

TOPOLOGY OF A CLASS OF $p2$ -CRYSTALLOGRAPHIC REPLICATION TILES

BENOÎT LORIDANT AND SHU-QIN ZHANG

ABSTRACT. We study the topological properties of a class of planar crystallographic replication tiles. Let $M \in \mathbb{Z}^{2 \times 2}$ be an expanding matrix with characteristic polynomial $x^2 + Ax + B$ ($A, B \in \mathbb{Z}$, $B \geq 2$) and $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. Then the equation

$$MT + \frac{B-1}{2}\mathbf{v} = T \cup (T + \mathbf{v}) \cup (T + 2\mathbf{v}) \cup \dots \cup (T + (B-2)\mathbf{v}) \cup (-T)$$

defines a unique nonempty compact set T satisfying $\overline{T^o} = T$. Moreover, T tiles the plane by the crystallographic group $p2$ generated by the π -rotation and the translations by integer vectors. It was proved by Leung and Lau in the context of self-affine lattice tiles with collinear digit set that $T \cup (-T)$ is homeomorphic to a closed disk if and only if $2|A| < B + 3$. However, this characterization does not hold anymore for T itself. In this paper, we completely characterize the tiles T of this class that are homeomorphic to a closed disk.

1. INTRODUCTION

A *crystallographic replication tile* with respect to a crystallographic group $\Gamma \subset \text{Isom}(\mathbb{R}^n)$ is a nonempty compact set $T \subset \mathbb{R}^n$ that is the closure of its interior ($\overline{T^o} = T$) and satisfies the following properties.

- (i) There is an expanding affine mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$, and a finite collection $\mathcal{D} \subset \Gamma$ called *digit set* such that

$$(1.1) \quad g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

- (ii) The family $\{\gamma(T); \gamma \in \Gamma\}$ is a *tiling* of \mathbb{R}^n . In other words, $\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} \gamma(T)$ and $\gamma(T^o) \cap \gamma'(T^o) = \emptyset$ for distinct elements $\gamma, \gamma' \in \Gamma$.

There is a vast literature dealing with the lattice case, *i.e.*, when Γ is isomorphic to \mathbb{Z}^n : criteria exist to check basic properties, such as the tiling property [11], connectedness [9] or, in the planar case ($n = 2$), homeomorphy to a closed disk (*disk-likeness*). For instance, Bandt and Wang recognize disk-like self-affine lattice tiles by the number and location of the neighbors in the tiling [4], and Lau and Leung characterize all the disk-like tiles among the class of self-affine lattice tiles with collinear digit set [12]. A powerful tool in the study of topological properties is the *neighbor graph*: it gives a precise description of the boundary of the tile in terms of a *graph directed iterated function system (GIFS)*. Akiyama and the first author elaborated a boundary parametrization method by making extensive use of the neighbor graph [2]. Algorithms allow to determine the neighbor graph for any given tile T [17], while it is usually difficult to deal with infinite classes of tiles. However, Akiyama and Thuswaldner computed the neighbor graph for an infinite

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class of planar self-affine lattice tiles associated with canonical number systems and used it to characterize the disk-like tiles among this class [3]. Methods relying on the neighbor graph were extended to crystallographic replication tiles in [15, 16].

If T is a crystallographic replication tile, the associated digit set \mathcal{D} must be a complete set of right coset representatives of the subgroup $g \circ \Gamma \circ g^{-1}$. On the other hand, if $T \subset \mathbb{R}^n$ is a nonempty compact set satisfying (1.1) and \mathcal{D} is a complete set of right coset representatives of the subgroup $g \circ \Gamma \circ g^{-1}$, Gelbrich proves that there is a subset $\Gamma' \subset \Gamma$ called *tiling set* such that the family $\{\gamma(T); \gamma \in \Gamma'\}$ is a tiling of \mathbb{R}^n . Under these conditions, it is not known in general whether the tiling set Γ' is a *subgroup* of the crystallographic group Γ , contrary to the lattice case (see [10]). However, the first author defined in [14] the *crystallographic number systems*, in analogy to the canonical number systems from the lattice case (see *e.g.* [8]). This gives a way to produce classes of crystallographic replication tiles whose tiling set is the whole group Γ . An infinite class of examples given in [14] reads as follows. Let $p2$ be the planar crystallographic group generated by the translations $a(x, y) = (x + 1, y)$, $b(x, y) = (x, y + 1)$ and the π -rotation $c(x, y) = (-x, -y)$. Moreover, for $A, B \in \mathbb{Z}$ satisfying $|A| \leq B \geq 2$, let g be the expanding mapping defined on \mathbb{R}^2 by

$$(1.2) \quad g(x, y) = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}.$$

Then the equation

$$(1.3) \quad g(T) = T \cup \left(T + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \dots \cup \left(T + \begin{pmatrix} B-2 \\ 0 \end{pmatrix} \right) \cup (-T)$$

defines a crystallographic replication tile whose tiling set is the whole group $p2$. This tiling property follows from the crystallographic number system property only for $A \geq -1$, as stated in [14], but we will deduce it for all values of A (see Proposition 2.5). Moreover, we will obtain topological information on T by comparing it with the self-affine lattice tile T^ℓ defined by

$$(1.4) \quad \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} T^\ell = T^\ell \cup \left(T^\ell + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \dots \cup \left(T^\ell + \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right).$$

In fact, for fixed A and B , the tile T^ℓ is a translation of $T \cup (-T)$, as shown in [14].

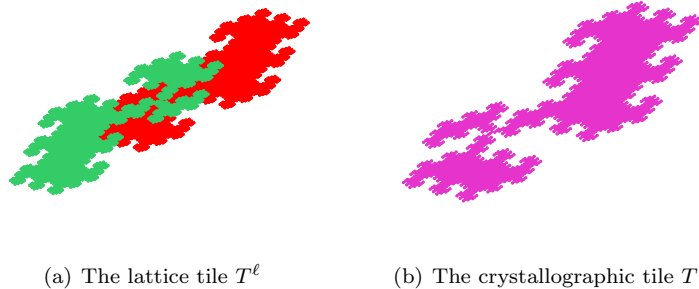


FIGURE 1. Lattice tile and Crystile for $A = 2, B = 3$.

It follows from Leung and Lau's result [12] on self-affine tiles with collinear digit set that T^ℓ is disk-like if and only if $2|A| - B < 3$. However, it was noticed in [14] that it can happen that T^ℓ is disk-like while T is not disk-like (see Figure 1 and Figure 10). The current paper will establish exactly for which parameters A, B this phenomenon occurs. For $2|A| - B < 3$, the associated lattice tile T^ℓ is disk-like and a result of

Akiyama and Thuswaldner [3] on canonical number system tiles will allow us to estimate the set of neighbors of T . Finding out the disk-like tiles for parameters satisfying $2|A| - B < 3$ will then rely on the construction of the associated neighbor graphs for the whole class. For $2|A| - B \geq 3$, a purely topological argument will enable us to prove that the associated tiles are not disk-like.

Our results easily generalize to a broader class of crystallographic replication tiles, closely related to the class of self-affine tiles with consecutive collinear digit set as studied by Leung and Lau in [12]. Therefore, we are able to show the following classification theorem.

Theorem 1.1. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $B \geq 2$, $M \in \mathbb{Z}^{2 \times 2}$ a matrix with characteristic polynomial $x^2 + Ax + B$ and let $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. Let T be the crystallographic replication tile defined by*

$$(1.5) \quad MT + \frac{B-1}{2}\mathbf{v} = T \cup (T + \mathbf{v}) \cup (T + 2\mathbf{v}) \cup \dots \cup (T + (B-2)\mathbf{v}) \cup (-T).$$

Then T is disk-like if and only if $-2 \leq A \leq 1$ and $B \geq 2$ or $A = B = 2$.

This class is obtained from Leung and Lau's class by replacing the last translation digit $(B-1)\mathbf{v}$ by the π -rotation around the origin. In this way, the digit set remains "almost consecutive" and the digit tiles $-T, T, T + \mathbf{v}, \dots, T + (B-2)\mathbf{v}$ form a connected chain, so that T itself is still connected. The original expanding mapping $(x, y)^t \mapsto M(x, y)^t$ of Leung and Lau is adjusted by a translation vector $\frac{B-1}{2}\mathbf{v}$ in order for the digit set to be a complete set of right coset representatives of $g \circ p2 \circ g^{-1}$. Note that there might be other choices for the digit set, but they may not preserve the connectedness of the tiles (see [13]).

The result tells us that only a few tiles of the class are disk-like. For larger values of A , the tiles become thinner, so that adjacent neighbors from both sides of the tile happen to meet, creating cut points (local or global).

We will see in Lemma 2.3 that, to prove Theorem 1.1, it suffices to prove the result for T given by (1.3). Then the proof of the theorem will be completed by Theorem 5.3 and 6.3.

The paper is organized as follows. In Section 2, we give basic definitions on crystallographic groups and general properties of the class of crystallographic replication tiles under consideration. Sections 3 and 4 are devoted to the construction of the neighbor graphs for part of this class. They will be the main tool for our topological study. In Section 5 and Section 6, we characterize the disk-like tiles among our class for the range of parameters A, B satisfying $2|A| - B < 3$. In Section 7, we show that T is not disk-like for all parameters satisfying $2|A| - B \geq 3$. Finally, Section 8 illustrates the theorem by examples.

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2. PRELIMINARIES

2.1. Basic definitions. Let us recall some definitions and facts about planar tilings and crystallographic replication tiles (*crystiles* for short).

A tiling of \mathbb{R}^2 is a cover of the space by nonoverlapping sets, *i.e.*, such that the interiors of two distinct sets of the cover are disjoint. We consider tilings using a single tile T with $T^\circ = T$ and a family Γ of isometries of \mathbb{R}^2 such that

$$\mathbb{R}^2 = \bigcup_{\gamma \in \Gamma} \gamma(T).$$

Assume that Γ contains id , the identity map of \mathbb{R}^2 . Then $T = id(T)$ is called the *central tile* of the tiling. Given two isometries $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$, we say that $\gamma(T), \gamma'(T)$ (or simply γ, γ') are *neighbors* if $\gamma'(T) \cap \gamma(T) \neq \emptyset$. The neighbor set of T is the set of neighbors of id , *i.e.*,

$$\mathcal{S} = \{\gamma \in \Gamma \setminus \{id\}; \gamma(T) \cap T \neq \emptyset\}.$$

It is symmetric and it generates Γ . The tiles considered in this paper will be compact and the tilings locally finite, *i.e.*, every compact set intersects finitely many tiles of the tiling. Therefore, \mathcal{S} will always be a finite set. The neighbor set of a tile $\gamma(T)$ ($\gamma \in \Gamma$) is equal to $\gamma\mathcal{S}$.

We will deal with families Γ of isometries that are *crystallographic groups* in dimension 2, *i.e.*, discrete cocompact subgroups Γ of the group $\text{Isom}(\mathbb{R}^2)$ of all isometries on \mathbb{R}^2 with respect to some metric. By a theorem of Bieberbach (see [5]), a crystallographic group Γ in dimension 2 contains a group Λ of translations isomorphic to the lattice \mathbb{Z}^2 , and the quotient group Γ/Λ , called *point group*, is finite. There are 17 nonisomorphic such groups. However, in this paper, we will mainly consider the following crystallographic $p2$ -groups.

Definition 2.1. Let $a(x, y) = (x + 1, y)$, $b(x, y) = (x, y + 1)$, $c(x, y) = (-x, -y)$. Then a crystallographic $p2$ -group is a group of isometries of \mathbb{R}^2 isomorphic to the subgroup of $\text{Isom}(\mathbb{R}^2)$ generated by the translations a, b and the π -rotation c .

In particular, the standard $p2$ -group Γ has the form

$$(2.1) \quad \Gamma = \{a^p b^q c^r; p, q \in \mathbb{Z}, r \in \{0, 1\}\}.$$

We will call a tiling with respect to a $p2$ -group a $p2$ -tiling, and a tiling with respect to a lattice group (*i.e.*, for which the point group only contains the class of the identity map of \mathbb{R}^2) a *lattice tiling*.

We will be concerned with self-replicating tiles constructed in the following way. We refer the reader to [6, 16] for further information about these tiles.

Definition 2.2. A planar *crystallographic replication tile* with respect to a crystallographic group Γ is a compact nonempty set $T \subset \mathbb{R}^2$ with the following properties:

- The family $\{\gamma(T); \gamma \in \Gamma\}$ is a tiling of \mathbb{R}^2 .
- There is an expanding affine map $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$ and there exists a finite collection $\mathcal{D} \subset \Gamma$ called *digit set* such that (1.1) is satisfied.

2.2. Lattice tiling and $p2$ -tiling. Let Γ be the standard $p2$ -group defined in (2.1). We recall that an expanding affine map g in \mathbb{R}^n has the form $g(x) = Mx + t$, where $t \in \mathbb{R}^n$ and M is an $n \times n$ expanding matrix, *i.e.*, all its eigenvalues have modulus greater than 1.

We consider a special class of $p2$ -crystallographic replication tiles, closely related to the class of self-affine tiles with collinear digit set studied by Leung and Lau in [12]. For $A, B \in \mathbb{Z}$, $B \geq 2$, let $M \in \mathbb{Z}^{2 \times 2}$ be a matrix with characteristic polynomial $x^2 + Ax + B$. Then M is expanding if and only if $|A| \leq B$. Moreover, let $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. The purpose of this paper is to study the topology of the crystallographic replication tiles defined by (1.5). A change of coordinate system will simplify the proof of Theorem 1.1, as stated in the following lemma.

Lemma 2.3. *Let $A, B \in \mathbb{Z}$ with $|A| \leq B$ and $B \geq 2$. Then the crystallographic replication tile defined by (1.5) is homeomorphic to the tile defined by (1.3).*

Proof. The expanding matrices used in (1.5) and (1.3) are similar via the transfer matrix $C = (\mathbf{v}, M\mathbf{v})$. It follows that the tiles defined by these equations only differ from the linear transformation associated with C .

□

From now on, given $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $B \geq 2$, we denote by g the expanding affine map (1.2), by \mathcal{D} the digit set

$$(2.2) \quad \mathcal{D} = \{id, a, \dots, a^{B-2}, c\}$$

where a, c are defined in Definition 2.1, and by $T = T(A, B)$ the associated tile satisfying (1.3), *i.e.*, $g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T)$.

The relation to self-affine tiles with collinear digit set reads as follows. Let

$$(2.3) \quad \mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}.$$

We denote by $T^\ell(A, B) = T^\ell$ the associated lattice tile satisfying (1.4), *i.e.*, $MT^\ell = \bigcup_{d \in \mathcal{N}} (T^\ell + d)$.

Lemma 2.4 ([14]). *We have*

$$(2.4) \quad T^\ell = T \cup (-T) + (M - I_2)^{-1} \begin{pmatrix} B-1 \\ 2 \\ 0 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix.

Hereafter, we denote the crystallographic tile and lattice tile associated with the above data (p_2, g, \mathcal{D}) and $(\mathbb{Z}^2, M, \mathcal{N})$ by T and T^ℓ , respectively.

Proposition 2.5. *T is a p_2 -crystallographic replication tile.*

Proof. By a result of Gelbrich [6], since \mathcal{D} is a complete set of right coset representatives of $g \circ \Gamma \circ g^{-1}$, we know that T has nonempty interior and the family $\{\gamma(T); \gamma \in \Gamma\}$ is a cover of \mathbb{R}^2 . Thus we only need to prove that this cover is in fact a tiling of \mathbb{R}^2 . For $A \geq -1$, the family $\{T^\ell + z; z \in \mathbb{Z}^2\}$ is a tiling of \mathbb{R}^2 , since the tile T^ℓ is associated to a quadratic canonical number system (see *e.g.* [3]). This also holds for the tiles T^ℓ with $A \leq 0$, as it is mentioned in [1] that changing A to $-A$, for a fixed B , results in an isometric transformation for the associated tiles T^ℓ (see Equation (6.1)). Therefore, by Lemma 2.4, we just need to show that T and $c(T) = -T$ do not overlap. This follows from the fact that T has nonempty interior and satisfies the set equation (1.3). Indeed, each of the B sets on the right side of this equation has two-dimensional Lebesgue measure α/B , where $\alpha > 0$ is the two-dimensional Lebesgue measure of T . The total measure of the right side being equal to α , the sets can not overlap. □

Note that for $-1 \leq A \leq B$, the above proposition is also a consequence of the crystallographic number system property [14].

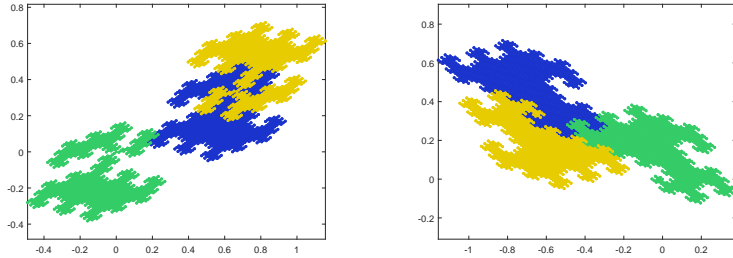


FIGURE 2. $B = 2$. For $A = 2$ on the left, T is not disk-like and for $A = -2$ on the right, T is disk-like

Remark 2.6. In the above proof, we mentioned the simple relation (6.1) between the lattice tiles T^ℓ associated to A and $-A$. It turns out that no such easy relation can be found for the corresponding tiles T , and the topology may become different when changing A to $-A$ (see Figure 2, Section 6 for detail).

For the lattice data $(\mathbb{Z}^2, M, \mathcal{N})$, the following proposition is proved by Leung and Lau [12].

Proposition 2.7. *Let A and B satisfy $|A| \leq B$ and $B \geq 2$. Then T^ℓ is homeomorphic to a closed disk if and only if $2|A| < B + 3$.*

2.3. Neighbor graph. Finally, we introduce an important tool for our study, namely, the neighbor graph.

Definition 2.8. ([16]) For $\Omega \subset \Gamma$ we define the graph $G(\Omega)$ as follows. The states of $G(\Omega)$ are the elements of Ω , and there is an edge

$$\gamma \xrightarrow{\delta|\delta'} \gamma' \quad \text{iff } \delta^{-1}g\gamma g^{-1}\delta' = \gamma' \text{ with } \gamma, \gamma' \in \Omega \text{ and } \delta, \delta' \in \mathcal{D}.$$

The *neighbor graph* $G(\mathcal{S})$ is very important in the present paper.

Recall that the neighbor set of T is defined by $\mathcal{S} = \{\gamma \in \Gamma \setminus \{id\}; T \cap \gamma(T) \neq \emptyset\}$. Set $B_\gamma = T \cap \gamma(T)$ for $\gamma \in \Gamma$. The nonoverlapping property yields for the boundary of T that $\partial T = \bigcup_{\gamma \in \mathcal{S}} B_\gamma$. Moreover using the above notation, the sets B_γ satisfy the set equation ([16])

$$B_\gamma = \bigcup \left\{ g^{-1}\delta(B_{\gamma'}); \delta \in \mathcal{D}, \gamma' \in \mathcal{S}, \exists \delta' \in \mathcal{D}, \gamma \xrightarrow{\delta|\delta'} \gamma' \in G(\mathcal{S}) \right\}.$$

The following characterization is from [16].

Characterization 2.9. Let t be a point in \mathbb{R}^2 , $(\delta_j)_{j \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ and $\gamma \in \mathcal{S}$. Then the following assertions are equivalent.

- $x = \lim_{n \rightarrow \infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(t) \in B_\gamma$.
- There is an infinite walk in $G(\mathcal{S})$ of the shape

$$\gamma \xrightarrow{\delta_1|\delta'_1} \gamma_1 \xrightarrow{\delta_2|\delta'_2} \gamma_2 \xrightarrow{\delta_3|\delta'_3} \dots$$

for some $\gamma_i \in \mathcal{S}$ and $\delta'_i \in \mathcal{D}$.

This means that for each $\gamma \in \mathcal{S}$, there is at least one infinite walk in $G(\mathcal{S})$ starting from the state γ . This will provide a method to construct the neighbor graph.

3. THE NEIGHBOR SET OF T FOR $A \geq -1$ AND $2A < B + 3$

For the sake of simplicity, in Sections 3, 4 and 5 we will restrict to the case $A \geq -1$ and $2A < B + 3$ and indicate in Section 6 the method to get the results for $A \leq -2$. Let T be the crystile and T^ℓ be the lattice tile defined by (1.3) and (1.4), and let $\mathcal{S}, \mathcal{S}^\ell$ be the neighbor sets of T, T^ℓ , respectively. $G(\mathcal{S})$ is the neighbor graph of T .

In this section, we will derive an ‘‘approximation’’ of the neighbor set \mathcal{S} for $A \geq -1$, $2A < B + 3$ from the relationship between the neighbor set of T and the neighbor set of T^ℓ . Akiyama and Thuswaldner prove the following characterization of the neighbors of T^ℓ in [3].

Proposition 3.1. *If $2A < B + 3$ and $A \neq 0$, then $\#\mathcal{S}^\ell = 6$. In particular,*

- (1) *If $A > 0$, then $\mathcal{S}^\ell = \{a^A b, a^{A-1} b, a, a^{-1}, a^{-A} b^{-1}, a^{-A+1} b^{-1}\}$;*
- (2) *If $A = -1$, we have $\mathcal{S}^\ell = \{a^{-1} b, b, a, a^{-1}, ab^{-1}, b^{-1}\}$;*
- (3) *If $A = 0$, we have $\mathcal{S}^\ell = \{a, a^{-1}, ab, a^{-1} b, ab^{-1}, a^{-1} b^{-1}, b, b^{-1}\}$.*

The following lemma gives a first coarse estimate of the neighbor set of T in terms of the neighbor set of T^ℓ .

Lemma 3.2. \mathcal{S} is a subset of $\mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$, where $\mathcal{S}^\ell c = \{s \circ c; s \in \mathcal{S}^\ell\}$.

Proof. Using Lemma 2.4, we know that the lattice tile is a translation of the union $T \cup c(T)$. Then it is easy to see that all possible neighbors of T are included in the union of the neighbor set of T^ℓ , the π -rotation of the neighbor set of T^ℓ and the π -rotation itself. \square

From the above lemma, we know an upper bound for the number of neighbors of the $p2$ -tile T . We deduce from [15] a lower bound for this number.

Lemma 3.3. In a lattice tiling or a $p2$ -tiling of the plane, each tile has at least six neighbors. This implies $\#\mathcal{S} \geq 6$ and $\#\mathcal{S}^\ell \geq 6$.

We use Characterization 2.9 to refine the estimate of the neighbor set of T (compare with Lemma 3.2).

Lemma 3.4. Let $\mathcal{S}' = \mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$. Then the following statements hold.

- (1) For $A > 0$, $\mathcal{S} \subset \mathcal{S}' \setminus \{a^A b, a^{-A} b^{-1}, a^{-A} b^{-1} c\}$;
- (2) For $A = -1$, $\mathcal{S} \subset \mathcal{S}' \setminus \{a^{-1} b, b, ab^{-1}, b^{-1}, ab^{-1} c, b^{-1} c\}$;
- (3) For $A = 0$, $\mathcal{S} \subset \mathcal{S}' \setminus \{ab, a^{-1} b^{-1}, ab^{-1}, a^{-1} b, b, b^{-1}, a^{-1} b^{-1} c, ab^{-1} c, b^{-1} c\}$.

In particular, \mathcal{S} has at least 6 but not more than 10 elements.

Proof. We know that $G(\mathcal{S})$ is a subgraph of $G(\mathcal{S}')$ by Lemma 3.2. The definition of the edges requires to calculate $g\mathcal{S}'g^{-1} = \{g\gamma g^{-1}; \gamma \in \mathcal{S}'\}$ at first. Let p and q be arbitrary elements in \mathbb{Z} . Recall that g has the form (1.2). Then

$$(3.1) \quad ga^p b^q g^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -qB \\ p - qA \end{pmatrix},$$

$$(3.2) \quad ga^p b^q c g^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1-q)B - 1 \\ p - qA \end{pmatrix}.$$

Thus the following relations hold:

$$ga^A b g^{-1} = a^{-B}, \quad ga^{-A} b^{-1} g^{-1} = a^B, \quad ga^{-A} b^{-1} c g^{-1} = a^{2B-1} c.$$

We claim that there are no edges starting from the states $a^A b$, $a^{-A} b^{-1}$, and $a^{-A} b^{-1} c$ for $A > 0$.

Indeed, for $\delta, \delta' \in \mathcal{D}$,

$$\delta^{-1} ga^A b g^{-1} \delta' = \delta^{-1} a^{-B} \delta' = \begin{cases} a^{-B}, & \delta = \delta' = id; \\ a^B, & \delta = \delta' = c; \\ a^{-B} c, & \delta = id, \delta' = c; \\ a^B c, & \delta = c, \delta' = id; \\ a^{B-p+q}, & \delta = a^p, \delta' = a^q, 1 \leq p, q \leq B-2. \end{cases}$$

Therefore, $\delta^{-1} ga^A b g^{-1} \delta'$ is not an element of \mathcal{S}' , which means that there is no edge starting from $a^A b$. The computation is similar for $a^{-A} b^{-1}$, $a^{-A} b^{-1} c$. Hence, we obtain that $a^A b$, $a^{-A} b^{-1}$, $a^{-A} b^{-1} c$ are not elements of \mathcal{S} by Characterization 2.9, which proves Item (1).

For $A = -1$ and $A = 0$, similar computations as above show that there is no edge starting from the states removed from \mathcal{S}' in Item (2) and Item (3).

Finally, by Lemma 3.3 and the above discussion, we obtain that the neighbor set of the crystal has at least 6 but not more than 10 elements because $\#\mathcal{S}' = 13$ by Lemma 3.2. \square

4. THE NEIGHBOR GRAPH OF T FOR $A \geq -1$ AND $2A < B + 3$

In this section, we explicitly construct the neighbor graph. Throughout the whole section, we restrict to the case $A \geq -1$ and $2A < B + 3$. In Lemma 3.4, we denoted by \mathcal{S}' the set $\mathcal{S}' = \mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$. Now for $A > 0$, let $\mathcal{S}'' = \mathcal{S}' \setminus \{a^A b, a^{-A} b^{-1}, a^{-A} b^{-1} c\}$, that is,

$$(4.1) \quad \mathcal{S}'' = \{a^{A-1} b, a, a^{-1}, b^{-1}, a^{1-A} b^{-1}, c, a^A b c, a^{A-1} b c, a c, a^{-1} c, a^{1-A} b^{-1} c\}.$$

For $A = 0$, we set

$$(4.2) \quad \mathcal{S}'' = \{a, a^{-1}, c, a c, a^{-1} c, a^{-1} b c, b c, a b c\},$$

and for $A = -1$,

$$(4.3) \quad \mathcal{S}'' = \{a, a^{-1}, c, a c, a^{-1} c, a^{-1} b c, b c\}.$$

By Lemma 3.4, we know that $\mathcal{S} \subset \mathcal{S}''$. We call the graph $G(\mathcal{S}'')$ the *pseudo-neighbor graph*. Tables 1 and 2 show all information on $G(\mathcal{S}'')$. The last column indicates the parameters A, B for which these edges exist. Furthermore, the pseudo-neighbor graphs for the cases $A \geq 3, B \geq 5$ are depicted in Figure 3. The edges named by (1), ..., (13) are listed in Tables 1 and 2.

Edge	Labels ($\delta \delta'$)	Name	Condition
$c \rightarrow a c$	$a^{B-2} id, a^{B-3} a, \dots, id a^{B-2}$	(1)	$B \geq 2$ and $A \geq -1$
$c \rightarrow a^{-1} c$	$a^{B-2} a^2, a^{B-3} a^3, \dots, a^2 a^{B-2}$	(13)	$B \geq 4$ and $A \geq -1$
$c \rightarrow c$	$a^{B-2} a, a^{B-3} a^2, \dots, a a^{B-2}$	(2)	$B \geq 3$ and $A \geq -1$
$c \rightarrow a^{-1}$	$c a^{B-2}$		$B \geq 2, A \geq -1$
$c \rightarrow a$	$a^{B-2} c$		$B \geq 2, A \geq -1$
$a \rightarrow a^{A-1} b$	$id a^{A-1}, a a^A, \dots, a^{B-A-1} a^{B-2}$	(3)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a^{-1} \rightarrow b c$	$c id$		$B \geq 3, A \in \{-1, 0, 1\}$
$a \rightarrow a^{1-A} b^{-1} c$	$c a^{A-1}$		$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a \rightarrow b c$	$id c$		$B \geq 2, A \in \{-1, 0, 1\}$
$a^{A-1} b c \rightarrow a b c$	$c c$		$B \geq 2, A \in \{-1, 0, 1\}$
$a \rightarrow a^{-1} b c$	$a c$		$B \geq 3, A \in \{0, -1\}$
$a^{-1} \rightarrow a^{-1} b c$	$c a$		$B \geq 3, A \in \{0, -1\}$
$a^{-1} \rightarrow a^{1-A} b^{-1}$	$a^{A-1} id, a^A a, \dots, a^{B-2} a^{B-A+1}$	(4)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a^{-1} \rightarrow a^{1-A} b^{-1} c$	$a^{A-1} c$		$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a b c \rightarrow a^{-1} b c$	$id id$		$B \geq 2, A = 0$
$a^{A-1} b c \rightarrow a^{1-A} b^{-1} c$	$a^{A-2} id, a^{A-3} a, \dots, id a^{A-2}$	(5)	$B \geq 2$ and $A \geq 2$

Table 1. Edges of $G(\mathcal{S}'')$ (Case $A \geq -1$ and $2A < B + 3$)

Edge	Labels ($\delta \delta'$)	Name	Condition
$a^{A-1}bc \rightarrow a^{A-1}b$	$c a^{A-2}$		$B \geq 2, A \geq 2$
$a^{A-1}bc \rightarrow a^{1-A}b^{-1}$	$a^{A-2} c$		$B \geq 2, A \geq 2$
$ac \rightarrow a^A bc$	$a^{B-A-1} id, a^{B-A-2} a,$ $\dots, id a^{B-A+1}$	(6)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$ac \rightarrow a^{-1}bc$	$a^{B-2} a^2, a^{B-3} a^3, \dots, id a^{B-2}$	(6)'	$B \geq 4, A \in \{0, -1\}$
$ac \rightarrow abc$	$a^{B-2} id, a^{B-3} a, \dots, id a^{B-2}$	(14)	$B \geq 2, A = 0$
$ac \rightarrow a^{A-1}bc$	$a^{B-A} id, a^{B-A-1} a, \dots, id a^{B-A}$	(7)	$B \geq 2$ and $A \geq 2$
$ac \rightarrow bc$	$a^{B-2} a, a^{B-3} a^2, \dots, a a^{B-2}$	(7)'	$B \geq 3, A \in \{-1, 0, 1\}$
$ac \rightarrow a^{A-1}b$	$a^{B-A} c$		$B \geq 2, A \geq 2$
$ac \rightarrow a^{1-A}b^{-1}$	$c a^{B-A}$		$B \geq 2, A \geq 2$
$a^A bc \rightarrow a^{-1}c$	$id id$		$B \geq 2, A \geq -1$
$a^A bc \rightarrow a$	$c id$		$B \geq 2, A \geq -1$
$a^A bc \rightarrow a^{-1}$	$id c$		$B \geq 2, A \geq -1$
$a^A bc \rightarrow ac$	$c c$		$B \geq 2, A \geq -1$
$bc \rightarrow a^{-1}bc$	$id id$		$B \geq 2, A = -1$
$a^{-1}c \rightarrow a^{1-A}b^{-1}c$	$a^{B-2} a^A, a^{B-3} a^{A+1}, \dots, a^A a^{B-2}$	(8)	$B \geq A+2, A > 0$
$a^{-1}c \rightarrow a^{-1}bc$	$c c$		$B = 2, A = -1$
$a^{1-A}b^{-1}c \rightarrow a^A bc$	$a^{B-2} a^{B-A+1}, a^{B-3} a^{B-A+2},$ $\dots, a^{B-A+1} a^{B-2}$	(9)	$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1}c \rightarrow a^{A-1}bc$	$a^{B-2} a^{B-A+2}, a^{B-3} a^{B-A+3},$ $\dots, a^{B-A+2} a^{B-2}$	(10)	$B \geq 6$ and $A \geq 4$
$a^{A-1}b \rightarrow a^{1-A}b^{-1}$	$id a^{B-A+1}, a a^{B-A+2}, \dots, a^{A-3} a^{B-2}$	(11)	$B \geq 4$ and $A \geq 3$
$a^{A-1}b \rightarrow a^A bc$	$c a^{B-A}$		$B \geq 2$ and $A \geq 2$
$a^{A-1}b \rightarrow a^{A-1}bc$	$c a^{B-A+1}$		$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1} \rightarrow a^{A-1}b$	$a^{B-A+1} id, a^{B-A+2} a, \dots, a^{B-2} a^{A-3}$	(12)	$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1} \rightarrow a^A bc$	$a^{B-A} c$		$B \geq 2$ and $A \geq 2$
$a^{1-A}b^{-1} \rightarrow a^{A-1}bc$	$a^{B-A+1} c$		$B \geq 4$ and $A \geq 3$

Table 2. Edges of $G(\mathcal{S}'')$ (Case $A \geq -1$ and $2A < B + 3$)

Since $\mathcal{S} \subset \mathcal{S}''$, it is clear that the neighbor graph $G(\mathcal{S})$ is a subgraph of the pseudo-neighbor graph. We will see that Characterization 2.9 will play an important role in the relationship between the neighbor graph $G(\mathcal{S})$ and the pseudo-neighbor graph $G(\mathcal{S}'')$.

Theorem 4.1. *Let \mathcal{S} be the neighbor set of T and \mathcal{S}'' be defined as in (4.1), (4.2) and (4.3). The following results hold for A, B satisfying $-1 \leq A \leq B$, $B \geq 2$ and $2A < B + 3$.*

- (1) For $A \geq 3$ and $B \geq 5$, $\mathcal{S} = \mathcal{S}''$.
- (2) For $A = 3$ and $B = 4$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c\}$.

- (3) For $A = 2$ and $B = 2$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c, a, a^{-1}\}$.
(4) For $A = 2$ and $B \geq 3$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c\}$.
(5) For $A = 1$ and $B \geq 2$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c, a^{A-1}b, a^{1-A}b^{-1}\}$.
(6) For $A = 0$ and $B \geq 2$, $\mathcal{S} = \{a, a^{-1}, c, ac, a^{-1}bc, bc, abc\}$.
(7) For $A = -1$ and $B = 2$, $\mathcal{S} = \{a, a^{-1}, c, a^{-1}c, a^{-1}bc, bc\}$;
For $A = -1$ and $B \geq 3$, $\mathcal{S} = \{a, a^{-1}, c, ac, a^{-1}bc, bc\}$.

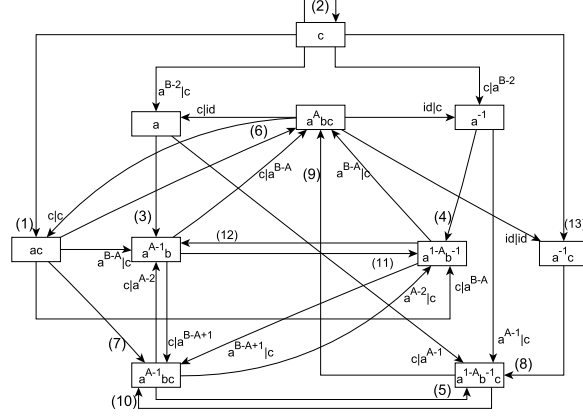
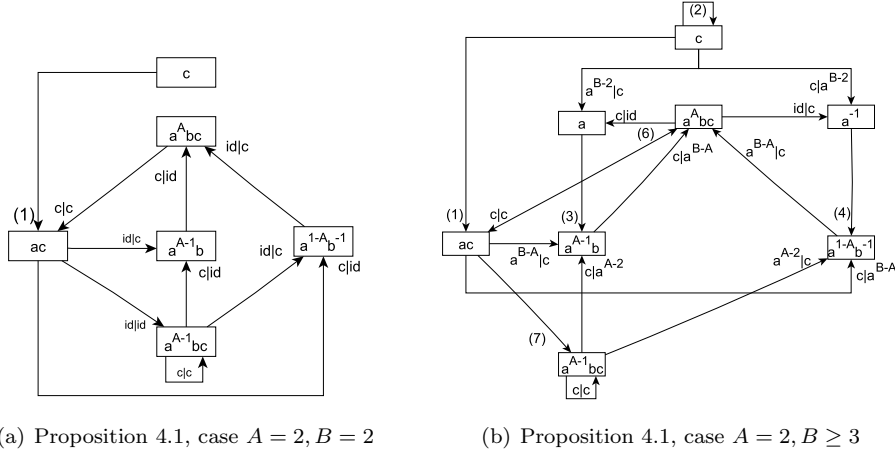


FIGURE 3. The graph $G(\mathcal{S}'')$ for $A \geq 3$ and $B \geq 5$ and $2A < B + 3$.



(a) Proposition 4.1, case $A = 2, B = 2$

(b) Proposition 4.1, case $A = 2, B \geq 3$

FIGURE 4. The neighbor graph $G(\mathcal{S})$ of T

Proof. By Characterization 2.9, the neighbor graph $G(\mathcal{S})$ is obtained from the pseudo-neighbor graph $G(\mathcal{S}'')$ by deleting the states that are not the starting state of an infinite walk. For $A \geq 3, B \geq 5$, from Figure 3, it is clear that there is an infinite walk starting from each state of $G(\mathcal{S}'')$. For $A = 3, B = 4$, from Table 1 and Table 2, we know that there is exactly one state $a^{-1}c$ from which there is no outgoing edge. For Item (3),(4),(5), and (6), see Figure 4(a), Figure 4(b), Figure 5 and Figure 7, respectively. For Item (7), it is easy to check that $a^{-1}c$ is the starting state of an infinite walk if and only if $B = 2$ and ac is the starting state of an infinite walk if and only if $B \geq 3$. Since the neighbor set has at least six elements by Lemma 3.3, we get the results of Item (7) (see Figure 6 for more details). \square

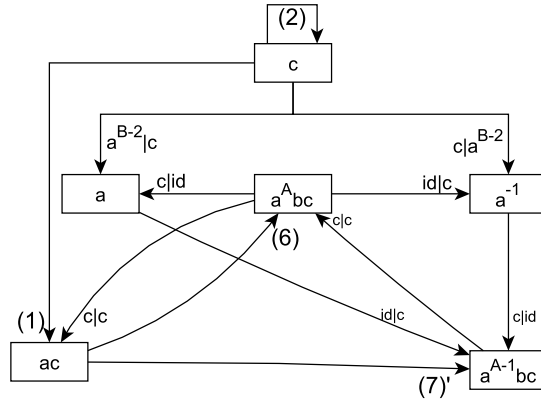


FIGURE 5. Theorem 4.1, Case $A = 1, B \geq 2$. We refer to Tables 1 and Table 2 for the conditions on the edges.

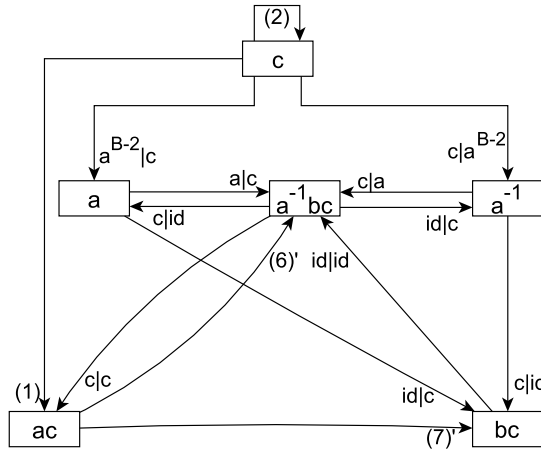


FIGURE 6. Theorem 4.1, case $A = -1, B \geq 3$. For the case $B = 2$, we only need to replace ac by $a^{-1}c$ and change the incoming and outgoing edges according to Tables 1 and Table 2.

5. CHARACTERIZATION OF THE DISK-LIKE TILES FOR $A \geq -1$ AND $2A < B + 3$

We are now in a position to study the topological properties of our family of $p2$ -tiles under the conditions $A \geq -1, 2A < B + 3$. We will characterize the disk-like tiles of the family under this condition. Loridant and Luo in [15] provided necessary and sufficient conditions for a $p2$ -tile to be disk-like. Before stating the theorem, we need a definition.

Definition 5.1. ([15]) If \mathcal{P} and \mathcal{F} are two sets of isometries in \mathbb{R}^2 , we say that \mathcal{P} is \mathcal{F} -connected iff for every disjoint pair (d, d') of elements in \mathcal{P} , there exist $n \geq 1$ and elements $d := d_0, d_1, \dots, d_{n-1}, d_n := d'$ of \mathcal{P} such that $d_i^{-1}d_{i+1} \in \mathcal{F}$ for $i = 0, 1, \dots, n - 1$.

The following statement is from [15]. In fact, the necessary part is due to the classification of Grünbaum and Shephard [7].

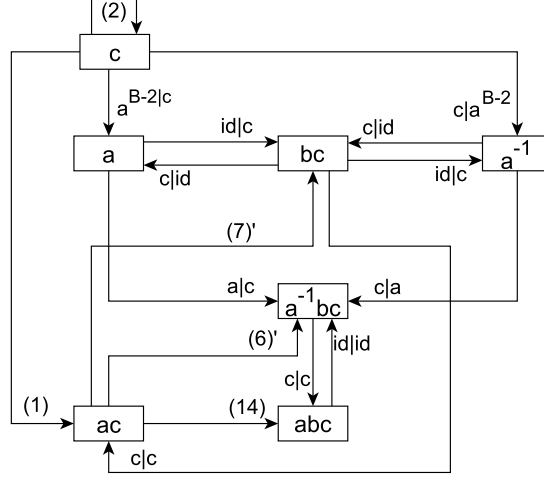


FIGURE 7. Theorem 4.1, the case $A = 0, B \geq 2$. We refer to Tables 1 and Table 2 for the conditions on the edges.

Proposition 5.2. *Let K be a crystal that tiles the plane by a $p2$ -group. Let \mathcal{F} be the corresponding digit set. Let a, b be translations, and c be a π -rotation.*

- (1) *Suppose that the neighbor set \mathcal{S} of K has six elements. Then K is disk-like iff \mathcal{F} is \mathcal{S} -connected.*
- (2) *Suppose that the neighbor set \mathcal{S} of K has seven elements*

$$\{b^{\pm 1}, c, bc, a^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}.$$

Then K is disk-like iff \mathcal{F} is $\{b^{\pm 1}, c, bc, a^{-1}c\}$ -connected.

- (3) *Suppose that the neighbor set \mathcal{S} of K has eight elements*

$$\{b^{\pm 1}, (a^{-1}b)^{\pm 1}, c, bc, ac, ab^{-1}c\}$$

$$(\text{resp. } \{b^{\pm 1}, c, bc, a^{-1}c, b^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}),$$

Then K is disk-like iff \mathcal{F} is $\{c, bc, ac, ab^{-1}c\}$ - (resp. $\{b^{\pm 1}, c, a^{-1}c\}$ -) connected.

- (4) *Suppose that the neighbor set \mathcal{S} of K has twelve elements*

$$\{a^{\pm 1}, b^{\pm 1}, (ab)^{\pm 1}, c, a^{-1}c, bc, abc, a^{-1}bc, a^{-1}b^{-1}c\},$$

Then K is disk-like iff \mathcal{F} is $\{c, a^{-1}c, bc\}$ -connected.

Applying this result, we obtain the following theorem.

Theorem 5.3. *Let $A, B \in \mathbb{Z}$ satisfy $-1 \leq A \leq B$, $B \geq 2$ and $2A < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (1.2) and (1.3). Then the following statements hold.*

- (1) *If $A \in \{-1, 0, 1\}, B \geq 2$ or $A = 2, B = 2$, then T is disk-like.*
- (2) *If $A \geq 2, B \geq 3$, then T is non-disk-like.*

Proof. Let \mathcal{S} be the neighbor set of T . By Theorem 4.1, we know that in the assumption of $A \in \{-1, 1\}, B \geq 2$ and $A = 2, B = 2$, the neighbor sets of T all have six elements. Let us check the case $A = 1, B \geq 2$ by showing that \mathcal{D} is \mathcal{S} -connected and applying Proposition 5.2 (1). Then $A = -1, B \geq 2$ and $A = 2, B = 2$ can be checked in the same way.

For $A = 1, B \geq 2$, the digit set is $\mathcal{D} = \{id, a, \dots, a^{B-2}, c\}$ and the neighbor set is $\mathcal{S} = \{a, a^{-1}, c, abc, bc, ac\}$. It is easy to find that the disjoint pairs (d, d') in $\mathcal{D} \times \mathcal{D}$ are the following ones:

$$(5.1) \quad (id, a^\ell), (a^\ell, id), (id, c), (c, id), (a^k, a^{k'})(a^j, c), \text{ or } (c, a^j),$$

where $\ell, k, k', j \in \{1, 2, \dots, B-2\}$.

We will check the pair $(a^k, a^{k'})$ at first. If $k < k'$, then let $n = k' - k$, and

$$d_0 = a^k, d_1 = a^{k+1}, \dots, d_{n-1} = a^{k'+1}, d_n = a^{k'}.$$

hence $d_i^{-1}d_{i+1} = a$ is in \mathcal{S} for $0 \leq i \leq n-1$. If $k > k'$, $d_i^{-1}d_{i-1} = a^{-1}$ is also in \mathcal{S} for $0 \leq i \leq n-1$. To check (id, a^ℓ) and (a^j, c) , it suffices to check (id, a) and (a, c) . It is clear for (id, a) . For (a, c) , let $n = 2$, and $d_0 = a, d_1 = id, d_2 = c$. Hence, we have proved that \mathcal{D} is \mathcal{S} -connected. By Proposition 5.2 (1), T is disk-like.

For $A = 0$ and $B \geq 2$ and the neighbor set

$$\mathcal{S} = \{a, a^{-1}, c, a^{-1}bc, bc, ac, abc\}$$

has seven elements. By Proposition 5.2 (2), we need to prove that \mathcal{D} is $\{a, a^{-1}, c, ac, bc\}$ -connected. This is achieved in the same way as above.

We now prove Item (2). For $A = 2, B \geq 3$ and by Theorem 4.1, we know that

$$\mathcal{S} = \{a, a^{-1}, ab, a^{-1}b^{-1}, c, abc, a^2bc, ac\}.$$

Let $a' = a^2b, b' = ab$, then \mathcal{S} has the form

$$\Upsilon := \{b', b'^{-1}, a'^{-1}b', a'b'^{-1}, c, b'c, a'c, a'b'^{-1}c\}$$

of Proposition 5.2 (3). However, it is easily checked that \mathcal{D} is not $\{c, abc, ab^2c, ac\}$ -connected. By Proposition 5.2 (3), T is not disk-like.

For $A \geq 3, B \geq 4$, we have $\#\mathcal{S} = 9$ if $A = 3, B = 4$, and $\#\mathcal{S} = 10$ if $A \geq 3, B \geq 5$ by Theorem 4.1. According to Grünbaum and Shephard's classification of isohedral tilings (see [7, Sect. 6.2, p.285]), the cases in Proposition 5.2 are the only ones leading to disk-like $p2$ -tiles in the plane. So T is non-disk-like for $A \geq 3, B \geq 4$. \square

6. CHARACTERIZATION OF THE DISK-LIKE TILES FOR $A \leq -2$ AND $2|A| < B + 3$

We now deal with the case $A \leq -2$ and $2|A| < B + 3$. Let us recall a statement in [1, Equation (2.11), p. 2177]. Let T^ℓ be the lattice tile associated with M and the digit set \mathcal{N} (see (2.3)) and \bar{T}^ℓ the lattice tile associated with $\bar{M} = \begin{pmatrix} 0 & -B \\ 1 & A \end{pmatrix}$ and \mathcal{N} . Then we have

$$(6.1) \quad \bar{T}^\ell = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T^\ell + \sum_{k=1}^{\infty} \bar{M}^{-2k} \begin{pmatrix} B-1 \\ 0 \end{pmatrix}.$$

It follows that T^ℓ and \bar{T}^ℓ have the same topology. It is remarkable that this does not hold for the associated crystiles T and \bar{T} , as is illustrated below.

By [3], we know all the information on the neighbor set of the lattice tile T^ℓ for $A \geq -1$, hence we can derive the neighbor set of \bar{T}^ℓ immediately.

Lemma 6.1. *If $2A < B + 3$ and $A > 0$, then the neighbor set of \bar{T}^ℓ is*

$$(6.2) \quad \{(-A, 1), (-A+1, 1), (-1, 0), (1, 0), (A, -1), (A-1, -1)\},$$

or, using translation mappings rather than vectors,

$$(6.3) \quad \{a^{-A}b, a^{-A+1}b, a^{-1}, a, a^A b^{-1}, a^{A-1}b^{-1}\}.$$

Proof. $\gamma = a^p b^q \in \Gamma$ ($p, q \in \mathbb{Z}$) is a neighbor of T^ℓ iff $T^\ell \cap \gamma(T^\ell) \neq \emptyset$. Let $\gamma' = a^{-p} b^q$, then this is equivalent to $\bar{T}^\ell \cap \gamma'(\bar{T}^\ell) \neq \emptyset$ by (6.1). Thus, using Proposition 3.1, we get the neighbor set (6.2) of \bar{T}^ℓ . \square

For $-1 \leq A \leq B \geq 2$, the data $(g, \mathcal{D}, p2)$ is a crystallographic number system, hence, the tiling group is the whole crystallographic group $p2$ [14]. It follows from Proposition 2.5 that this property still holds for $A \leq -2$. Now, by Lemma 6.1, to obtain the neighbor set of $p2$ -crystiles for $A \leq -2$, we only need to repeat the methods in Section 3 and 4, dealing with similar estimates and computations. We come to the following theorem for $A \leq -2$ (we do not reproduce the computations).

Theorem 6.2. *Let $A, B \in \mathbb{Z}$ satisfy $2 \leq -A \leq B$ and $2|A| < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (1.2) and (1.3). Then the following statements hold.*

- (1) *For $A = -2$ and $B = 2$ or 3 , the neighbor set of the crystile T is*

$$\mathcal{S} = \{a, a^{-1}, c, a^{-1}c, a^{-2}bc, a^{-1}bc\};$$

- (2) *For $A = -2, B \geq 4$, the neighbor set of the crystile T is*

$$\mathcal{S} = \{a, a^{-1}, c, ac, a^{-2}bc, a^{-1}bc\}.$$

- (3) *For $A = -3, B = 4$, the neighbor set of the crystile T is*

$$\mathcal{S} = \{a, a^{-1}, a^{-2}b, a^2b^{-1}, c, a^{-1}c, a^{-2}bc, a^{-3}bc\}.$$

- (4) *For $A = -3, B \geq 5$, the neighbor set of the crystile T is*

$$\mathcal{S} = \{a, a^{-1}, a^{A+1}b, a^{-1-A}b^{-1}, c, ac, a^{A+1}bc, a^A bc, a^{-1}c\}.$$

- (5) *For $A \leq -4, B \geq 6$, the neighbor set of the crystile T is*

$$\mathcal{S} = \{a, a^{-1}, a^{A+1}b, a^{-1-A}b^{-1}, c, a^{-1}c, ac, a^{A+1}bc, a^A bc, a^{-A-1}b^{-1}c\}.$$

Consequently, we can infer from Lemma 5.2 the following theorem.

Theorem 6.3. *Let $A, B \in \mathbb{Z}$ satisfy $2 \leq -A \leq B$ and $2|A| < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given by (1.2) and (1.3). Then the following statements hold.*

- (1) *If $A = -2, B \geq 2$, then T is disk-like.*
(2) *If $A \leq -3, B \geq 4$, then T is not disk-like.*

Proof. For Item (1), we know from Theorem 6.2 that the neighbor set of T has six neighbors. Thus, by Proposition 5.2 Item (1), T is disk-like.

For $A = -3, B = 4$, the neighbor set is

$$\mathcal{S} = \{a, a^{-1}, a^{-2}b, a^2b^{-1}, c, a^{-2}bc, a^{-3}bc, a^{-1}c\}.$$

Let $a' = a^{-3}b, b' = a^{-1}$, then \mathcal{S} has the form

$$\Upsilon := \{b', b'^{-1}, a'^{-1}b', a'b'^{-1}, c, b'c, a'c, a'b'^{-1}c\}$$

of Proposition 5.2 (3). However, it is easily checked that \mathcal{D} is not $\{c, a^{-2}bc, ab^{-3}c, a^{-1}c\}$ -connected. By Proposition 5.2 Item (3), T is not disk-like.

For the cases $A = -3, B \geq 5$ and $A \leq -4, B \geq 6$, T has 9 and 10 neighbours, respectively. Thus T is not disk-like as we have discussed in Theorem 5.3. \square

7. NON-DISK-LIKENESS OF TILES FOR $2|A| \geq B + 3$

So far, we have dealt with the case $2|A| < B + 3$ and characterized the disk-like $p2$ -tiles in Theorem 5.3 and Theorem 6.3. If $2|A| \geq B + 3$, it was proved in [12] that the lattice tiles T^ℓ are not disk-like. We prove that this also holds for the corresponding $p2$ -tiles T .

Recall that the $p2$ -tile T satisfies the equation

$$(7.1) \quad T = \bigcup_{i=1}^B f_i(T),$$

where

$$f_1 = g^{-1} \circ id, f_i = g^{-1} \circ a^{i-1} \quad (2 \leq i \leq B-1), f_B = g^{-1} \circ c,$$

g is the expanding map given by (1.2), and \mathcal{D} is the digit set defined as (2.2). We denote the fixed point of a mapping f by $\text{Fix}(f)$ and the linear part of g by M . Then we have the following facts:

$$(7.2) \quad \text{Fix}(f_i) = (M - I_2)^{-1} \begin{pmatrix} i-1 - \frac{B-1}{2} \\ 0 \end{pmatrix} \quad \text{for } 1 \leq i \leq B-1,$$

$$(7.3) \quad \text{Fix}(f_B) = (M + I_2)^{-1} \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}.$$

By (7.1), the fixed points given by (7.2) and (7.3) all belong to T . First of all, we give a key lemma for the main result.

Lemma 7.1. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $2|A| \geq B + 3$, and let T be the p 2-crystile defined by the data (g, \mathcal{D}) given in (1.2) and (1.3) and $c(T)$ be the π -rotation of T . Then $\sharp(T \cap c(T)) \geq 2$.*

Proof. By (7.2), we notice that for $2 \leq p, q \leq B-2$

$$\text{Fix}(f_p) = -\text{Fix}(f_q) \quad \text{if } p+q = B-1.$$

This means that $\text{Fix}(f_p)$ and $\text{Fix}(f_q)$ are both in T and $c(T)$. If $B > 3$, these points are different and we are done. If $B \leq 3$, we only need to consider the case $|A| = 3, B = 3$ since we assume that $2|A| \geq B + 3$. Since $B = 3$, by (7.2), $\text{Fix}(f_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is in $T \cap c(T)$. And for the case $A = 3, B = 3$, there exists an eventually periodic sequence of edges (see Figure 8).

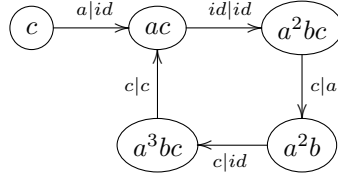


FIGURE 8. An eventually periodic sequence of edges for $A = 3, B = 3$.

The edges of this figure are defined in the same way as in Definition 2.8 and it follows that

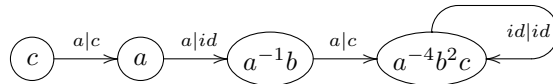
$$x_0 = \lim_{n \rightarrow \infty} g^{-1} a \circ (g^{-1} \circ g^{-1} c \circ g^{-1} c \circ g^{-1} c)^n(t) \in T \cap c(T),$$

(see also Characterization 2.9). Here, $t \in \mathbb{R}^2$ is arbitrary. Note that

$$x_0 = g^{-1} a \left(\text{Fix}(g^{-1} \circ g^{-1} c \circ g^{-1} c \circ g^{-1} c) \right),$$

and it is easy to compute that $x_0 = \begin{pmatrix} -\frac{13}{73} \\ \frac{16}{219} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For the case $A = -3, B = 3$, we find the eventually periodic sequence of edges



So we have

$$x'_0 = \lim_{n \rightarrow \infty} g^{-1} a \circ g^{-1} a \circ g^{-1} a \circ (g^{-1})^n(t) \in T \cap c(T),$$

and it is easy to verify that $x'_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. \square

Theorem 7.2. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $2|A| \geq B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (1.2) and (1.3). Then T is not disk-like.*

Proof. By a result of [12], we know that if $2|A| \geq B + 3$, then T^ℓ is not disk-like. Suppose that T is disk-like. By Lemma 7.1, we have $\sharp(T \cap c(T)) \geq 2$. By [16, Proposition 4.1 item (2), p. 127], this implies that $T \cap c(T)$ is a simple arc. Therefore $T \cup c(T)$ is disk-like, as the union of two topological disks whose intersection is a simple arc is again a topological disk. However, by Lemma 2.4, T^ℓ is a translation of $T \cup c(T)$, therefore T^ℓ must be disk-like. This contradicts the assumption $2|A| \geq B + 3$. \square

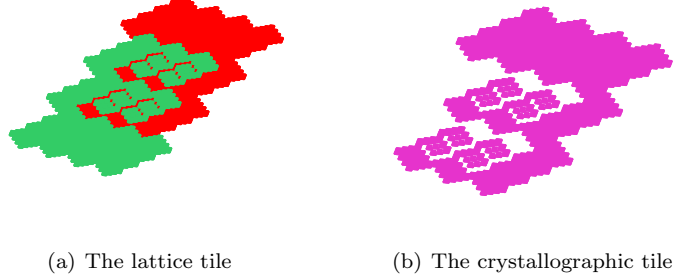


FIGURE 9. $A = 1, B = 4$.

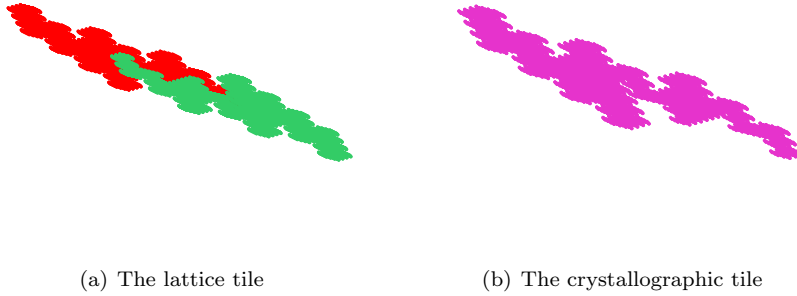
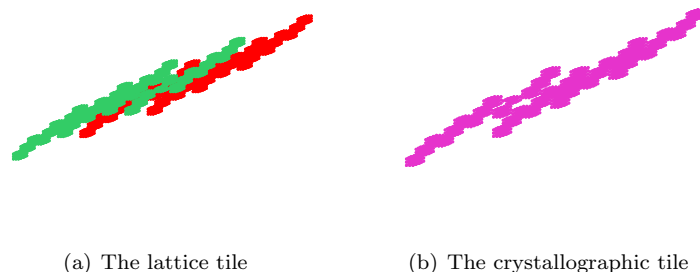


FIGURE 10. Lattice tile and Crystile for $A = -3, B = 4$.

8. EXAMPLES

Now we provide some examples. For fixed A and B , even though the lattice tile T^ℓ is a translate of $T \cup (-T)$, T and T^ℓ may have completely different topological behaviour. We give the following examples to illustrate this phenomenon. In Figure 9, $A = 1, B = 4$, T and T^ℓ are both disk-like. For Figure 1 and Figure 10, T^ℓ is disk-like while T is not. In Figure 11, T and T^ℓ are both not disk-like.

FIGURE 11. Lattice tile and Crystile for $A = 3, B = 3$.

REFERENCES

- [1] S. AKIYAMA AND B. LORIDANT, *Boundary parametrization of planar self-affine tiles with collinear digit set*, Sci. China Math., 53 (2010), pp. 2173–2194.
- [2] ———, *Boundary parametrization of self-affine tiles*, J. Math. Soc. Japan, 63 (2011), pp. 525–579.
- [3] S. AKIYAMA AND J. M. THUSWALDNER, *The topological structure of fractal tilings generated by quadratic number systems*, Comput. Math. Appl., 49 (2005), pp. 1439–1485.
- [4] C. BANDT AND Y. WANG, *Disk-like self-affine tiles in \mathbb{R}^2* , Discrete Comput. Geom., 26 (2001), pp. 591–601.
- [5] J. J. BURCKHARDT, *Die Bewegungsgruppen der Kristallographie*, vol. 13 of Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Verlag Birkhäuser, Basel, 1947.
- [6] G. GELBRICH, *Crystallographic reptiles*, Geom. Dedicata, 51 (1994), pp. 235–256.
- [7] B. GRÜNBAUM AND G. C. SHEPHARD, *Tilings and patterns*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Company, New York, 1989. An introduction.
- [8] I. KÁTAL, *Number systems and fractal geometry*, University of Pécs, (1995).
- [9] I. KIRAT AND K.-S. LAU, *On the connectedness of self-affine tiles*, J. London Math. Soc. (2), 62 (2000), pp. 291–304.
- [10] J. C. LAGARIAS AND Y. WANG, *Integral self-affine tiles in \mathbb{R}^n . I. Standard and nonstandard digit sets*, J. London Math. Soc. (2), 54 (1996), pp. 161–179.
- [11] ———, *Self-affine tiles in \mathbf{R}^n* , Adv. Math., 121 (1996), pp. 21–49.
- [12] K.-S. LEUNG AND K.-S. LAU, *Disklikeness of planar self-affine tiles*, Trans. Amer. Math. Soc., 359 (2007), pp. 3337–3355.
- [13] K.-S. LEUNG AND J. J. LUO, *Connectedness of planar self-affine sets associated with non-consecutive collinear digit sets*, J. Math. Anal. Appl., 395 (2012), pp. 208–217.
- [14] B. LORIDANT, *Crystallographic number systems*, Monatsh Math, 167 (2012), pp. 511–529.
- [15] B. LORIDANT AND J. LUO, *On a theorem of Bandt and Wang and its extension to p_2 -tiles*, Discrete Comput. Geom., 41 (2009), pp. 616–642.
- [16] B. LORIDANT, J. LUO, AND J. M. THUSWALDNER, *Topology of crystallographic tiles*, Geom. Dedicata, 128 (2007), pp. 113–144.
- [17] K. SCHEICHER AND J. M. THUSWALDNER, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc., 133 (2002), pp. 163–182.

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