, $a^n \in \mathcal{L}(\sigma)$, i.e. the language of σ;
3. $\exists k \in \mathbb{N}$ s.t. $\forall b \in A$, $\forall c \in A' := A \setminus \{ a \}$, b occurs in $\sigma^k(c)$.

Remark. If $p > 1$, then Condition 2 is redundant.
An associated subshift:

$$X_\sigma = \left\{ x = (x_i)_i \in A^{\mathbb{Z}} \;\middle|\; x_{[i,j]} := x_i x_{i+1} \ldots x_j \in \mathcal{L}(\sigma)\ \forall i, j \in \mathbb{Z} \right\}.$$

Define an incidence matrix M_σ by for each $(b, c) \in A \times A$,

$$(M_\sigma)_{b,c} = \text{ the number of occurrences of } c \text{ in } \sigma(b).$$

Then, $M_\sigma = \begin{pmatrix} p & O \\ * & Q \end{pmatrix}$ and Q is a primitive $(A' \times A')$-matrix.
Let θ denote Perron-Frobenius eigenvalue of Q.

Remark.

1 $\theta = 1 \Leftrightarrow X_\sigma$ is finite or countable. Consider only the case $\theta > 1$.

2 $\theta > 1 \Leftrightarrow X_\sigma$ is a Cantor set.

3 $\exists \omega \in X_\sigma \setminus \{ a^\infty \}, \exists k \in \mathbb{N}, \exists \ell \in \mathbb{Z}$ s.t.

$$T_\sigma{}^\ell \sigma^k(\omega) = \omega.$$

If $k = 1$, then such ω is called a quasi-fixed point of σ.

4 $\mathcal{L}(\omega) = \mathcal{L}(\sigma)$. Hence, $\overline{\mathrm{Orb}_{T_\sigma}(\omega)} = X_\sigma$.

Lemma 2 (Y, 2007)

If ω is a quasi-fixed point of an almost primitive substitution σ, then

1 $\forall n \in \mathbb{N}, \exists L \in \mathbb{N}$ *s.t.* a^n *occurs in* $\omega_{[i,i+L)}$ $\forall i \in \mathbb{Z}$;

2 $\forall u \in \mathcal{L}(\omega), \exists n \in \mathbb{N}$ *s.t. if* $\omega_{[i,i+n)} = a^n$ *and* $\omega_{i+n} \neq a$ *then*

$$i < \exists j < i + 2n \ s.t. \ \omega_{[j,j+|u|)} = u.$$

Proposition 3 (Y, 2007)

The subshift X_σ is almost minimal, in other words, by definition,

1 X_σ *has a unique fixed point, i.e.* a^∞;

2 $\overline{\mathrm{Orb}_{T_\sigma}(x)} = X_\sigma, \forall x \in X_\sigma \setminus \{ a^\infty \}$.

Definition 4 (Return words)

Let $u, v \in \mathcal{L}(\omega)$ be so that $uv \in A^+$. A word $w \in A^+$ is called a return word to $u.v$ in the quasi-fixed point $\omega \overset{\text{def}}{\Leftrightarrow}$

1. $uwv \in \mathcal{L}(\omega)$;
2. the word uv occurs exactly twice in uwv as prefix and suffix.

Notation.

- Let $\mathcal{R}_{u.v}$ denote the set of return words to $u.v$.
- Set $\mathcal{R}_n = \mathcal{R}_{a^n.a^n}$ and $\mathcal{R}_v = \mathcal{R}_{\Lambda.v}$.

Eg (Cantor substitution). $\sigma : a \mapsto aaa, b \mapsto bab$. Then,

$$\mathcal{R}_b = \{ ba, ba^3, ba^9, ba^{27}, \dots \} \text{ and } \mathcal{R}_{a.a} = \{ a, ababa \} .$$

Facts. $\forall n \in \mathbb{N}$,

- $\#\mathcal{R}_n < \infty$, $a \in \mathcal{R}_n$ and $\mathcal{R}'_n := \mathcal{R}_n \setminus \{ a \} \neq \emptyset$;
- $\forall n \in \mathbb{N}$, each word in $\mathcal{R}_{n+1}$ can be uniquely written as a concatenation of words in $\mathcal{R}_n$.

Assume $p > 1$. Set $\alpha = \log_p \theta$. Set

$$f_\alpha(n) = \begin{cases} n & \text{if } \alpha < 1; \\ n \log n & \text{if } \alpha = 1; \\ n^\alpha & \text{if } \alpha > 1. \end{cases}$$

Notation. For $u \in A^*$, set $|u|_{\neg a} = \# \{ 1 \le i \le |u| \mid u_i \ne a \}$.

Lemma 5 (Y)

$\exists C > 0$ *s.t.*

1. $\forall v \in \mathcal{L}(\omega)^+$, $\sup \{ |w|_{\neg a} \mid w \in \mathcal{R}_v \} \le C|v|^\alpha$;
2. $\forall n \in \mathbb{N}$, $\forall u \in \mathcal{R}'_n$,
   $C^{-1} n^\alpha \le |u|_{\neg a} \le C n^\alpha$ *and* $C^{-1} f_\alpha(n) \le |u| \le C f_\alpha(n)$.

Theorem 6 (Y)

If $\alpha < 1$, then the quasi-fixed point ω is generated by a proper and almost primitive sequence of finitely many morphisms.

A sequence $S = \{ \sigma_n : A_{n+1} \to {A_n}^+ \mid n \geq 0 \}$ of morphisms is almost primitive under a constant $n_0 \overset{\text{def}}{\Leftrightarrow} \exists \{ a_n \}_n \in \prod_{n=0}^{\infty} A_n$ s.t.

1. $\forall n \geq 0$, $\exists p_n \geq 1$ s.t. $\sigma_n(a_{n+1}) = {a_n}^{p_n}$;
2. ${a_0}^p \in \mathcal{L}(S)$, $\forall p \geq 1$;
3. $\forall n \geq 0$, $\forall b \in A'_{n+1} := A_{n+1} \setminus \{ a_{n+1} \}$, some letter in A'_n occurs in $\sigma_n(b)$;
4. $\forall n \geq 0$, $\forall b \in A_n$, $\forall c \in A'_{n+n_0}$, b occurs in $\sigma_n \sigma_{n+1} \ldots \sigma_{n+n_0-1}(c)$.

An almost primitive sequence $S = \{ \sigma_n : A_{n+1} \to {A_n}^+ \mid n \geq 0 \}$ of morphisms is proper $\overset{\text{def}}{\Leftrightarrow}$

5. $\forall n \geq 0$, $\forall b \in A_{n+1}$, the first and last letters of $\sigma_n(b)$ are both a_n;
6. $\forall n \geq 0$, $p_n = 1$.

Let σ denote Cantor substitution: $a \mapsto aaa, b \mapsto bab$.

$$M_\sigma = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, p = 3, \theta = 2 \text{ and } \alpha = \log_3 2 < 1;$$

$$\mathcal{R}_n = \{ a, u_n \}, \text{ where } u_1 = ababa, \text{ and for } n \geq 2,$$

$$u_{n+1} = \begin{cases} au_n au_n a & \text{if } n \in I := \left\{ \frac{3^k - 1}{2} \;\middle|\; k \in \mathbb{N} \right\}; \\ au_n a & \text{if } n \in \mathbb{N} \setminus I. \end{cases}$$

Put $B = \{ 1, 2 \}$. Define $\tau_0 : B \to A^+, \tau_n : B \to B^+ \ (n \in \mathbb{N})$ by

$$\tau_0 : \begin{array}{l} 1 \mapsto a \\ 2 \mapsto ababa \end{array}; \ \tau_n : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 12121 \end{array} (n \in I); \ \tau_n : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 121 \end{array} (n \in \mathbb{N} \backslash I).$$

Then, $\tau_n = \begin{cases} \tau_1 & \text{if } n \in I; \\ \tau_2 & \text{if } n \in \mathbb{N} \setminus I \end{cases}$, and $\forall n \in \mathbb{N}$,

$$\omega_{[-n, |\tau_0 \tau_1 \ldots \tau_{n-1}(2)| - n)} = \tau_0 \tau_1 \ldots \tau_{n-1}(2) = \tau_0 \tau_1 {\tau_2}^2 \tau_1 {\tau_2}^8 \tau_1 {\tau_2}^{26} \ldots \tau_{n-1}(2).$$

Lemma 7 (Y)

- $\forall v \in \mathcal{L}(\omega)$, $\exists N \in \mathbb{N}$ *s.t. if* $\left|\omega_{[i,i+N)}\right|_{\neg a} \geq N$ *then* v *occurs in* $\omega_{[i,i+N)}$;
- $\forall b \in A'$, $|\sigma^k(b)|_{\neg a} = O(\theta^k)$ *and* $|\sigma^k(b)| = \begin{cases} O(p^k) & \textit{if } \alpha < 1; \\ O(k\theta^k) & \textit{if } \alpha = 1; \\ O(\theta^k) & \textit{if } \alpha > 1. \end{cases}$
- $\min\limits_{b \in A} |\sigma^k(b)| = O(p^k)$.