Real sets induced by endomorphisms of the free monoid

Paul Surer

Institut für Mathematik
Universität für Bodenkultur

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Definitions and Notations

$\mathcal{A} = \{1, 2, \ldots, m\}$	finite alphabet
$\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$	free monoid over $\mathcal{A}$ consisting of the set of non-empty words $\mathcal{A}^+$ and the empty word ε (neutral element)
$\overline{\mathcal{A}} = \{\bar{1}, \bar{2}, \ldots, \bar{m}\}$	set of inverse letters
$\overline{\mathcal{A}}^* := \overline{\mathcal{A}}^+ \cup \{\varepsilon\}$	free monoid over $\overline{\mathcal{A}}$
$(\mathcal{A} \cup \overline{\mathcal{A}})^*$	finite words over $\mathcal{A} \cup \overline{\mathcal{A}}$
$\sim$	equivalence relation on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the cancellation law $x\bar{x} \sim \varepsilon \sim \bar{x}x$ (for $x \in \mathcal{A}$)

Observe that $(\mathcal{A} \cup \overline{\mathcal{A}})^*/\sim$ is the free group over $\mathcal{A}$. For $X = x_1 x_2 \ldots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ the word $\overline{X} := \bar{x}_m \ldots \bar{x}_1 \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ is the inverse of X (modulo $\sim$) where $\bar{\bar{x}} = x$ for all $x \in \mathcal{A}$.

Definitions and Notations

For $X = x_1 x_2 \ldots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ and $y \in \mathcal{A}$ define

$$|X|_y = \#\{j \in \{1,\ldots,n\} : x_j = y\} - \#\{j \in \{1,\ldots,n\} : x_j = \bar{y}\}$$
$$|X| = \sum_{y \in \mathcal{A}} |X|_y$$
$$\mathbf{l}(X) = (|X|_1, |X|_2, \ldots, |X|_m)\,.$$

We denote by $\preceq$ the partial ordering on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the prefix-condition:

$$X \preceq Y \Longleftrightarrow \overline{X}Y \in \mathcal{A}^* \pmod{\sim},$$
$$X \prec Y \Longleftrightarrow \overline{X}Y \in \mathcal{A}^+ \pmod{\sim}.$$

Note that for $A, B \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $A \prec B$ the set

$$\{X \in (\mathcal{A} \cup \overline{\mathcal{A}})^* / \sim : A \preceq X \preceq B\}$$

is a $\prec$-chain.

Morphisms of the free monoid

Denote by $\sigma : \mathcal{A}^* \longmapsto \mathcal{A}^*$ a primitive morphism (substitution), hence, there exists a positive integer n such that $|\sigma^n(y)|_x \geq 1$ for all $x, y \in \mathcal{A}$.
We define

$\mathbf{M} := (|\sigma(y)|_x)_{1 \leq x,y \leq m}$	the incidence matrix of σ;
$\theta > 1$	the dominant Perron-Frobenius eigenvalue of $\mathbf{M}$;
$\mathbf{v} = (v_1, \ldots, v_m)$	a strictly positive left eigenvector of $\mathbf{M}$ with respect to θ.

We can extend σ easily to $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ by defining $\sigma(\bar{x}) = \overline{\sigma(x)}$ for all $x \in \mathcal{A}$.

Coding prescriptions

Definition

A *coding prescription (with respect to σ)* is a function c with domain $\mathcal{A}^2$ that assigns to each pair of letters a finite set of integers such that

1. for each $x \in \mathcal{A}$ the set $c(xx)$ is a complete residue system modulo $|\sigma(x)|$ such that for all $k \in c(xx)$ we have $-|\sigma(x)| < k < |\sigma(x)|$;
2. for all $ab \in \mathcal{A}^2$ we have $c(ab) = \{k \in c(aa) : k < 0\} \cup \{0\} \cup \{k \in c(bb) : k > 0\}$.

We call a coding prescription *continuous* if $c(ab)$ consists of consecutive integers for each $ab \in \mathcal{A}^2$.

Graph associated to a coding prescription

We associate to a coding prescription a finite graph $H_{\sigma,c}$ with vertex set $\mathcal{A}^2$ and an edge from ab to $a_1 b_1$ labelled by $(D, a_1 b_1)$ with $D \in \mathcal{A}^* \cup \overline{A}^*$ whenever $|D| \in c(ab)$ and $\overline{\sigma(a)} \preceq D\bar{a}_1 \prec D \prec Db_1 \preceq \sigma(b)$.

Observe that each each vertex ab has exactly $|c(ab)|$ outgoing edges.

For each positive integer n we denote by $H^n_{\sigma,c}(ab)$ the set of paths of length n that start in ab. Analogously $H^\infty_{\sigma,c}(ab)$ is the set of infinite walks that start in ab.

Finite paths

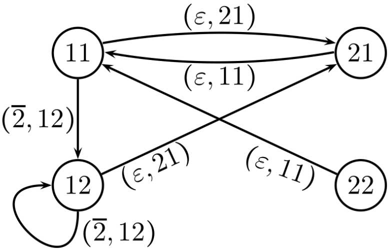
Theorem 1
Let n be a positive integers and define the function $c^{(n)}$ on $\mathcal{A}^2$ by

$$c^{(n)}(ab) := \left\{ \sum_{j=1}^{n} |\sigma^{n-j}(D_j)| : (D_j, a_j b_j)_{j=1}^{n} \in H_{\sigma,c}^{n}(ab) \right\} \subset \mathbb{Z}.$$

Then $c^{(n)}$ is a coding prescription with respect to σ^n.

Example (Fibonacci substitution)

$$\begin{aligned} \sigma(1) &= 12, \\ \sigma(2) &= 1. \end{aligned} \qquad \begin{aligned} c(11) = c(12) &= \{-1, 0\}, \\ c(21) = c(22) &= \{0\}. \end{aligned}$$



We have $\sigma^2 : 1 \mapsto 121, 2 \mapsto 12$ and $\sigma(\overline{2}) = \overline{1}$, therefore

$$c^{(2)}(11) = c^{(2)}(12) = \{-2, -1, 0\}, c^{(2)}(21) = c^{(2)}(22) = \{-1, 0\}.$$

Infinite paths

For each $ab \in \mathcal{A}^2$ define

$$I_{ab} := \left\{ \sum_{j\geq 1} \langle \mathbf{l}(D_j), \mathbf{v} \rangle \theta^{-j} : (D_j, a_j b_j)_{j\geq 1} \in H^{\infty}_{\sigma,c}(ab) \right\} \subset \mathbb{R}.$$

Note that always $0 \subseteq I_{ab}$ where equality may hold.

Problem

Characterise the sets I_{ab} (in terms of c and σ).

Some fundamental properties

Lemma 2

There exists a positive constant $K \in \mathbb{R}$ such that for each $ab \in \mathcal{A}^2$

$$I_{ab} = K \cdot \lim_{n \to \infty} \theta^{-n} c^{(n)}(ab)$$

(the limit with respect to the Hausdorff metric).

Proposition 3

For each $ab \in \mathcal{A}^2$

1. I_{ab} is a compact set;
2. $I_{ab} \subseteq [-v_a, v_b]$;
3. $I_{ab} = (I_{aa} \cap (-\infty, 0)) \cup \{0\} \cup (I_{bb} \cap (0, \infty))$;
4. $$I_{ab} = \bigcup_{(D, a_1 b_1) \in H^1_{\sigma,c}(ab)} \theta^{-1}(\langle \mathbf{l}(D), \mathbf{v} \rangle + I_{a_1 b_1}).$$

Continuous coding prescriptions

Proposition 4

Suppose that c is continuous.

- For each $ab \in \mathcal{A}^2$ we have $I_{ab} = [-v_a^-, v_b^+]$ with $v_a^-, v_b^+ \geq 0$.
- $(v_1^-, \ldots, v_m^-) + (v_1^+, \ldots, v_m^+) = (v_1, \ldots, v_m)$.
- For each $ab \in \mathcal{A}^2$ the particular subsets in the union

$$I_{ab} = \bigcup_{(D, a_1 b_1) \in H^1_{\sigma,c}(ab)} \theta^{-1}(\langle \mathbf{l}(D), \mathbf{v} \rangle + I_{a_1 b_1})$$

have pairwise disjoint interior.

A special case

Proposition 5

Suppose that $|\sigma(x)| \equiv 1 \pmod 2$ for all $x \in \mathcal{A}$ and $c(ab) \subset 2\mathbb{Z}$ for all $ab \in \mathcal{A}^2$. Then the following items hold for all $ab \in \mathcal{A}^2$.

- $I_{ab} = [-v_a, v_b]$.
- The particular subsets in the union

$$I_{ab} = \bigcup_{(D,a_1b_1)\in H^1_{\sigma,c}(ab)} \theta^{-1}(\langle \mathbf{l}(D), \mathbf{v}\rangle + I_{a_1b_1})$$

  have pairwise disjoint interior.

Questions

Question 1

For which setting is I_{ab} an interval?

Question 2

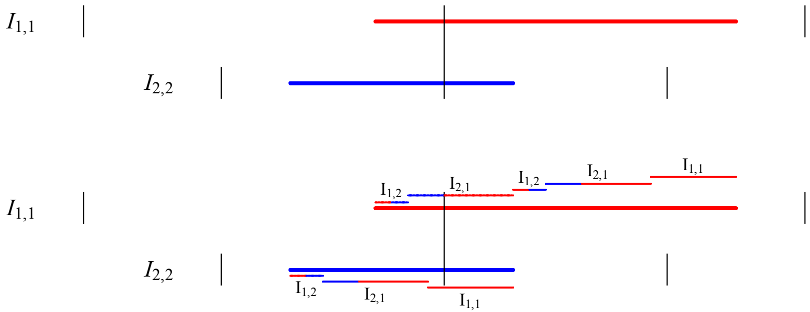
For which setting do the subsets in the union (4. of Proposition 3) have disjoint interior? Alternatively, which setting corresponds to a graph directed iterated function system?

Question 3

What is the Lebesuge measure of I_{ab}?

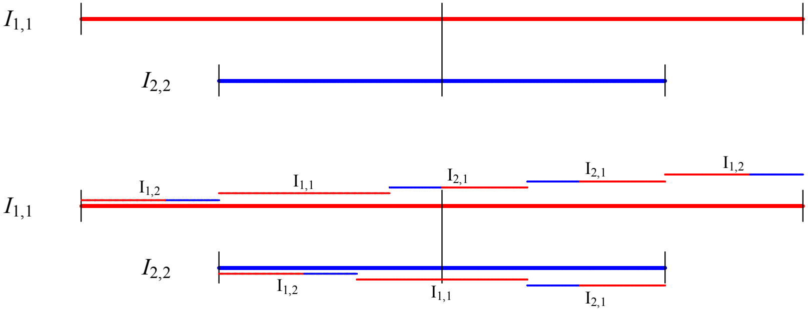
Example 1

$$\begin{array}{llll} \sigma: & 1 \mapsto 12112, & 2 \mapsto 121 \\ c: & 11 \mapsto \{-1,0,1,2,3\}, & 22 \mapsto \{-2,1,0\} \end{array}$$



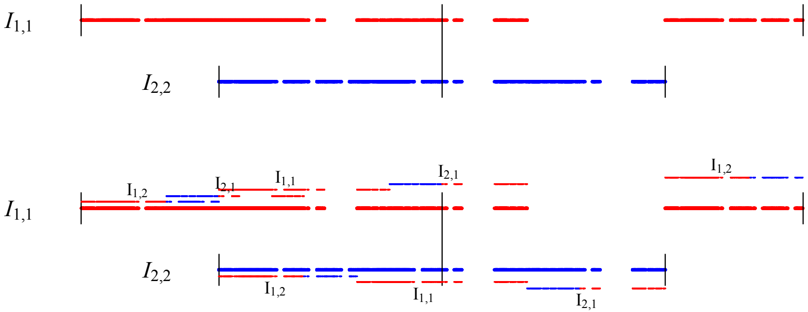
Example 2

$$\begin{aligned} \sigma: \quad & 1 \mapsto 12112, & 2 &\mapsto 121 \\ c: \quad & 11 \mapsto \{-4,-2,0,2,4\}, & 22 &\mapsto \{-2,0,2\} \end{aligned}$$



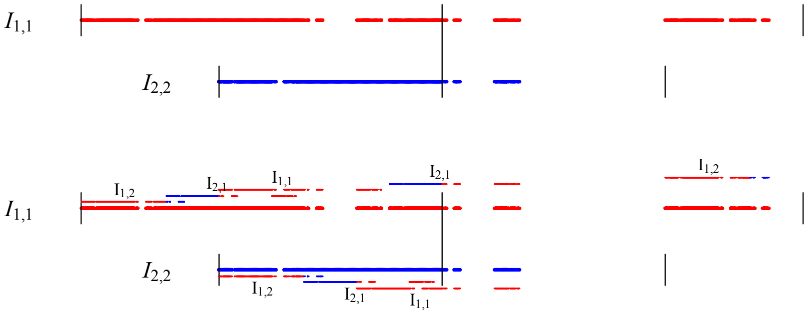
Example 3

$$\sigma: \quad 1 \mapsto 12112, \qquad 2 \mapsto 121$$
$$c: \quad 11 \mapsto \{-4,-3,-2,0,4\}, \quad 22 \mapsto \{-2,0,2\}$$



Example 4

$$\sigma: \quad 1 \mapsto 12112, \qquad 2 \mapsto 121$$
$$c: \quad 11 \mapsto \{-4,-3,-2,0,4\}, \quad 22 \mapsto \{-2,-1,0\}$$



Example 5

$$\begin{array}{llll} \sigma: & 1 \mapsto 121, & 2 & \mapsto 12 \\ c: & 11 \mapsto \{-2,-1,0\}, & 22 & \mapsto \{0,1\} \end{array}$$