Tilings induced by a class of cubic Rauzy fractals

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Abstract

We study aperiodic and periodic tilings induced by the Rauzy fractal and its subtiles associated to beta-substitutions related to the polynomial $x^3 - ax^2 - bx - 1$ for $a \geq b \geq 1$. In particular, we compute the corresponding boundary graphs, describing the adjacencies in the tilings. These graphs are a valuable tool for more advanced studies of the topological properties of the Rauzy fractals. As an example, we show that the Rauzy fractals are not homeomorphic to a closed disk as soon as $a \leq 2b - 4$. The methods presented in this paper may be used to obtain similar results for other classes of substitutions.

Keywords: substitution, Rauzy fractal, self-replicating tiling, lattice tiling

1. Introduction

In 1982 Gérard Rauzy [22] studied the symbolic dynamical system over the alphabet $\{1, 2, 3\}$ induced by the substitution

$$1 \mapsto 12, \quad 2 \mapsto 13, \quad 3 \mapsto 1$$

and associated to it a set known as Rauzy fractal. It is a compact set equal to the closure of its interior and it decomposes naturally into three subsets (subtiles). Moreover, the Rauzy fractal induces two types of tilings: a periodic tiling whose central tile is the Rauzy fractal, and an aperiodic tiling generated by the three subtiles. In [17, 19, 20], topological properties of the Rauzy fractal were studied and the Hausdorff dimension of its boundary was computed.

Generalisations of this dynamical system and results concerning the associated fractal sets can be found in the literature. In [3], the considerations of Rauzy are formulated in a general way for primitive Pisot substitutions. The interiors of the subtiles associated to a primitive unimodular Pisot substitution do not overlap provided that the substitution satisfies the so called strong coincidence condition [3, 14]. Several classes of substitutions were shown to satisfy this condition. For example, in [5] it was proven that every primitive irreducible Pisot substitution over an alphabet consisting of two letters satisfies it. It is conjectured that this is true for alphabets of arbitrary size but a general proof is still outstanding.

Rauzy fractals associated to primitive unimodular Pisot substitutions have been studied in various articles [8, 10, 12, 13, 16, 21, 24, 26]. They appear naturally in connection to many topics as numeration systems, geometrical representation of symbolic dynamical systems, multidimensional continued fractions and simultaneous approximations, self-similar tilings and Markov partitions of hyperbolic automorphisms of the torus.

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In [14, 18] it was shown that the subtiles induce an aperiodic multiple tiling of the space, called self-replicating multiple tiling. If the substitution is irreducible, the Rauzy fractals also provide a periodic (or lattice) multiple tiling (see [3, 12]). Actually, a lattice multiple tiling even exists in some reducible cases. A necessary and sufficient graph-based condition can be found in [23]. For large classes of substitutions these multiple tilings are shown to be proper tilings, *i.e.*, two different tiles have disjoint interiors. Even if there is no known counterexample in the irreducible case, it is up to now not possible to prove this *tiling property* in general without requiring additional conditions like the super coincidence condition or the finiteness property.

The aim of this paper is to study the aperiodic (self-replicating) and periodic (lattice) tilings induced by the substitutions

$$\sigma_{a,b}: \begin{array}{rcl} 1 & \mapsto & \underbrace{1\dots1}_{a \text{ times}}2 \\ 2 & \mapsto & \underbrace{1\dots1}_{b \text{ times}}3 \\ 3 & \mapsto & 1 \end{array}$$

over the alphabet $\{1,2,3\}$, where $1 \leq b \leq a$. For every such pair (a,b), $\sigma_{a,b}$ is an irreducible primitive unimodular Pisot substitution. Moreover, it satisfies the super coincidence condition (see [6, 25]). Therefore, all the tilings are proper tilings.

The class of Rauzy fractals (central tile in the periodic tiling) associated to $\sigma_{a,b}$ was first studied by Sh. Ito and M. Kimura in [17]. They showed that for $a = b = 1$, the boundary of the Rauzy fractal is a Jordan curve and they also computed its Hausdorff dimension. Later, for the same case, A. Messaoudi [19] constructed a finite state automaton that generates the boundary of the Rauzy fractal. This helped to prove that this boundary is a quase-circle. In [19], analog results were obtained for the case $a \geq 1$ and $b = 1$.

In [26], J. Thuswaldner gave an explicit formula for the fractal dimension of the boundary of the Rauzy fractal in the case $a \geq b \geq 1$. This result was based on the self replicating tiling.

In our work, we will describe the boundary of the tiles by determining their neighbours in the tilings. The results will be presented as self-replicating and lattice *boundary graphs*, recently introduced in the context of Rauzy fractals by Siegel and Thuswaldner in [23]. The boundary graphs are of great help in the topological study of a Rauzy fractal. Indeed, the topological behaviour of a fractal tile is mainly determined by the number and configuration of the neighbours of the tile in the tiling. For a given substitution, the computation of the boundary graphs is algorithmic, but the treatment of a whole class is usually not possible. We manage to compute the self-replicating boundary graph for the whole class of substitutions $\sigma_{a,b}$. Moreover, we compute the whole lattice boundary graph for a subclass, and conjecture the shape of this graph for the rest of the class. Also, we obtain a lower bound (depending on a, b) for the number of neighbours of the Rauzy fractal in the lattice tiling. As a consequence, we deduce that, if $a \leq 2b - 4$, then the Rauzy fractal is not homeomorphic to a topological disk. For restricted values of a, b, we are even able to compute the whole lattice boundary graph. Although our analysis is restricted to the class of substitutions $\sigma_{a,b}$, we are convinced that our considerations can be extended to other classes of substitutions.

The paper is organised as follows. In Section 2, we present the substitution class and define the Rauzy fractal, the different types of tilings and the boundary graphs. Interestingly, the characterization of these graphs involves a combination of algebraic properties and graph-based topological studies. In Section 3 we state the main theorems of this paper and give some examples. Section 4 contains some preparations for the proofs of the main results in Sections 5 and 6. In Section 7, we present some ideas to solve our conjecture concerning the lattice boundary graphs of our substitution class.

2. The class of substitutions $\sigma_{a,b}$

2.1. Notations and Definitions

Let $\mathcal{A} := \{1,2,3\}$ be the *alphabet*. We denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$, including the empty word ε. For a word $w \in \mathcal{A}^*$ we write $|w|$ for its *length* and the number of occurrences of a letter i in w is denoted by $|w|_i$. This allows us to define the *abelianisation mapping*

$$\begin{array}{rccc} \mathbf{l}: & \mathcal{A}^* & \to & \mathbb{N}^3 \\ & w & \mapsto & (|w|_i)_{i\in\mathcal{A}}. \end{array}$$

For $1 \leq b \leq a$, we call $\sigma = \sigma_{a,b} : \mathcal{A}^* \to \mathcal{A}^*$ the mapping

$$\begin{array}{rccl} \sigma: & 1 & \mapsto & \underbrace{1\dots1}_{a \text{ times}}2 \\ & 2 & \mapsto & \underbrace{1\dots1}_{b \text{ times}}3 \\ & 3 & \mapsto & 1, \end{array}$$

extended to $\mathcal{A}^*$ by concatenation. The *incidence matrix* $\mathbf{M}$ of the substitution σ is the 3×3 matrix obtained by abelianisation :

$$\mathbf{l}(\sigma(w)) = \mathbf{M}\mathbf{l}(w)$$

for all $w \in \mathcal{A}^*$. Thus we have

$$\mathbf{M} = \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$\mathbf{M}$ is a primitive matrix, *i.e.*, $\mathbf{M}^k$ has only strictly positive entries for some power $k \in \mathbb{N}$; we denote by β the corresponding dominant Perron-Frobenius eigenvalue, satisfying $\beta^3 = a\beta^2 + b\beta + 1$. The substitution σ has the following properties. It is

- *primitive*: the incidence matrix $\mathbf{M}$ is a primitive matrix;.
- *unimodular*: β is an algebraic unit;
- *irreducible*: the algebraic degree of β is exactly $|\mathcal{A}| = 3$;
- *Pisot*: the Galois conjugates α_1, α_2 of β have modulus strictly smaller than 1.

Observe that the substitutions $\sigma_{a,b}$ are so-called *beta-substitutions*, that is, the induced dynamical system is intimately related to beta-expansions. Details can be found, for example, in [8].

2.2. Associated Rauzy fractals

There are several equivalent ways of constructing the Rauzy fractal. For an overview of the different methods we refer to [7]. Here we will use the way via the so-called prefix-suffix graph presented in [12].

Let $\mathbf{u}_\beta$ be a strictly positive right eigenvector and $\mathbf{v}_\beta$ a strictly positive left eigenvector of $\mathbf{M}$ that correspond to the dominant eigenvalue β such that $\langle \mathbf{u}_\beta, \mathbf{v}_\beta \rangle = 1$. We set

$$\begin{aligned} \mathbf{v}_\beta =& (v_1, v_2, v_3) = (1, \beta - a, \beta^2 - a\beta - b) = (1, \beta^{-2} + b\beta^{-1}, \beta^{-1}), \\ \mathbf{u}_\beta =& \frac{1}{3\beta^2 - 2a\beta - b}(\beta^2, \beta, 1). \end{aligned}$$

Note that

$$1 = v_1 > v_2 > v_3 > 0. \tag{2.1}$$

Moreover, let $\mathbf{u}_{\alpha_i}$ be the eigenvectors for the Galois conjugates obtained by replacing β by α_i in the coordinates of the vector $\mathbf{u}_\beta$. We obtain the decomposition

$$\mathbb{R}^3 = \mathbb{H}_e \oplus \mathbb{H}_c,$$

where

- $\mathbb{H}_e$ is the *expanding line*, generated by $\mathbf{u}_\beta$.
- $\mathbb{H}_c$ is the *contracting space*, generated by $\mathbf{u}_{\alpha_1} + \mathbf{u}_{\alpha_2}$ and $-\alpha_2\mathbf{u}_{\alpha_1} - \alpha_1\mathbf{u}_{\alpha_2}$.

We denote by $\pi : \mathbb{R}^3 \to \mathbb{H}_c$ the projection onto $\mathbb{H}_c$ along $\mathbb{H}_e$ and by $\mathbf{h}$ the restriction of $\mathbf{M}$ on the contractive space $\mathbb{H}_c$. Note that if we define the norm

$$||\mathbf{x}|| = \max\left\{|\langle \mathbf{x}, \mathbf{v}_{\alpha_1}\rangle|, |\langle \mathbf{x}, \mathbf{v}_{\alpha_2}\rangle|\right\},$$

then $\mathbf{h}$ is a contraction with $\|\mathbf{h}\mathbf{x}\| \le \max\{|\alpha_1|, |\alpha_2|\}\, \|\mathbf{x}\| < \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{H}_c$.

Furthermore, we have

$$\forall w \in \mathcal{A}^*, \quad \mathbf{h}(\pi(\mathbf{l}(w))) = \pi(\mathbf{M}\mathbf{l}(w)) = \pi(\mathbf{l}(\sigma(w))). \tag{2.2}$$

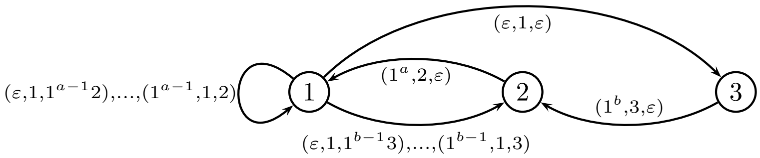
The *prefix-suffix graph* Γ_σ is defined as follows (see also [11]). Let $\mathcal{P}$ be the finite set

$$\mathcal{P} = \{(p, i, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^* \,|\, \exists\, j \in \mathcal{A}, \sigma(j) = pis\}.$$

Then Γ_σ is the directed graph with

- vertices : the letters of the alphabet $\mathcal{A}$;
- edges : $i \xrightarrow{(p,i,s)} j$ if and only if $\sigma(j) = pis$, where $(p, i, s) \in \mathcal{P}$.

The prefix-suffix graph $\Gamma_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is



. (2.3)

Here, for a letter $i \in \mathcal{A}$, i^k stands for $\underbrace{i \ldots i}_{k \text{ times}}$.

The Rauzy fractal and its subtiles are geometric representations of the infinite walks in the prefix-suffix graph [12]:

$$\mathcal{T} = \left\{ \sum_{k \ge 0} \mathbf{h}^k \pi(\mathbf{l}(p_k)) \;\middle|\; \begin{array}{l} j_0 \xrightarrow{(p_0,j_0,s_0)} j_1 \xrightarrow{(p_1,j_1,s_1)} j_2 \xrightarrow{(p_2,j_2,s_2)} \cdots \\ \text{is an infinite path of } \Gamma_\sigma \end{array} \right\}$$

and for $j \in \mathcal{A}$

$$\mathcal{T}(j) = \left\{ \sum_{k \ge 0} \mathbf{h}^k \pi(\mathbf{l}(p_k)) \;\middle|\; \begin{array}{l} j_0 = j \xrightarrow{(p_0,j_0,s_0)} j_1 \xrightarrow{(p_1,j_1,s_1)} j_2 \xrightarrow{(p_2,j_2,s_2)} \cdots \\ \text{is an infinite path of } \Gamma_\sigma \end{array} \right\}. \tag{2.4}$$

Since σ is a primitive unimodular Pisot substitution satisfying the strong coincidence condition, the subtiles have disjoint interiors (*e.g.*, see [8]). Moreover, by [24] each subtile is the closure of its interior, as it is the solution of a graph-based iterated function system.

Due to the connection with beta-expansions, the Rauzy fractals for our class coincide with beta-tiles treated, for example, in [1, 9].

2.3. Tilings

For a substitution $\sigma_{a,b}$ of our class, the Rauzy fractal gives rise to two types of tilings of the contracting space $\mathbb{H}_c$: an aperiodic tiling and a periodic tiling, obtained as follows.

The *self-replicating translation set* is

$$\Gamma_{srs} := \left\{[\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^3) \times \mathcal{A} \,|\, 0 \le \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_i \right\}. \tag{2.5}$$

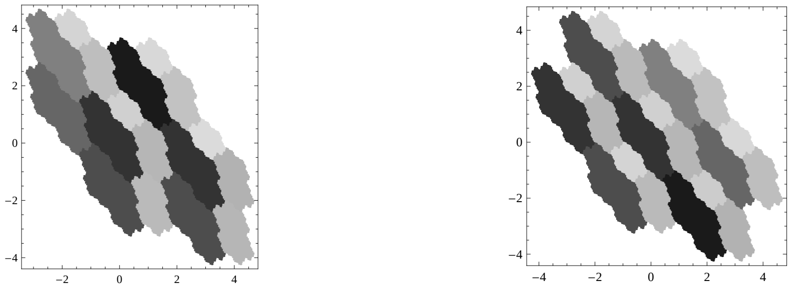


Figure 1: Clippings around the origin of the self-replicating tiling (left) and the lattice tiling (right) associated to the substititution $\sigma_{1,1}$.

Then $\{\mathcal{T}(i)+\gamma | \, [\gamma,i] \in \Gamma_{srs}\}$ is the *self-replicating tiling* of the contracting space (see [18]).

The *lattice translation set* is

$$\Gamma_{lat} = \left\{ [\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^3) \times \mathcal{A} \,|\, \mathbf{x} = (x_1, x_2, x_3), x_1 + x_2 + x_3 = 0 \right\}.$$

Then $\{\mathcal{T}(i)+\gamma | \, [\gamma,i] \in \Gamma_{lat}\}$ is the *lattice tiling* of the contracting space (see [3, 12]) .

By [2] the tilings have the following properties:

- *covering property:* $\mathbb{H}_c = \bigcup_{[\gamma,i]\in\Gamma_{srs}} \mathcal{T}(i)+\gamma = \bigcup_{[\gamma,i]\in\Gamma_{lat}} \mathcal{T}(i)+\gamma$;
- *tiling property:* the interiors of two tiles $\mathcal{T}(i)+\gamma$, $\mathcal{T}(j)+\gamma'$ with $[\gamma,i] \neq [\gamma',j] \in \Gamma_{srs}$ or $[\gamma,i] \neq [\gamma',j] \in \Gamma_{lat}$ are disjoint;
- *local finiteness:* for each compact subset B of $\mathbb{H}_c$, the subsets $\{[\gamma,i] \in \Gamma_{srs} | \, (\mathcal{T}(i)+\gamma) \cap B \neq \emptyset\}$ and $\{[\gamma,i] \in \Gamma_{lat} | \, (\mathcal{T}(i)+\gamma) \cap B \neq \emptyset\}$ are finite.

Figure 1 shows the self-replicating tiling (left) and the lattice tiling (right) for the Tribonacci substitution $\sigma_{1,1}$. The lattice tiling and topological properties of $\mathcal{T}$ have been already studied in [19, 20].

2.4. Boundary graphs

Graphs that describe the intersection of two tiles in the above tilings were introduced by Siegel and Thuswaldner [23]. The aim of this paper is the computation of these graphs for the whole class $\sigma_{a,b}$ introduced in Subsection 2.1. We recall briefly their definitions in terms of our class $\sigma_{a,b}$.

We call *neighbours* two subtiles of the self-replicating (or lattice) tiling if their intersection is non-empty. The intersection $\mathcal{T}(i) \cap (\mathcal{T}(j)+\gamma)$ will be described by the vertex $[i,\gamma,j]$ in the boundary graph. Since $[j,-\gamma,i]$ would correspond to the same intersection translated by $-\gamma$, we impose the vertices to belong to

$$\mathcal{D} = \left\{ [i,\gamma,j] \in \mathcal{A} \times \pi\left(\mathbb{Z}^3\right) \times \mathcal{A} \,|\, \gamma = \pi(\mathbf{x}), (\langle \mathbf{x}, \mathbf{v}_\beta \rangle > 0) \text{ or } (\gamma = \mathbf{0} \text{ and } i < j) \right\}.$$

Definition 2.1 (*cf.* [23])**.** The *self-replicating boundary graph* $\mathcal{G}^{(B)}_{srs}$ (*lattice boundary graph* $\mathcal{G}^{(B)}_{lat}$, respectively) is the largest graph with the following properties.

1. The vertices $[i,\gamma,j]$ are elements of $\mathcal{D}$ such that

$$\|\gamma\| \leq \frac{2\max_{(p,j,s)\in\mathcal{P}_\sigma} \|\pi \mathbf{l}(p)\|}{1-\max\{|\alpha_1|,|\alpha_2|\}}. \tag{2.6}$$

2. There is an edge from $[i,\gamma,j]$ to $[i',\gamma',j']$ if and only if there exist $[\overline{i},\overline{\gamma},\overline{j}] \in \mathcal{A}\times\pi(\mathbb{Z}^3)\times\mathcal{A}$ and $(p_1,a_1,s_1),(p_2,a_2,s_2)\in\mathcal{P}$ such that

$$\begin{cases} [i',\gamma',j'] = [\overline{i},\overline{\gamma},\overline{j}] \text{ (Type 1) or } [i',\gamma',j'] = [\overline{j},-\overline{\gamma},\overline{i}] \text{ (Type 2)},\\ a_1 = i \text{ and } p_1a_1s_1 = \sigma(\overline{i}),\\ a_2 = j \text{ and } p_2a_2s_2 = \sigma(\overline{j}),\\ \mathbf{h}\overline{\gamma} = \gamma + \pi(\mathbf{l}(p_2) - \mathbf{l}(p_1)). \end{cases}$$

The edge is labelled by

$$\eta = \begin{cases} \pi\mathbf{l}(p_1), & \text{if } \langle \mathbf{l}(p_1),\mathbf{v}_\beta\rangle \leq \langle \mathbf{l}(p_2)+\mathbf{x},\mathbf{v}_\beta\rangle,\\ \pi\mathbf{l}(p_2)+\gamma, & \text{otherwise,}\end{cases}$$

where $\mathbf{x}\in\mathbb{Z}^3$ such that $\pi(\mathbf{x}) = \gamma$.

3. Each vertex belongs to an infinite walk starting from a vertex $[i,\gamma,j]$ with $[\gamma,j]\in\Gamma_{srs}$ ($[\gamma,j]\in\Gamma_{lat}\setminus(\{\mathbf{0}\}\times\mathcal{A})$, respectively).

There exist algorithms to compute $\mathcal{G}^{(B)}_{srs}$ and $\mathcal{G}^{(B)}_{lat}$ for any given substitution (see [23]). These algorithms mainly consist in enumerating nodes satisfying Condition 1. and pruning them according to Conditions 2. and 3. However, the bound (2.6) in Definition 2.1 is inconvenient when working with a whole class like $\sigma_{a,b}$. We will formulate an equivalent definition for $\mathcal{G}^{(B)}_{srs}$ in Section 4 without this bound (see Theorem 4.1).

The following three propositions contain information on the structure and the use of the boundary graphs. The proofs can be found in [23].

Proposition 2.2 (*cf.* [23, Proposition 5.5])**.** *The self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$ and the lattice boundary graph $\mathcal{G}^{(B)}_{lat}$ are well defined and finite.*

Proposition 2.3 (*cf.* [23, Theorem 5.7])**.** *Let $[i,\gamma,j]$ be a vertex in the self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$. Then $[\gamma,j]\in\Gamma_{srs}$.*

Unfortunately, an analogue assertion for the lattice boundary graph does not hold. For this reason, our results concerning the lattice boundary graph will be weaker than for the self-replicating boundary graph.

Proposition 2.4 (*cf.* [23, Corollary 5.9])**.** *Let $[i,\gamma,j]\in\mathcal{D}$. A point $\xi\in\mathbb{H}_c$ belongs to the intersection $\mathcal{T}(i)\cap(\mathcal{T}(j)+\gamma)$ with $[\gamma,j]\in\Gamma_{srs}$ if and only if there exists an infinite walk in $\mathcal{G}^{(B)}_{srs}$ starting from $[i,\gamma,j]$ and labelled by $(\eta^{(k)})_{\geq 0}$ such that*

$$\xi = \sum_{k\geq 0}\mathbf{h}^k\eta^{(k)}.$$

3. Main theorems

The main results of this paper consist in a description of the boundary graphs associated to the substitutions of the class defined in Subsection 2.1. For convenience, we set

$$m(a,b) := \max\left\{1, \left\lfloor\frac{a}{a-b+2}\right\rfloor\right\}.$$

Note that $m(a,b) = 1$ if and only if $a \geq 2b-3$.

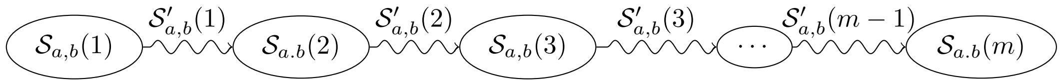


Figure 2: The subgraphs $\mathcal{S}_{a,b}(t)$ $(t = 1 \cdots m = m(a,b))$ of the self-replicating boundary graph are linked by the edges of $\mathcal{S}'_{a,b}(t)$ $(t = 1 \cdots m-1)$.

3.1. The self-replicating boundary graph

Let $1 \le b \le a$. For $1 \le t \le m(a,b)$, we call $\mathcal{S}_{a,b}(t)$ the graph whose nodes and edges are given in Adjacency Table 1 ($t = 1$) and Adjacency Table 2 ($t \ge 2$). For $1 \le t \le m(a,b) - 1$, we denote by $\mathcal{S}'_{a,b}(t)$ the graph described by Adjacency Table 3. Finally, let $\mathcal{S}(a,b)$ denote the union of these graphs:

$$\mathcal{S}(a,b) = \bigcup_{t=1}^{m(a,b)} \mathcal{S}_{a,b}(t) \;\cup \bigcup_{t=1}^{m(a,b)-1} \mathcal{S}'_{a,b}(t).$$

Remark 3.1. This subdivision of $\mathcal{S}(a,b)$ into several subgraphs has technical reasons that will become apparent in the proof. As can be seen in the Adjacency Tables, the subgraphs $\mathcal{S}_{a,b}(t)$ for the different values of t have very similar structures. Schematically, $\mathcal{S}(a,b)$ has the shape depicted in Figure 2. It turns out that the size of $\mathcal{S}(a,b)$ associated with $\sigma_{a,b}$ grows with the number $m(a,b)$. Mainly, if $m(a,b) \le m(a',b')$, then the graph associated with $\sigma_{a,b}$ is a subgraph of the graph associated with $\sigma_{a',b'}$. In particular, $\mathcal{S}_{a,b}(1)$ is common to all the self-replicating boundary graphs of our class. Properly speaking, only the vertices of the graph are entirely determined by $m = m(a,b)$. This almost holds for the edges too: the exceptions are listed in the arrow Condition of the adjacency tables. The labels and the number of edges strongly depend on the specific values of a and b.

Theorem 3.2. *The self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$ related to the substitution $\sigma_{a,b}$ is equal to the graph $\mathcal{S}(a,b)$. Its nodes and edges can be read off from Adjacency Tables 1, 2 and 3.*

Proof. The complete proof can be found in Section 5. We give here a short overview of the strategy. First we will give an alternative definition of the self-replicating boundary graph in Theorem 4.1. In this way, we will be able to prove that $\mathcal{S}(a,b)$ is a subgraph of $\mathcal{G}^{(B)}_{srs}$. This part of the proof, relying on Lemma 5.1, is purely algebraic. The proof of the reverse inclusion mixes algebraic and algorithmic methods (Lemmata 4.2, 4.5 and 5.2). In Lemma 5.2, we will show that the vertices of the strongly connected components of $\mathcal{G}^{(B)}_{srs}$ are included in $\mathcal{S}(a,b)$. With this result we will obtain that, in fact, all vertices of $\mathcal{G}^{(B)}_{srs}$ are included in $\mathcal{S}(a,b)$ and, finally, that $\mathcal{G}^{(B)}_{srs}$ is a subgraph of $\mathcal{S}(a,b)$. □

3.2. The lattice boundary graph

As already mentioned, we are not able to characterise $\mathcal{G}^{(B)}_{lat}$ completely for all possible values of a and b. We give a complete description of the lattice boundary graph for the case $m(a,b) = 1$ and conjecture the shape of $\mathcal{G}^{(B)}_{lat}$ for the other cases. Let $1 \le b \le a$. For $1 \le t \le m(a,b)$, we call $\mathcal{L}_{a,b}(t)$ the graph whose nodes and edges are given in Adjacency Table 4 ($t = 1$) and Adjacency Table 5 ($t \ge 2$). For $1 \le t \le m(a,b) - 1$, we denote by $\mathcal{L}'_{a,b}(t)$ the graph described by Adjacency Tables 6 and 7. Note that in these tables the vertices that correspond to elements of the lattice translation set Γ_{lat} are highlighted in grey. We denote by $\mathcal{L}(a,b)$ the union of all these graphs:

$$\mathcal{L}(a,b) = \bigcup_{t=1}^{m(a,b)} \mathcal{L}_{a,b}(t) \;\cup \bigcup_{t=1}^{m(a,b)-1} \mathcal{L}'_{a,b}(t).$$

Vertex			Edge(s)			
#	Name	Condition	to	Label(s)	Type	Condition
A_1	$[1, \pi(0,0,1), 1]$		C_1	$\{\pi(e,0,0) \mid 0 \leq e \leq a-b\}$	1	
			D_1	$\{\pi(0,0,0)\}$	1	
			O_1	$\{\pi(0,0,0)\}$	1	
			N_1	$\{\pi(e,0,0) \mid 0 \leq e \leq a-b-1\}$	1	$a \neq b$
B_1	$[1, \pi(0,0,1), 2]$		N_1	$\{\pi(a-b,0,0)\}$	1	
			C_1	$\{\pi(a-b+1,0,0)\}$	1	$b \neq 1$
C_1	$[1, \pi(0,1,-1), 1]$		P_1	$\{\pi(e,0,0) \mid 0 \leq e \leq b-1\}$	1	
			H_1	$\{\pi(e,0,0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
			I_1	$\{\pi(e,0,0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
D_1	$[1, \pi(0,1,-1), 2]$		H_1	$\{\pi(b-1,0,0)\}$	1	
			I_1	$\{\pi(b-1,0,0)\}$	1	$b \geq 2$
E_1	$[2, \pi(1,0,-1), 1]$		C_1	$\{\pi(a-b+1,0,-1)\}$	2	
			N_1	$\{\pi(a-b,0,-1)\}$	2	$a \neq b$
F_1	$[3, \pi(1,0,-1), 1]$		D_1	$\{\pi(1,0,-1)\}$	2	
			O_1	$\{\pi(1,0,-1)\}$	2	
G_1	$[1, \pi(1,0,-1), 1]$	$a \neq b$ or $a \geq 4$	C_1	$\{\pi(e,0,-1) \mid 1 \leq e \leq a-b\}$	2	$a \neq b$
			N_1	$\{\pi(e,0,-1) \mid 1 \leq e \leq a-b-1\}$	2	$a \geq b+2$
H_1	$[2, \pi(1,-1,1), 1]$		P_1	$\{\pi(b,-1,1)\}$	2	
			H_1	$\{\pi(b-1,-1,1)\}$	2	$b \geq 2$
			I_1	$\{\pi(b-1,-1,1)\}$	2	$b \geq 2$
I_1	$[1, \pi(1,-1,1), 1]$	$b \geq 2$	P_1	$\{\pi(e,-1,1) \mid 1 \leq e \leq b-1\}$	2	$b \geq 2$
			H_1	$\{\pi(e,-1,1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$
			I_1	$\{\pi(e,-1,1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$
J_1	$[1, \pi(0,0,0), 2]$		A_1	$\{\pi(a-1,0,0)\}$	1	
K_1	$[1, \pi(0,0,0), 3]$		$B_!$	$\{\pi(b-1,0,0)\}$	1	
			J_1	$\{\pi(b,0,0)\}$	1	$a \neq b$
			M_1	$\{\pi(b-1,0,0)\}$	1	$b = 1$
L_1	$[2, \pi(0,0,0), 3]$	$a = b$	J_1	$\{\pi(a,0,0)\}$	1	$a = b$
M_1	$[2, \pi(0,0,1), 2]$	$b = 1$	C_1	$\{\pi(a,0,0)\}$	1	$b = 1$
N_1	$[1, \pi(0,1,0), 1]$		E_1	$\{\pi(0,0,0)\}$	1	
			F_1	$\{\pi(0,0,0)\}$	1	
			G_1	$\{\pi(0,0,0)\}$	1	$a \neq b$
O_1	$[3, \pi(0,1,-1), 2]$		P_1	$\{\pi(b,0,0)\}$	1	
P_1	$[2, \pi(1,-1,0), 1]$		E_1	$\{\pi(1,-1,0)\}$	2	
			F_1	$\{\pi(1,-1,0)\}$	2	
			G_1	$\{\pi(1,-1,0)\}$	2	$a \neq b$ or $a \geq 4$

Adjacency Table 1: The subgraph $\mathcal{S}_{a,b}(1)$ of the self-replicating boundary graph.

Remark 3.3. The graph $\mathcal{L}(a,b)$ has a similar structure as the graph $\mathcal{S}(a,b)$, with subgraphs $\mathcal{L}_{a,b}(t)$ and edges between them described by $\mathcal{L}'_{a,b}(t)$ (see Remark 3.1).

Theorem 3.4. *For all a, b, the graph $\mathcal{L}(a,b)$ is a subgraph of the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to $\sigma_{a,b}$.*

We will prove the theorem at the beginning of Section 6. For $m(a,b) = 1$, we will also prove the reverse inclusion, as stated below.

Theorem 3.5. *Let $a \geq 2b-3$. Then the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to the substitution $\sigma_{a,b}$ equals $\mathcal{L}(a,b)$.*

The proof can be found in Section 6. Unfortunately, we cannot use the same strategy as for Theorem 3.2, although the lattice boundary graph and the self-replicating boundary graph have very similar definitions. The main problem is that an equivalent of Proposition 2.3 does not hold for the lattice boundary graph.

Proof. The proof is mainly topological and will run as follows. We make use of the results concerning the self-replicating boundary graph in order to construct a *tube* around the Rauzy fractal (central tile). We will show that six Γ_{lat}-translates of the Rauzy fractal are sufficient to cover all this tube and deduce from topological arguments that no other Γ_{lat}-translate can intersect the central tile. □

The following conjecture remains.

Conjecture 3.6. $\mathcal{L}(a,b)$ coincides with the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to the substitution $\sigma_{a,b}$ for all $a \geq b \geq 1$.

Vertex			Edge(s)			
#	Name	Condition	to	Label(s)	Type	Condition
A_t	$[1, \pi(t-1, 1-t, t), 1]$		C_t	$\{\pi(e,0,0) \mid 0 \leq e \leq \delta_t - 2\}$	1	
			D_t	$\{\pi(e,0,0) \mid 0 \leq e \leq \delta_t + b - a - 2\}$	1	
B_t	$[1, \pi(t-1, 1-t, t), 2]$		C_t	$\{\pi(\delta_t - 1, 0, 0)\}$	1	
C_t	$[1, \pi(1-t, t, -t), 1]$		H_t	$\{\pi(e,0,0) \mid 0 \leq e \leq a - \delta_t\}$	1	
			I_t	$\{\pi(e,0,0) \mid 0 \leq e \leq a - \delta_t\}$	1	$a \geq \delta_t + 1$
D_t	$[1, \pi(1-t, t, -t), 2]$		H_t	$\{\pi(a - \delta_t + 1, 0, 0)\}$	1	
			I_t	$\{\pi(a - \delta_t + 1, 0, 0)\}$	1	$a \geq \delta_t + 1$
E_t	$[2, \pi(2-t, t-1, -t), 1]$		C_t	$\{\pi(\delta_t - t, t-1, -t)\}$	2	
F_t	$[3, \pi(2-t, t-1, -t), 1]$		D_t	$\{\pi(\delta_t - a + b - t, t-1, -t)\}$	2	
G_t	$[1, \pi(2-t, t-1, -t), 1]$		C_t	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 3\}$	2	
			D_t	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_{t-1} - 1\}$	2	
H_t	$[2, \pi(t, -t, t), 1]$		H_t	$\{\pi(a - \delta_t + t, -t, t)\}$	2	
			I_t	$\{\pi(a - \delta_t + t, -t, t)\}$	2	$a \geq \delta_t + 1$
I_t	$[1, \pi(t, -t, t), 1]$	$a \geq \delta_t + 1$	H_t	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 1\}$	2	$a \geq \delta_t + 1$
			I_t	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 1\}$	2	$a \geq \delta_t + 1$
Q_t	$[3, \pi(t-1, 1-t, t-1), 1]$		B_t	$\{\pi(a - \delta_t + t, 1-t, t-1)\}$	2	

Adjacency Table 2: The subgraph $\mathcal{S}_{a,b}(t)$ of the self-replicating boundary graph, where $2 \leq t \leq m(a,b)$ ($\delta_t = t(a-b+2)$).

Edges			
from	to	Label(s)	Type
A_t	E_{t+1}	$\{\pi(e,0,0) \mid 0 \leq e \leq \delta_t - 1\}$	1
	F_{t+1}	$\{\pi(0,0,0)\}$	1
	G_{t+1}	$\{\pi(e,0,0) \mid 0 \leq e \leq \delta_t - 1\}$	1
B_t	E_{t+1}	$\{\pi(\delta_t, 0, 0)\}$	1
	G_{t+1}	$\{\pi(\delta_t, 0, 0)\}$	1
C_t	A_{t+1}	$\{\pi(e,0,0) \mid 0 \leq e \leq a - \delta_t - 1\}$	1
	B_{t+1}	$\{\pi(e,0,0) \mid 0 \leq e \leq b - \delta_t - 1\}$	1
	Q_{t+1}	$\{\pi(0,0,0)\}$	1
D_t	A_{t+1}	$\{\pi(a - \delta_t, 0, 0)\}$	1
E_t	E_{t+1}	$\{\pi(\delta_t + 1 - t, t-1, -t)\}$	2
	G_{t+1}	$\{\pi(\delta_t + 1 - t, t-1, -t)\}$	2
G_t	E_{t+1}	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 2\}$	2
	F_{t+1}	$\{\pi(2-t, t-1, -t)\}$	2
	G_{t+1}	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 2\}$	2
H_t	A_{t+1}	$\{\pi(a - \delta_t - 1 + t, -t, t)\}$	2
I_t	A_{t+1}	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 2\}$	2
	B_{t+1}	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq b - \delta_t - 2\}$	2
	Q_{t+1}	$\{\pi(t, -t, t)\}$	2

Adjacency Table 3: The subgraph $\mathcal{S}'_{a,b}(t)$ of the self-replicating boundary graph, where $1 \leq t \leq m(a,b) - 1$ ($\delta_t = t(a-b+2)$).

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Theorem 3.4 shows that for $a \leq 2b - 4$, *i.e.*, $m(a,b) \geq 2$, each tile in the lattice tiling has 10 or more neighbours. Using a classical result concerning lattice tilings (see, for example, [4, Lemma 5.1] or [15]), we conclude that the Rauzy fractals in these cases are not disk-like.

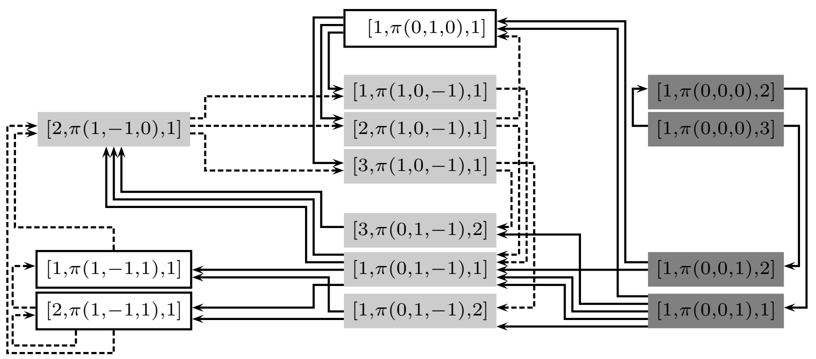
Theorem 3.7. *If $a \leq 2b - 4$ then the Rauzy fractal $\mathcal{T}$ induced by the substitution $\sigma_{a,b}$ is not homeomorphic to a topological disk.*

3.3. Examples

Example 3.8. Let $a = 3$ and $b = 2$. Then $m(3,2) = 1$. The self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$ of $\sigma_{3,2}$ consists of 14 vertices. The graph is shown in (3.1). Edges of Type 1 are drawn solid while those of Type 2 are dashed. The labels can be obtained from Adjacency Table 1. The lattice boundary graph $\mathcal{G}^{(B)}_{lat}$ has a similar shape. Indeed, it can be obtained from (3.1) by removing the dark grey vertices (and the associated edges). The vertices that correspond to elements of Γ_{lat} are the light grey ones. Figure 3 shows the Rauzy fractal and its neighbours in the self-replicating tiling (left) and the lattice tiling (right). $\mathcal{T}(1)$ is the biggest subtile, followed by $\mathcal{T}(2)$ and $\mathcal{T}(3)$. The numbers inside show the corresponding translation. The boundaries between subtiles with respect to the same translation are grey.

Vertex			Edge(s)			
#	Name	Condition	to	Label(s)	Type	Condition
C_1	$[1, \pi(0,1,-1), 1]$		P_1	$\{\pi(e,0,0) \mid 0 \le e \le b-1\}$	1	
			H_1	$\{\pi(e,0,0) \mid 0 \le e \le b-2\}$	1	$b \ge 2$
			I_1	$\{\pi(e,0,0) \mid 0 \le e \le b-2\}$	1	$b \ge 2$
D_1	$[1, \pi(0,1,-1), 2]$		H_1	$\{\pi(b-1,0,0)\}$	1	
			I_1	$\{\pi(b-1,0,0)\}$	1	$b \ge 2$
E_1	$[2, \pi(1,0,-1), 1]$		C_1	$\{\pi(a-b+1,0,-1)\}$	2	
			N_1	$\{\pi(a-b,0,-1)\}$	2	$a \ne b$
F_1	$[3, \pi(1,0,-1), 1]$		D_1	$\{\pi(1,0,-1)\}$	2	
			O_1	$\{\pi(1,0,-1)\}$	2	
G_1	$[1, \pi(1,0,-1), 1]$	$a \ne b$ or $a \ge 4$	C_1	$\{\pi(e,0,-1) \mid 1 \le e \le a-b\}$	2	$a \ne b$
			N_1	$\{\pi(e,0,-1) \mid 1 \le e \le a-b-1\}$	2	$a \ge b+2$
H_1	$[2, \pi(1,-1,1), 1]$		P_1	$\{\pi(b,-1,1)\}$	2	
			H_1	$\{\pi(b-1,-1,1)\}$	2	$b \ge 2$
			I_1	$\{\pi(b-1,-1,1)\}$	2	$b \ge 2$
I_1	$[1, \pi(1,-1,1), 1]$	$b \ge 2$	P_1	$\{\pi(e,-1,1) \mid 1 \le e \le b-1\}$	2	$b \ge 2$
			H_1	$\{\pi(e,-1,1) \mid 1 \le e \le b-2\}$	2	$b \ge 3$
			I_1	$\{\pi(e,-1,1) \mid 1 \le e \le b-2\}$	2	$b \ge 3$
N_1	$[1, \pi(0,1,0), 1]$		E_1	$\{\pi(0,0,0)\}$	1	
			F_1	$\{\pi(0,0,0)\}$	1	
			G_1	$\{\pi(0,0,0)\}$	1	$a \ne b$
O_1	$[3, \pi(0,1,-1), 2]$		P_1	$\{\pi(b,0,0)\}$	1	
P_1	$[2, \pi(1,-1,0), 1]$		E_1	$\{\pi(1,-1,0)\}$	2	
			F_1	$\{\pi(1,-1,0)\}$	2	
			G_1	$\{\pi(1,-1,0)\}$	2	$a \ne b$ or $a \ge 4$

Adjacency Table 4: The subgraph $\mathcal{L}_{a,b}(1)$ of the lattice boundary graph. The vertices highlighted in grey correspond to elements of the lattice translation set Γ_{lat}.



(3.1)

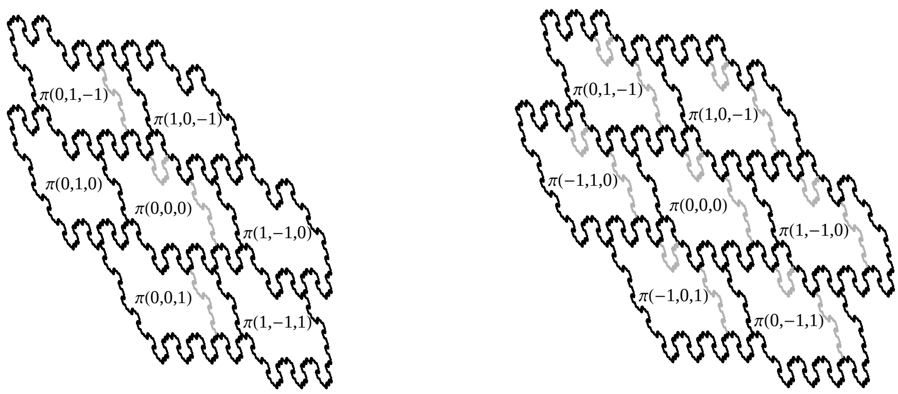


Figure 3: The neighbours of the Rauzy fractal in the self-replicated (left) and the lattice (right) tiling for $\sigma_{3,2}$.

Example 3.9. From Adjacency Table 1 we see that the case $b = 1$ has a special behaviour. Indeed, vertex $[1, \pi(1,-1,1), 1]$ does not appear but a vertex $[2, \pi(0,0,1), 2]$ exists. Analogously

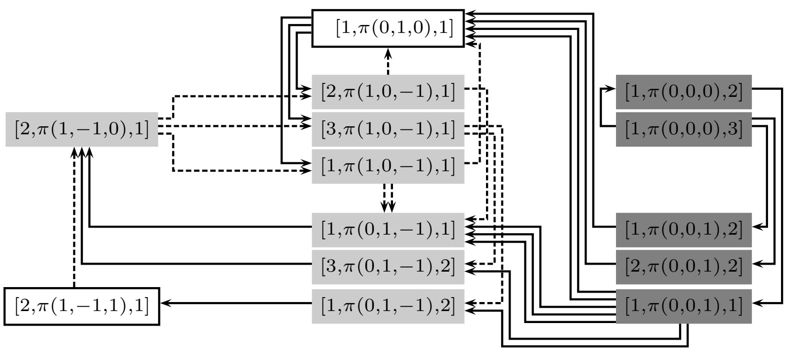
Vertex			Edge(s)			
#	Name	Condition	to	Label(s)	Type	Condition
A_t	$[1,\pi(t-1,1-t,t),1]$		C_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le \delta_t-2\}$	1	
			D_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le \delta_t+b-a-2\}$	1	
B_t	$[1,\pi(t-1,1-t,t),2]$		C_t	$\{\pi(\delta_t-1,0,0)\}$	1	
C_t	$[1,\pi(1-t,t,-t),1]$		H_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le a-\delta_t\}$	1	
			I_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le a-\delta_t\}$	1	$a\ge \delta_t+1$
D_t	$[1,\pi(1-t,t,-t),2]$		H_t	$\{\pi(a-\delta_t+1,0,0)\}$	1	
			I_t	$\{\pi(a-\delta_t+1,0,0)\}$	1	$a\ge \delta_t+1$
E_t	$[2,\pi(2-t,t-1,-t),1]$		C_t	$\{\pi(\delta_t-t,t-1,-t)\}$	2	
E'_t	$[2,\pi(1,t-1,-t),1]$		C_t	$\{\pi(\delta_t-t,t-1,-t)\}$	2	
F_t	$[3,\pi(2-t,t-1,-t),1]$		D_t	$\{\pi(\delta_t-a+b-t,t-1,-t)\}$	2	
F'_t	$[3,\pi(1,t-1,-t),1]$		D_t	$\{\pi(\delta_{t-1}-t+2,t-1,-t)\}$	2	
G_t	$[1,\pi(2-t,t-1,-t),1]$		C_t	$\{\pi(e+2-t,t-1,-t)\,\vert\, 0\le e\le \delta_t-3\}$	2	
			D_t	$\{\pi(e+2-t,t-1,-t)\,\vert\, 0\le e\le \delta_{t-1}-1\}$	2	
G'_t	$[1,\pi(1,t-1,-t),1]$		C_t	$\{\pi(e+1,t-1,-t)\,\vert\, 0\le e\le \delta_t-t-2\}$	2	
			D_t	$\{\pi(e+1,t-1,-t)\,\vert\, 0\le e\le \delta_{t-1}-t\}$	2	
H_t	$[2,\pi(t,-t,t),1]$		H_t	$\{\pi(a-\delta_t+t,-t,t)$	2	
			I_t	$\{\pi(a-\delta_t+t,-t,t)\}$	2	$a\ge \delta_t+1$
H'_t	$[1,\pi(0,t,-t),2]$		H_t	$\{\pi(a-\delta_t+t,0,0)\}$	1	
			I_t	$\{\pi(a-\delta_t+t,0,0)\}$	1	$a\ge \delta_t+1$
I_t	$[1,\pi(t,-t,t),1]$	$a\ge \delta_t+1$	H_t	$\{\pi(e+t,-t,t)\,\vert\, 0\le e\le a-\delta_t-1\}$	2	$a\ge \delta_t+1$
			I_t	$\{\pi(e+t,-t,t)\,\vert\, 0\le e\le a-\delta_t-1\}$	2	$a\ge \delta_t+1$
I'_t	$[1,\pi(0,t,-t),1]$		H_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le a-1-\delta_t+t\}$	1	
			I_t	$\{\pi(e,0,0)\,\vert\, 0\le e\le a-1-\delta_t+t\}$	1	
Q_t	$[3,\pi(t-1,1-t,t-1),1]$		B_t	$\{\pi(a-\delta_t+t,1-t,t-1)\}$	2	
Q'_t	$[1,\pi(0,t-1,1-t),3]$		B_t	$\{\pi(b-2-\delta_{t-1}+t,0,0)\}$	1	

Adjacency Table 5: The subgraph $\mathcal{L}_{a,b}(t)$ of the lattice boundary graph, where $2\le t\le m(a,b)$ ($\delta_t=t(a-b+2)$). The vertices highlighted in grey correspond to elements of the lattice translation set Γ_{lat}.

Edges			
from	to	Label(s)	Type
C_1	A_2	$\{\pi(e,0,0)\,\vert\, 0\le e\le b-3\}$	1
	B_2	$\{\pi(e,0,0)\,\vert\, 0\le e\le 2b-a-3\}$	1
	Q_2	$\{\pi(0,0,0)\}$	1
D_1	A_2	$\{\pi(b-2,0,0)\}$	1
E_1	E_2	$\{\pi(a-b+2,0,-1)\}$	2
	G_2	$\{\pi(a-b+2,0,-1)\}$	2
G_1	E_2	$\{\pi(e+1,0,-1)\,\vert\, 0\le e\le a-b\}$	2
	F_2	$\{\pi(1,0,-1)\}$	2
	G_2	$\{\pi(e+1,0,-1)\,\vert\, 0\le e\le a-b\}$	2
H_1	A_2	$\{\pi(b-2,-1,1)\}$	2
I_1	A_2	$\{\pi(e+1,-1,1)\,\vert\, 0\le e\le a-4\}$	2
	B_2	$\{\pi(e+1,-1,1)\,\vert\, 0\le e\le 2b-a-4\}$	2
	Q_2	$\{\pi(1,-1,1)\}$	2

Adjacency Table 6: The subgraph $\mathcal{L}'_{a,b}(1)$ of the lattice boundary graph.

to the previous example, (3.2) shows $\mathcal{G}^{(B)}_{srs}$ for $\sigma_{3,1}$. When we compare the self-replicating tiling induced by $\sigma_{3,1}$ (Figure 4, left) with the self-replicating tiling induced by $\sigma_{3,2}$ (Figure 3, left) we see what this change of vertices means for the tiling. Indeed, the $\pi(1,-1,1)$-translate of $\mathcal{T}(1)$ is no longer neighbour of of $\mathcal{T}(1)$ but the $\pi(0,0,1)$-translate of $\mathcal{T}(2)$ is adjacent to $\mathcal{T}(2)$. The lattice tiling is shown on the right in Figure 4. The Rauzy fractal has 6 neighbours.



(3.2)

Edges			
from	to	Label(s)	Type
A_t	E_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le \delta_t - 1\}$	1
	F_{t+1}	$\{\pi(0,0,0)\}$	1
	G_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le \delta_t - 1\}$	1
B_t	E_{t+1}	$\{\pi(\delta_t,0,0)\}$	1
	G_{t+1}	$\{\pi(\delta_t,0,0)\}$	1
C_t	A_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le a - \delta_t - 1\}$	1
	B_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le b - \delta_t - 1\}$	1
	Q_{t+1}	$\{\pi(0,0,0)\}$	1
D_t	A_{t+1}	$\{\pi(a - \delta_t,0,0)\}$	1
E_t	E_{t+1}	$\{\pi(\delta_t + 1 - t, t - 1, -t)\}$	2
	G_{t+1}	$\{\pi(\delta_t + 1 - t, t - 1, -t)\}$	2
E'_t	E_{t+1}	$\{\pi(\delta_t - t + 1, t - 1, -t)\}$	2
	G_{t+1}	$\{\pi(\delta_t - t + 1, t - 1, -t)\}$	2
G_t	E_{t+1}	$\{\pi(e + 2 - t, t - 1, -t) \mid 0 \le e \le \delta_t - 2\}$	2
	F_{t+1}	$\{\pi(2 - t, t - 1, -t)\}$	2
	G_{t+1}	$\{\pi(e + 2 - t, t - 1, -t) \mid 0 \le e \le \delta_t - 2\}$	2
G'_t	E_{t+1}	$\{\pi(e + 1, t - 1, -t) \mid 0 \le e \le \delta_t - t - 1\}$	2
	F_{t+1}	$\{\pi(1, t - 1, -t)\}$	2
	G_{t+1}	$\{\pi(e + 1, t - 1, -t) \mid 0 \le e \le \delta_t - t - 1\}$	2
H_t	A_{t+1}	$\{\pi(a - \delta_t - 1 + t, -t, t)\}$	2
H'_t	A_{t+1}	$\{\pi(a - 1 - \delta_t + t, 0, 0)\}$	1
I_t	A_{t+1}	$\{\pi(e + t, -t, t) \mid 0 \le e \le a - \delta_t - 2\}$	2
	B_{t+1}	$\{\pi(e + t, -t, t) \mid 0 \le e \le b - \delta_t - 2\}$	2
	Q_{t+1}	$\{\pi(t, -t, t)\}$	2
I'_t	A_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le a - 2 - \delta_t + t\}$	1
	B_{t+1}	$\{\pi(e,0,0) \mid 0 \le e \le b - 2 - \delta_t + t\}$	1
	Q_{t+1}	$\{\pi(0,0,0)\}$	1

Adjacency Table 7: The subgraph $\mathcal{L}'_{a,b}(t)$ of the lattice boundary graph for $2 \le t \le m(a,b) - 1$ ($\delta_t = t(a - b + 2)$).

Example 3.10. The case $a = b$ also has a special behaviour. Here a vertex $[2, \pi(0,0,0), 3]$ shows up. For the Rauzy fractal this means that the subtiles $\mathcal{T}(2)$ and $\mathcal{T}(3)$ have common points. An example can be found in [23], where the substitution $\sigma_{4,4}$ is extensively studied (see [23, Figure 5.7] for $\mathcal{G}^{(B)}_{srs}$ and [23, Figure 3.2] for the tilings). We have $m(4,4) = 2$ and, hence, the self-replicating boundary graph has 24 vertices.

By Theorem 3.4, the lattice boundary graph for this example has at least 24 vertices and the Rauzy fractal has 10 neighbours in the lattice tiling. Thus, it is not homeomorphic to a disk. The actual graph can be found in [23, Figure 5.3]. We see that in this case Conjecture 3.6 holds.

4. Some preparations

We collect here several theorems and lemmata needed for the proof of the main results of the paper. The methods in this section are purely algebraic and similar results may be derived for other types of substitutions.

As already mentioned at the end of Section 2, we first give an alternative definition of the self-replicating boundary graph. Actually, the following theorem applies to each primitive unimodular Pisot substitution. Note that the first two conditions 1. and 2. are the same as in Definition 2.1.

Theorem 4.1. *The self replicated boundary graph $\mathcal{G}^{(B)}_{srs}$ equals the largest graph with*

1. *vertex set that consists of elements $[i, \gamma, j] \in \mathcal{D}$ with $[\gamma, j] \in \Gamma_{srs}$;*
2. *an edge from $[i, \gamma, j]$ to $[i', \gamma', j']$ if and only if there exist $[\bar{i}, \overline{\gamma}, \bar{j}] \in \mathcal{A} \times \pi(\mathbb{Z}^3) \times \mathcal{A}$ and $(p_1, a_1, s_1), (p_2, a_2, s_2) \in \mathcal{P}$ such that*

$$\begin{cases} [i', \gamma', j'] = [\bar{i}, \overline{\gamma}, \bar{j}] \text{ (Type 1) or } [i', \gamma', j'] = [\bar{j}, -\overline{\gamma}, \bar{i}] \text{ (Type 2)}, \\ a_1 = i \text{ and } p_1 a_1 s_1 = \sigma(\bar{i}), \\ a_2 = j \text{ and } p_2 a_2 s_2 = \sigma(\bar{j}), \\ \mathbf{h}\overline{\gamma} = \gamma + \pi(\mathbf{l}(p_2) - \mathbf{l}(p_1)). \end{cases}$$

The edge is labelled by

$$\eta = \begin{cases} \pi \mathbf{l}(p_1), & \text{if } \langle \mathbf{l}(p_1), \mathbf{v}_\beta \rangle \le \langle \mathbf{l}(p_2) + \mathbf{x}, \mathbf{v}_\beta \rangle, \\ \pi \mathbf{l}(p_2) + \gamma, & \text{otherwise}, \end{cases}$$

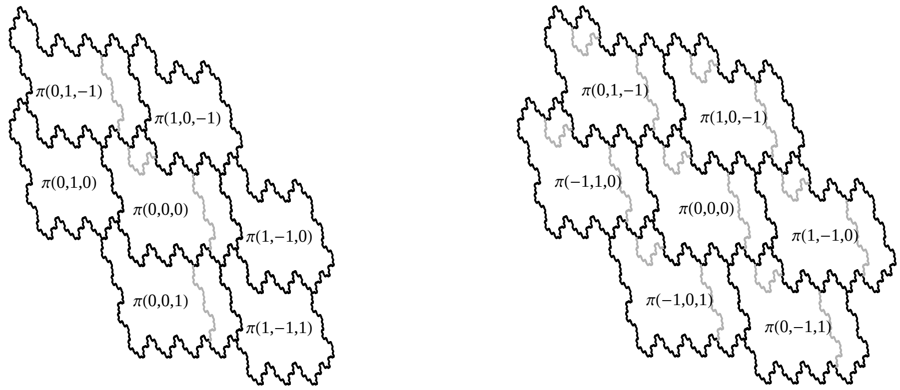


Figure 4: The neighbours of the Rauzy fractal in the self-replicated (left) and the lattice (right) tiling for $\sigma_{3,1}$.

where $\mathbf{x} \in \mathbb{Z}^3$ *such that* $\pi(\mathbf{x}) = \gamma$*;*

3. *every vertex lies on a path that ends up in a strongly connected component.*

Proof. Denote by G the largest graph fulfilling Items 1., 2. and 3. from above. We want to prove that $G = \mathcal{G}^{(B)}_{srs}$. The edges are defined in the same way, hence, it suffices to prove that both graphs have the same set of vertices.

By Proposition 2.3 for every vertex $[i, \gamma, j]$ of $\mathcal{G}^{(B)}_{srs}$ we have $[\gamma, j] \in \Gamma_{srs}$. Furthermore, $\mathcal{G}^{(B)}_{srs}$ is finite by Proposition 2.2 and every vertex of $\mathcal{G}^{(B)}_{srs}$ lies on an infinite walk by Definition 2.1. This implies that every vertex lies on a path ending up in a strongly connected component. Hence, $\mathcal{G}^{(B)}_{srs}$ is a subgraph of G.

Now consider $[i, \gamma, j] \in G$. Obviously, $[i, \gamma, j] \in \mathcal{G}^{(B)}_{srs}$ as soon as γ satisfies (2.6). Indeed, the other items of Definition 2.1 are easily seen to be fulfilled.

By Item 3. there exists a (finite) path from $[i, \gamma, j]$ to a vertex belonging to a strongly connected component of G. Therefore, there is an infinite path

$$[i, \gamma, j] \to [i_1, \gamma_1, j_1] \to [i_2, \gamma_2, j_2] \to \dots$$

in G going through finitely many vertices. Using the relation $\mathbf{h}\gamma_{k+1} = \pm\gamma_k \pm \pi(\mathbf{l}(p_2^{(k)}) - \mathbf{l}(p_1^{(k)}))$ that holds for each edge of this walk and the fact that $\mathbf{h}$ is a contraction, one obtains that γ satisfies (2.6). □

In the following lemma we estimate the number and shape of the predecessors of a given vertex in the boundary graph. The computations are valid for every irreducible beta-substitution.

Lemma 4.2. *Consider an edge from* $[i, \pi(\mathbf{x}), j]$ *to* $[i', \pi(x', y', z'), j']$ *in the self-replicating boundary graph* $\mathcal{G}^{(B)}_{srs}$*. Then*

$$\mathbf{x} = \left(-\left\lfloor x'\left(\beta^{-2} + b\beta^{-1}\right) + y'\beta^{-1}\right\rfloor, x', y'\right) \tag{4.1}$$

if the edge is of Type 1 and

$$\mathbf{x} = \left(\left\lceil x'\left(\beta^{-2} + b\beta^{-1}\right) + y'\beta^{-1}\right\rceil, -x', -y'\right) \tag{4.2}$$

if the edge is of Type 2.

Proof. If the edge is of Type 1, by the definition of $\mathcal{G}^{(B)}_{srs}$ and (2.2), we have

$$\mathbf{h}(\pi(x', y', z')) = \pi(\mathbf{M}(x', y', z')) = \pi(ax' + by' + z', x', y') = \pi(\mathbf{x}) - \pi(\mathbf{l}(p_1)) + \pi(\mathbf{l}(p_2)) \tag{4.3}$$

with $\sigma(i') = p_1 i s_2$ and $\sigma(j') = p_2 j s_2$.

Now observe that for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Q}^3$ we have

$$\pi(\mathbf{x}_1) = \pi(\mathbf{x}_2) \iff \langle \mathbf{x}_1, \mathbf{v}_\beta \rangle = \langle \mathbf{x}_2, \mathbf{v}_\beta \rangle \iff \mathbf{x}_1 = \mathbf{x}_2.$$

The first equivalence can be obtained by considering Galois conjugates (*cf.* [23, Equation (2.5)]), the second one is a consequence of the irreducibility of the substitution. In particular, this shows that π is injective for integer vectors. Therefore (4.3) can only hold if

$$\mathbf{x} = (ax' + by' + z' - e_2 + e_1, x', y') \tag{4.4}$$

where $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$ are integer vectors (by the shape of $\sigma_{a,b}$ the prefixes p_1 and p_2 consist of the symbols ε or 1 only). Since $[i, \pi(\mathbf{x}), j]$ is a vertex of $\mathcal{G}^{(B)}_{srs}$ we have $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_j < 1$ by Proposition 2.3. Applying this on (4.4) gives

$$\begin{aligned} 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle =& \langle (ax' + by' + z' - e_2 + e_1, x', y'), \mathbf{v}_\beta \rangle \\ =& ax' + by' + z' - e_2 + e_1 + x' \left(\beta^{-2} + b\beta^{-1} \right) + y'\beta^{-1} < 1. \end{aligned}$$

Since $ax' + by' + z' - e_2 + e_1$ is an integer we immediately obtain $ax' + by' + z' - e_2 + e_1 = -\left\lfloor x' \left(\beta^{-2} + b\beta^{-1} \right) + y'\beta^{-1} \right\rfloor$. Inserting this into (4.4) yields the assertion.

If the edge is of Type 2 we obtain, analogously to (4.4),

$$\mathbf{x} = (-ax' - by' - z' - e_2 + e_1, -x', -y') \tag{4.5}$$

with $\sigma(j') = p_1 i s_2$, $\sigma(i') = p_2 j s_2$, $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$. The same considerations as above yield

$$-ax' - by' - z' - e_2 + e_1 = -\left\lfloor -x' \left(\beta^{-2} + b\beta^{-1} \right) - y'\beta^{-1} \right\rfloor = \left\lceil x' \left(\beta^{-2} + b\beta^{-1} \right) + y'\beta^{-1} \right\rceil$$

which gives (4.2). □

An immediate consequence of Lemma 4.2 is the following corollary, that will be frequently used in the algorithmic part of the proof of Theorem 3.2.

Corollary 4.3. *For a predecessor* $[i, \pi(\mathbf{x}), j]$ *of a vertex* $[i', \gamma', j']$ *in the self-replicating boundary graph* $\mathcal{G}^{(B)}_{srs}$*, there are at most two possible choices for* $\mathbf{x}$*. One is connected via an edge of Type 1, another is connected via an edge of Type 2. Moreover, if a predecessor is of the form* $[i, \mathbf{0}, j]$*, then all predecessors are of this shape.*

Again, an analogue to Lemma 4.2, and therefore of Corollary 4.3, for $\mathcal{G}^{(B)}_{lat}$ does not exist and makes the proof of Theorem 3.5 more complicated.

Notation 4.4. Given an edge from $[i, \pi(x, y, z), j]$ to $[i', \pi(x', y', z')), j']$, we call the term $ax' + by' + z' - x$ (when the edge is of Type 1) or the term $ax' + by' + z' + x$ (when the edge is of Type 2), respectively, the *significant difference*.

We are interested in the pairs (i, j), (i', j') inducing significant differences. This is shown in Table 8.

There is an entry in the cell in row (i, j) and column (i', j') if there are $p_1, p_2, s_1, s_2 \in \mathcal{A}^*$ such that $\sigma(i') = p_1 i s_1$ and $\sigma(j') = p_2 j s_2$, *i.e.*, if there are edges $i \xrightarrow{(p_1, i, s_1)} i'$ and $j \xrightarrow{(p_2, j, s_2)} j'$ in Γ_σ. The corresponding entry is then a list of all possibilities for $e_2 - e_1$ with $(e_1, 0, 0) = \mathbf{l}(p_1)$ and $(e_2, 0, 0) = \mathbf{l}(p_2)$.

Note that, by construction, n is an element of the list in row (i, j) and column (i', j') if and only if $-n$ is an element of the list in row (j, i) and column (j', i').

The use of this table is enlightened in the following lemma.

	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$	$(3,1)$	$(3,2)$	$(3,3)$
$(1,1)$	$-a+1, \dots, a-1$	$-a+1, \dots, b-1$	$-a+1, \dots, 0$	$-b+1, \dots, a-1$	$-b+1, \dots, b-1$	$-b+1, \dots, 0$	$0, \dots, a-1$	$0, \dots, b-1$	0
$(1,2)$	$1, \dots, a$			$a-b+1, \dots, a$			a		
$(1,3)$		$b-a+1, \dots, b$			$1, \dots, b$			b	
$(2,1)$	$-a, \dots, -1$	$-a, \dots, -a+b-1$	$-a$						
$(2,2)$	0								
$(2,3)$		$b-a$							
$(3,1)$				$-b, \dots, a-b-1$	$-b, \dots, -1$	$-b$			
$(3,2)$				$a-b$					
$(3,3)$					0				

Table 8: The Table shows all possible differences for the prefixes of the labels of the product graph $\Gamma_\sigma \times \Gamma_\sigma$

Lemma 4.5. *Consider an edge from $[i, \pi(x,y,z), j]$ to $[i', \pi(x',y',z'), j']$ in the self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$. If the edge is of Type 1 then $ax' + by' + z' - x$ is contained in the list in row (i,j) and column (i',j') of Table 8. If the edge is of Type 2 then $ax' + by' + z' + x$ is contained in the list in row (j,i) and column (i',j') of Table 8.*

Proof. Let p_1, p_2 as in (2) of Definition 2.1. By the shape of the substitution we have $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$ with $e_1, e_2 \geq 0$. Suppose the edge is of Type 1. By construction of Table 8, $e_2 - e_1$ is an element of the list in row (i,j) and column (i',j') and by (4.4) we have that $e_2 - e_1 = ax' + by' + z' - x$.

If the edge is of Type 2, $e_1 - e_2$ is element of the list in row (j,i) and column (i',j'). On the other hand, (4.5) shows that $ax' + by' + z' + x = e_1 - e_2$. □

5. Proof of Theorem 3.2

The present section is devoted to the proof of Theorem 3.2. The first lemma (Lemma 5.1) shows that the vertices of $\mathcal{S}(a,b)$ defined in Section 3.1 are really related to the self-replicating translation set Γ_{srs}, *i.e.*, that $\mathcal{S}(a,b)$ satisfies Proposition 2.3. Its proof is purely algebraic. Afterwards, in Lemma 5.2, we characterise the strongly connected component of the self-replicating boundary graph. The proof uses alternatively algebraic and algorithmic methods. Finally we will use Theorem 4.1 to prove Theorem 3.2.

Lemma 5.1. *For each vertex $[i, \pi(\mathbf{x}), j]$ of $\mathcal{S}(a,b)$, $[\pi(\mathbf{x}), j] \in \Gamma_{srs}$.*

Proof. By definition, $[\pi(\mathbf{x}), j] \in \Gamma_{srs}$ if and only if $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_j$. Recall that $\mathbf{v}_\beta = (v_1, v_2, v_3) = (1, \beta - a, \beta^2 - a\beta - b)$.

For $\mathbf{x} = \mathbf{0}$ the statement is trivial. If $\mathbf{x} \neq \mathbf{0}$ we consider six cases.

Case 1. $\mathbf{x} = (0,1,0)$: Using (2.1) we see

$$0 \leq \langle (0,1,0), \mathbf{v}_\beta \rangle = v_2 < v_1.$$

Thus $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 2. $\mathbf{x} = (1,-1,0)$: Again, (2.1) immediately yields

$$0 \leq \langle (1,-1,0), \mathbf{v}_\beta \rangle = v_1 - v_2 < v_1,$$

hence $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 3. $\mathbf{x} = (t,-t,t)$ $(1 \leq t \leq m(a,b))$: We have $\langle (t,-t,t), \mathbf{v}_\beta \rangle = tc$ with

$$c := \langle (1,-1,1), \mathbf{v}_\beta \rangle = 1 - (\beta^{-2} + b\beta^{-1}) + \beta^{-1} = \frac{a-b+2}{\beta+1}.$$

Hence,

$$\langle (t,-t,t), \mathbf{v}_\beta \rangle = tc = t\frac{a-b+2}{\beta+1} \geq 0$$

for each $t \geq 0$. On the other hand, if $\frac{a}{a-b+2} \geq 1$, we have $t \leq \frac{a}{a-b+2}$. Then

$$\langle (t,-t,t), \mathbf{v}_\beta \rangle \leq \frac{a}{a-b+2}c = \frac{a}{\beta+1} < 1 = v_1$$

since $a < \beta$. If $\frac{a}{a-b+2} < 1$ (and therefore $t = 1$) we have $b < 2$ and, hence, $b = 1$. Thus

$$\langle (t,-t,t), \mathbf{v}_\beta \rangle = c = \frac{a-b+2}{\beta+1} = \frac{a+1}{\beta+1} < 1.$$

In both cases, $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 4. $\mathbf{x} = (2-t, t-1, -t)$ **(**$1 \leq t \leq m(a,b)$**):**

$$\langle (2-t, t-1, -t), \mathbf{v}_\beta \rangle = 1 - (t-1)\frac{a-b+2}{\beta+1} - \beta^{-1} < 1.$$

On the other hand,

$$\begin{aligned}\langle (2-t, t-1, -t), \mathbf{v}_\beta \rangle =& 2 - b\beta^{-1} - \beta^{-2} - t\frac{a-b+2}{\beta+1} \\ =& \underbrace{(1 - b\beta^{-1} - \beta^{-2})}_{>0} + \underbrace{(1 - t\frac{a-b+2}{\beta+1})}_{>0} > 0.\end{aligned}$$

Thus $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 5. $\mathbf{x} = (t-1, 1-t, t)$ **(**$1 \leq t \leq m(a,b)$**):** We use the previous case to estimate

$$\begin{aligned}0 < v_3 = \langle (0,0,1), \mathbf{v}_\beta \rangle \leq& \underbrace{\langle (t-1, 1-t, t-1), \mathbf{v}_\beta \rangle}_{\geq 0 \text{ by Case 3}} + \langle (0,0,1), \mathbf{v}_\beta \rangle = \langle (t-1, 1-t, t), \mathbf{v}_\beta \rangle \\ =& -1 + \langle (t,-t,t), \mathbf{v}_\beta \rangle + \langle (0,1,0), \mathbf{v}_\beta \rangle < \langle (0,1,0), \mathbf{v}_\beta \rangle = v_2.\end{aligned} \tag{5.1}$$

Hence $[\pi(\mathbf{x}), 1], [\pi(\mathbf{x}), 2] \in \Gamma_{srs}$.

Case 6. $\mathbf{x} = (1-t, t, -t)$ **(**$1 \leq t \leq m(a,b)$**):** By Case 3,

$$\langle (1-t, t, -t), \mathbf{v}_\beta \rangle = 1 - \langle (t,-t,t), \mathbf{v}_\beta \rangle > 0.$$

Lower estimation yields

$$\begin{aligned}\langle (1-t, t, -t), \mathbf{v}_\beta \rangle =& -(t-1)\langle (1,-1,1), \mathbf{v}_\beta \rangle + \langle (0,1,0), \mathbf{v}_\beta \rangle - \langle (0,0,1), \mathbf{v}_\beta \rangle \\ <& \langle (0,1,0), \mathbf{v}_\beta \rangle = v_2.\end{aligned}$$

This shows that $[\pi(\mathbf{x}), 1], [\pi(\mathbf{x}), 2] \in \Gamma_{srs}$.

□

Later we will use the above lemma to prove that $\mathcal{S}(a,b)$ is contained in $\mathcal{G}_{srs}^{(B)}$.

Showing the reverse inclusion will be more difficult. In the following lemma we start with the strongly connected components. One can check directly from the definition that the graph $\mathcal{S}(a,b)$

contains $m(a,b)$ strongly connected components. We will call their sets of vertices $C_{a,b}(1),\dots,C_{a,b}(m(a,b))$. In particular, we have

$$\begin{aligned}C_{a,b}(1) := \{&[1,\pi(0,1,-1),1],[1,\pi(0,1,-1),2],[3,\pi(0,1,-1),2],[2,\pi(1,0,-1),1],\\ &[3,\pi(1,0,-1),1],[2,\pi(1,-1,0),1],[2,\pi(1,-1,1),1]\}\\ &\cup\begin{cases}\{[1,\pi(0,1,0),1],[1,\pi(1,0,-1),1]\} & \text{if } a\neq b\\ \emptyset & \text{otherwise}\end{cases}\\ &\cup\begin{cases}\{[1,\pi(1,-1,1),1]\} & \text{if } b\geq 2\\ \emptyset & \text{otherwise}\end{cases}.\end{aligned}$$

and, for $t\in\{2,\dots,m(a,b)\}$,

$$C_{a,b}(t) = \{[2,\pi(t,-t,t),1]\}\cup\begin{cases}\{[1,\pi(t,-t,t),1]\} & \text{if } a>t(a-b+2)\\ \emptyset & \text{otherwise}\end{cases}.$$

The following lemma shows that the vertices of the strongly connected components of $\mathcal{G}^{(B)}_{srs}$ are contained in the sets $C_{a,b}(t)$. Actually, this forms the core of our further argumentation. The proof makes alternatively use of algebraic estimations and combinatorial algorithms.

Lemma 5.2. *The vertices of the strongly connected components of the self-replicating boundary graph $\mathcal{G}^{(B)}_{srs}$ are contained in $\bigcup_{t=1}^{m(a,b)} C_{a,b}(t)$.*

Proof. The vertices of the strongly connected components are exactly those vertices that are contained in cycles. Therefore, consider a cycle of the self-replicating boundary graph passing the vertices $[i_n,\pi(x_n,y_n,z_n),j_n]$, $n\in\{0,\dots,q-1\}$. By Proposition 2.3, for every $n\in\{0,\dots,q-1\}$ we have $[\pi(x_n,y_n,z_n),j_n]\in\Gamma_{srs}$ and, by definition, $[i_n,\pi(x_n,y_n,z_n),j_n]$ and $[i_{n+1},\pi(x_{n+1},y_{n+1},z_{n+1}),j_{n+1}]$ (indices modulo q) satisfy 2. of Definition 2.1. Let

$$t := \max_{n\in\{0,\dots,q-1\}} \|(x_n,y_n,z_n)\|_\infty .$$

1. At first suppose that $t\geq 2$. We start with proving that $t\leq m(a,b)$ and deduce that for all $n\in\{0,\dots,q-1\}$ we have $(x_n,y_n,z_n)=(t,-t,t)$. We first claim that $x_n\neq -t$ for all $n\in\{0,\dots,q-1\}$. Suppose x_n equals $-t$. Then, by the fact that $[i_n,\pi(x_n,y_n,z_n),j_n]\in\mathcal{D}$, we have
$$\langle(x_n,y_n,z_n),\mathbf{v}_\beta\rangle = -t+y_n(b\beta^{-1}+\beta^{-2})+z_n\beta^{-1}\geq 0.$$
Since $|y_n|\leq t$, $|z_n|\leq t$ and $t\geq 2$ we necessarily have that y_n and z_n are positive and at least one of them is strictly greater than 1. Furthermore, we have
$$\begin{aligned}y_n(b\beta^{-1}+\beta^{-2})+&z_n\beta^{-1}\geq t \Rightarrow\\ &y_n(a\beta^{-1}+b\beta^{-2}+\beta^{-3})+z_n(b\beta^{-1}+\beta^{-2})\geq t\beta^{-1}+y_na\beta^{-1}+z_nb\beta^{-1}.\end{aligned}$$
Using Lemma 4.2 we can estimate
$$\begin{aligned}|\langle(x_{n+1},y_{n+1},z_{n+1}),\mathbf{v}_\beta\rangle| =&|\langle(y_n,z_n,z_{n+1}),\mathbf{v}_\beta\rangle|\\ \geq&\left|y_n(a\beta^{-1}+b\beta^{-2}+\beta^{-3})+z_n(b\beta^{-1}+\beta^{-2})\right|-\left|z_{n+1}\beta^{-1}\right|\\ \geq& t\beta^{-1}+y_na\beta^{-1}+z_nb\beta^{-1}-t\beta^{-1}=y_na\beta^{-1}+z_nb\beta^{-1}>1,\end{aligned}$$
which contradicts the assumption that $[\pi(x_{n+1},y_{n+1},z_{n+1}),j_{n+1}]\in\Gamma_{srs}$. Therefore, for all $n\in\{0,\dots,q-1\}$, $x_n\neq -t$.

   By this consideration and Lemma 4.2, we may assume without loss of generality that $x_0=t$. We have
$$\langle(x_0,y_0,z_0),\mathbf{v}_\beta\rangle = t\underbrace{(a\beta^{-1}+b\beta^{-2}+\beta^{-3})}_{=1}+y_0(b\beta^{-1}+\beta^{-2})+z_0\beta^{-1}. \tag{5.2}$$

By the definition of t, it is clear that we have $-t \leq y_0, z_0 \leq t$. Suppose that $-t+1 \leq y_0 \leq t$. We will derive a contradiction. There are two cases.

Case 1. $0 \leq y_0 \leq t$**:** In this case (5.2) reduces to

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq t(a-1)\beta^{-1} + tb\beta^{-2} + t\beta^{-3}.$$

Our assumption that $t \geq 2$ implies that, if $a > 1$, we have $t(a-1) \geq a$. Therefore

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq a\beta^{-1} + tb\beta^{-2} + t\beta^{-3} > 1.$$

If $a = 1$ then $b = 1$, which is the classical Tribonacci case, it is easy to verify that

$$t(\beta^{-2} + \beta^{-3}) \geq 2(\beta^{-2} + \beta^{-3}) > 1,$$

too. In both cases, this contradicts the fact that $[\pi(x_0, y_0, z_0), j_0] \in \Gamma_{srs}$.

Case 2. $-t+1 \leq y_0 \leq -1$**:** Here (5.2) gives

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq 1 + (t-1)\left((a-b-1)\beta^{-1} + (b-1)\beta^{-2} + \beta^{-3}\right) + (z_0 + t - 1)\beta^{-1}.$$

If $a > b+1$ this expression is again greater than 1 since $z_0 \geq -t$. Also for $a = b+1$ and $z_0 \geq -t+1$ as well as for $a = b$ and $z_0 \geq 0$ we have $\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle > 1$. Thus we have two subcases left.

- Suppose $a = b+1$ and $z_0 = -t$. Whenever the edge from $[i_0, \pi(x_0, y_0, z_0), j_0]$ to $[i_1, \pi(x_1, y_1, z_1), j_1]$ is of Type 1, the significant difference is

$$ax_1 + by_1 + z_1 - x_0 \geq -a.$$

Using (4.4) we obtain

$$z_1 \geq -ax_1 - by_1 + x_0 - a = -ay_0 - bz_0 + x_0 - a \geq a + bt + t - a > t.$$

Similarly, if the edge is of Type 2, we deduce $z_1 < -t$. In both cases this contradicts the definition of t.

- The case $a = b$ and $-t \leq z_0 \leq -1$ is treated analogously.

Therefore the only possibility is $y_0 = -t$. By Lemma 4.2 this implies that $x_1 = \pm t$. Thus, by the beginning of this proof, $x_1 = t$. We can prove in a similar way that $y_1 = -t$. Now it follows from Lemma 4.2 that the edge is necessarily of Type 2 and, hence, $z_0 = t$. We infer that $(x_n, y_n, z_n) = (t, -t, t)$ for all $n \in \{0, \dots, q-1\}$.

Since all the edges are of Type 2, the significant difference is $at - bt + t + t = t(a-b+2) \leq a$ by (4.5). This yields $t \leq \frac{a}{a-b+2}$ and, since t is an integer, $t = \left\lfloor \frac{a}{a-b+2} \right\rfloor = m(a,b)$, as it was claimed.

Up to now we have proved that, for a cycle $([i_n, \pi(x_n, y_n, z_n), j_n])_{n \in \{0,\dots,q-1\}}$ in $\mathcal{G}_{srs}^{(B)}$, if we set $t := \max_{n \in \{0,\dots,q-1\}} \|(x_n, y_n, z_n)\|_\infty$ and suppose $t \geq 2$, then $t \leq m(a,b)$ and for all n, $(x_n, y_n, z_n) = (t, -t, t)$.

To determine the exact set of vertices we use Lemma 4.5. More precisely, for an edge of Type 2 to exist from $[i, \pi(t, -t, t), j]$ to $[i', \pi(t, -t, t), j']$, the list in row (j, i) and column (i', j') in Table 8 must necessarily contain $t(a-b+2)$. Thus, we search in Table 8 for values between $2(a-b+2)$ $(t = 2)$ and a.

The cells of Table 8 contain all such pairs $(i, j) \to (i', j')$. Each cell represents a possible edge. Note that several edges are possible only for special conditions on a, b and t - these cells are marked with $*$ (for example, $(1,1) \to (1,1)$ is possible only for $t(a-b+2) < a$). We do not have to take care of edges $(2,3) \to (2,1)$ and $(1,3) \to (2,1)$, respectively, since $2(a-b+2) > a-b > a-b-1$.

$(1,1) \to (1,1)*$	$(2,1) \to (1,1)$	$(1,1) \to (1,2)*$	$(3,1) \to (1,2)*$	$(1,1) \to (2,1)*$	$(2,1) \to (2,1)$
$(1,1) \to (2,2)*$	$(3,1) \to (2,2)*$	$(1,1) \to (3,1)*$	$(2,1) \to (3,1)*$	$(1,1) \to (3,2)*$	$(3,1) \to (3,2)*$

Table 9

Table 9 induces a graph with 6 vertices and 12 edges. We want to determine its strongly connected components since the vertices of our cycle are necessarily contained in them. This can be done algorithmically. Since each vertex of a strongly connected component necessarily has incoming as well as outgoing edges, we will successively remove vertices from our graph that do not have incoming or outgoing edges with respect to Table 9 (where we do not consider the $*$-tags for the moment). The complete process can be found in the Annex.

As a result of this algorithm we obtain the two vertices

$$[1, \pi(t, -t, t), 1] \text{ and } [2, \pi(t, -t, t), 1].$$

The respective edges in Table 9 are highlighted in grey. Two of these edges have a $*$-tag and we note that these edges are not possible for $t(a - b + 2) = a$. In this case, the vertex $[1, \pi(t, -t, t), 1]$ has no outgoing edge and thus cannot be a vertex of the strongly connected component. Therefore, we have shown that the vertices of our cycle are contained in $C_{a,b}(t)$ with $2 \leq t \leq \frac{a}{a-b+2}$.

2. We now treat the case of the possible strongly connected components whose vertices $[i, \pi(\mathbf{x}), j]$ satisfy $\|\mathbf{x}\|_\infty \leq 1$. There are 27 $\mathbb{Z}^3$-vectors whose maximum norm is less or equal to 1. For these vertices to belong to the self-replicating boundary graph we must have $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < 1$. Thus, we can restrict to

$$\mathbf{x} \in \{(0,0,0), (0,0,1), (0,1,0), (0,1,-1), (1,0,-1), (1,-1,0), (1,-1,1), (0,1,1), (-1,1,1)\}.$$

The vectors $(0, 1, 1)$ and $(-1, 1, 1)$ can be excluded. Their possible successors in a strongly connected component would have the form $[i, \pi(1, 1, z'), j]$ with $z' \geq -1$. But then

$$\langle (1, 1, z'), \mathbf{v}_\beta \rangle \geq 1 + v_2 - v_3 > 1,$$

thus $[\pi(1, 1, z'), j]$ is not in the set Γ_{srs}, a contradiction.

The point $\mathbf{x} = (0, 0, 0)$ does not give rise to a strongly connected component for the following reason. By Lemma 4.2, the only possible predecessor for a vertex of the form $[i', \mathbf{0}, j']$ is of the shape $[i, \mathbf{0}, j]$. By Lemma 4.5 the list in row (i, j) (row (j, i), respectively) and column (i', j') in Table 8 must contain 0. Since by definition $i \neq j$ and $i' \neq j'$ we easily see that such edges may only occur for $(i, j) = (1, 3)$ and $(i', j') = (1, 2)$ and for $(i, j) = (2, 3)$ and $(i', j') = (1, 2)$. This leads to $[1, \mathbf{0}, 3] \to [1, \mathbf{0}, 2]$ and $[2, \mathbf{0}, 3] \to [1, \mathbf{0}, 2]$ as the only possible edges in the strongly connected component. But combining these two edges does not give rise to a component. Therefore, $\mathbf{x} = (0, 0, 0)$ cannot show up in a strongly connected component.

Since $\mathbf{x} = (0, 0, 0)$ does not induce vertices of a strongly connected component we can exclude $\mathbf{x} = (0, 0, 1)$ completely (again, by Lemma 4.2).

By Lemma 4.2, the remaining five possibilities for $\mathbf{x}$ provide eight different shapes of edges shown in Table 10.

Similarly as before we use Table 8 and write down all possible edges. Table 11 consists of eight blocks corresponding to the eight shapes of edges, analogously to Table 9. The cells of a block contain all pairs $(i, j) \to (i', j')$ with suitable significant difference found in Table 8, according to Lemma 4.5. Again we mark a cell with $*$ when special conditions on a and b are needed (for example, in Block 4 all cells are tagged, since for $a = b$ the corresponding significant difference $b + 1 = a + 1$ does not appear in any list in Table 8). Note that $[\pi(0, 1, 0), 2], [\pi(0, 1, 0), 3] \notin \Gamma_{srs}$. Hence, we do not write pairs $(i, j) \to (i', j')$ involving these elements.

Shape	from	to	Type	significant difference
1	$[i,\pi(0,1,0),j]$	$[i',\pi(1,0,-1),j']$	Type 1	$a-1$
2	$[i,\pi(0,1,-1),j]$	$[i',\pi(1,-1,0),j']$	Type 1	$a-b$
3	$[i,\pi(0,1,-1),j]$	$[i',\pi(1,-1,1),j']$	Type 1	$a-b+1$
4	$[i,\pi(1,0,-1),j]$	$[i',\pi(0,1,0),j']$	Type 2	$b+1$
5	$[i,\pi(1,0,-1),j]$	$[i',\pi(0,1,-1),j']$	Type 2	b
6	$[i,\pi(1,-1,0),j]$	$[i',\pi(1,0,-1),j']$	Type 2	a
7	$[i,\pi(1,-1,1),j]$	$[i',\pi(1,-1,0),j']$	Type 2	$a-b+1$
8	$[i,\pi(1,-1,1),j]$	$[i',\pi(1,-1,1),j']$	Type 2	$a-b+2$

Table 10

Block 1		$[i,\pi(0,1,0),j] \longrightarrow [i',\pi(1,0,-1),j']$			
$(1,1)\to(1,1)$	$(1,1)\to(1,2)*$	$(1,1)\to(1,3)*$	$(1,1)\to(2,1)$	$(1,1)\to(2,2)*$	$(1,1)\to(2,3)*$
$(1,1)\to(3,1)$	$(1,1)\to(3,2)*$	$(1,1)\to(3,3)*$			
Block 2		$[i,\pi(0,1,-1),j] \longrightarrow [i',\pi(1,-1,0),j']$			
$(1,1)\to(1,1)$	$(1,2)\to(1,1)*$	$(2,2)\to(1,1)*$	$(1,1)\to(1,2)*$	$(1,3)\to(1,2)*$	$(2,3)\to(1,2)$
$(1,1)\to(1,3)*$	$(1,1)\to(2,1)$	$(3,2)\to(2,1)$	$(1,1)\to(2,2)*$	$(1,3)\to(2,2)*$	$(3,3)\to(2,2)*$
$(1,1)\to(2,3)*$	$(1,1)\to(3,1)$	$(1,1)\to(3,2)*$	$(1,3)\to(3,2)*$	$(1,1)\to(3,3)*$	
Block 3		$[i,\pi(0,1,-1),j] \longrightarrow [i',\pi(1,-1,1),j']$			
$(1,1)\to(1,1)*$	$(1,2)\to(1,1)$	$(1,1)\to(1,2)*$	$(1,3)\to(1,2)*$	$(1,1)\to(2,1)*$	$(1,2)\to(2,1)$
$(1,1)\to(2,2)*$	$(1,3)\to(2,2)*$	$(1,1)\to(3,1)*$	$(1,2)\to(3,1)*$	$(1,1)\to(3,2)*$	$(1,3)\to(3,2)*$
Block 4		$[i,\pi(1,0,-1),j] \longrightarrow [i',\pi(0,1,0),j']$			
$(1,1)\to(1,1)*$	$(2,1)\to(1,1)*$	$(1,1)\to(2,1)*$	$(1,3)\to(2,1)*$	$(2,1)\to(2,1)*$	$(2,3)\to(2,1)*$
$(1,1)\to(3,1)*$	$(2,1)\to(3,1)*$				
Block 5		$[i,\pi(1,0,-1),j] \longrightarrow [i',\pi(0,1,-1),j']$			
$(1,1)\to(1,1)*$	$(2,1)\to(1,1)$	$(3,1)\to(1,2)$	$(1,1)\to(2,1)*$	$(1,3)\to(2,1)*$	$(2,1)\to(2,1)*$
$(2,3)\to(2,1)*$	$(3,1)\to(2,2)$	$(1,1)\to(3,1)*$	$(2,1)\to(3,1)*$	$(3,1)\to(3,2)$	
Block 6		$[i,\pi(1,-1,0),j] \longrightarrow [i',\pi(1,0,-1),j']$			
$(2,1)\to(1,1)$	$(3,1)\to(1,2)*$	$(2,1)\to(2,1)$	$(3,1)\to(2,2)*$	$(2,1)\to(3,1)$	$(3,1)\to(3,2)*$
Block 7		$[i,\pi(1,-1,1),j] \longrightarrow [i',\pi(1,-1,0),j']$			
$(1,1)\to(1,1)*$	$(2,1)\to(1,1)$	$(1,1)\to(1,2)*$	$(3,1)\to(1,2)*$	$(1,1)\to(2,1)*$	$(2,1)\to(2,1)$
$(1,1)\to(2,2)*$	$(3,1)\to(2,2)*$	$(1,1)\to(3,1)*$	$(2,1)\to(3,1)*$	$(1,1)\to(3,2)*$	$(3,1)\to(3,2)*$
Block 8		$[i,\pi(1,-1,1),j] \longrightarrow [i',\pi(1,-1,1),j']$			
$(1,1)\to(1,1)*$	$(2,1)\to(1,1)*$	$(1,1)\to(1,2)*$	$(3,1)\to(1,2)*$	$(1,1)\to(2,1)*$	$(2,1)\to(2,1)*$
$(1,1)\to(2,2)*$	$(3,1)\to(2,2)*$	$(1,1)\to(3,1)*$	$(2,1)\to(3,1)*$	$(1,1)\to(3,2)*$	$(3,1)\to(3,2)*$

Table 11

Table 11 induces a graph with 36 vertices. Again, we determine the strongly connected component algorithmically by successively removing vertices that have no incoming or outgoing edges with respect to Table 11 (temporarily we ignore the $*$- tags). This is carefully executed in the Annex.

As a result of this algorithm, we obtain the following vertices:

$$[1,\pi(0,1,0),1], [1,\pi(0,1,-1),1], [1,\pi(0,1,-1),2], [3,\pi(0,1,-1),2], [1,\pi(1,0,-1),1],$$
$$[2,\pi(1,0,-1),1], [3,\pi(1,0,-1),1], [2,\pi(1,-1,0),1], [1,\pi(1,-1,1),1], [2,\pi(1,-1,1),1].$$

The corresponding edges in Table 11 are highlighted in grey.

We note that these vertices match with the vertices of $C_{a,b}(1)$ for $a > b > 1$. Hence, in this case the lemma is proved.

In the case $b = 1$, no type of edge that start in $[1,\pi(1,-1,1),1]$ (shapes 7 and 8) exists (observe the $*$-tags). Thus, the vertex $[1,\pi(1,-1,1),1]$ cannot be contained in the strongly connected components. This shows the lemma in the case $a > b = 1$.

Finally, suppose $a = b$ and consider the $*$-tags in Block 4: edges of Shape 4 do not appear in this case. Now notice that edges of this shape are the only incoming edges of the vertex $[1,\pi(0,1,0),1]$. Hence, we deduce that $[1,\pi(0,1,0),1]$ cannot be a vertex of the strongly connected components in this case. The same applies to the vertex $[1,\pi(1,0,-1),1]$ since it has no outgoing edge ($*$-tags in Block 4 and Block 5). Thus, the lemma holds also for all $a = b \geq 1$.

□

Proof of Theorem 3.2. From Lemma 5.1 we know that for all vertices $[i, \gamma, j]$ of $\mathcal{S}(a, b)$ we have $[\gamma, j] \in \Gamma_{srs}$. It is also easy to verify that all edges satisfy Item 2. of Theorem 4.1. Furthermore, each vertex lies on a path ending in a strongly connected component. Therefore, $\mathcal{S}(a, b)$ is a subgraph of $\mathcal{G}_{srs}^{(B)}$.

Now we are going to check that $\mathcal{G}_{srs}^{(B)}$ is a subgraph of $\mathcal{S}(a, b)$. In Lemma 5.2 we showed that the vertices of the strongly connected components of $\mathcal{G}_{srs}^{(B)}$ are contained in the sets $C_{a,b}(1), \ldots, C_{a,b}(m(a, b))$. Observe that $\mathcal{S}(a, b)$ contains all these vertices. The fact that every vertex in $\mathcal{G}_{srs}^{(B)}$ lies on a path that ends up in a strongly connected component immediately implies that such a path passes a vertex of $\mathcal{S}(a, b)$.

Claim. *No more edges (nor vertices) satisfying Items 1. and 2. of Theorem 4.1 can be added to* $\mathcal{S}(a, b)$.

Proving this claim will obviously prove the theorem. The proof of the claim runs along the following algorithm. We go through all types of vertices of $\mathcal{S}(a, b)$ and investigate the possible incoming edges:

- we use Lemma 4.2 to show that a predecessor $[i, \gamma, j]$ of a vertex $[i', \gamma', j']$ can have at most two different values for γ;
- we use Lemma 4.5 to determine i and j.

The computations are carried out in the Annex.

□

6. Proof of Theorem 3.4 and Theorem 3.5

The proof of Theorem 3.4 runs along the same line as the first part of the proof of Theorem 3.2.

Proof of Theorem 3.4. It is quite easy to see that $\mathcal{L}(a, b)$ is a subgraph of $\mathcal{G}_{lat}^{(B)}$. Indeed, all vertices lie on a finite path that ends in a vertex of $\mathcal{G}_{srs}^{(B)}$. These vertices satisfy (2.6). Using a similar argument as in the second part of the proof of Theorem 4.1, one can show that actually each vertex of $\mathcal{L}(a, b)$ satisfies (2.6). □

The lack of statements similar to Proposition 2.3 and Theorem 4.1 leads us to consider an other strategy to prove that $\mathcal{L}(a, b)$ coincides with $\mathcal{G}_{lat}^{(B)}$. The self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ gives us a list of all subtiles that intersect with the Rauzy fractal in the aperiodic tiling. We use this information to construct a tube around the central tile, that isolates the central tile from the remaining part of the tiling. To show that our graph $\mathcal{L}(a.b)$ is exactly the lattice boundary graph, we will prove that the neighbours occurring in $\mathcal{L}(a, b)$ cover the whole tube. As a consequence, the neighbour set cannot be bigger. In the last step we deduce the exact set of vertices and edges. We illustrate this proof on behalf of an example (Example 6.1). Since the computation in the general case seems to be difficult, we will then restrict to the most simple case, assuming that $m(a, b) = 1$.

According to Definition 2.4 and the prefix-suffix graph Γ_σ, the Rauzy fractal $\mathcal{T}$ is the solution of a graph directed function system: for every $i \in \mathcal{A}$,

$$\mathcal{T}(i) = \bigcup_{\sigma(j)=pis} \pi(\mathbf{l}(p)) + \mathbf{h}(\mathcal{T}(j)). \tag{6.1}$$

For convenience, we denote by $\mathcal{B}(i)$ the finite set

$$\mathcal{B}(i) := \{\pi(\mathbf{l}(p)) + \mathbf{h}(\mathcal{T}(j)) | \, \exists (p,i,s) \in \mathcal{P} : \sigma(j) = pis\}. \tag{6.2}$$

Note that each $\mathcal{B}(i)$ is a finite set, in particular, $|\mathcal{B}(1)| = a + b + 1$ and $|\mathcal{B}(2)| = |\mathcal{B}(3)| = 1$ (since $\mathcal{T}(2)$ and $\mathcal{T}(3)$ are, in fact, translated **h**-images of $\mathcal{T}(1)$ and $\mathcal{T}(2)$, respectively).

Now consider a vertex $[i, \gamma, j]$ of the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ with $\gamma \neq \{\mathbf{0}\}$.

By Proposition 2.4 (see also [23, Theorem 5.6]), this is equivalent to the fact that $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma) \neq \emptyset$. Now we may wonder for which $B \in \mathcal{B}(j)$ we have $\mathcal{T}(i) \cap (B + \gamma) \neq \emptyset$. In other words, we want a characterisation of the set

$$\mathcal{O}([i, \gamma, j]) := \{\gamma + B | \, B \in \mathcal{B}(j), \mathcal{T}(i) \cap (B + \gamma) \neq \emptyset\}.$$

Example 6.1. We explain the above definitions and the idea of the proof by considering the substitution $\sigma_{3,2}$, introduced in Example 3.8. Figure 5 shows again the central tile (subdivided into the three subtiles $\mathcal{T}(1)$, $\mathcal{T}(2)$ and $\mathcal{T}(3)$) and its neighbours in the self-replicating tiling (see also Figure 3 left). The neighbours are subdivided with respect to (6.1) (grey boundaries). In particular, we see that $\mathcal{B}(1)$ consists of 6 elements: 3 copies of smaller versions of $\mathcal{T}(1)$, 2 copies of smaller versions of $\mathcal{T}(2)$ and 1 copy of a smaller version of $\mathcal{T}(3)$. $\mathcal{B}(2)$ and $\mathcal{B}(3)$ consist only of a smaller version of $\mathcal{T}(1)$ and a smaller version of $\mathcal{T}(2)$, respectively.

Now we subdivide each neighbour $\gamma + \mathcal{T}(j)$ with respect to (6.1) and consider the subsets that are adjacent to the central tile. These sets are highlighted in light grey and form our tube. In words of our formalism, $\mathcal{O}([i, \gamma, j])$ consists exactly of those (γ-translated) sets of $\mathcal{B}(j)$ that intersect with $\mathcal{T}(i)$. For example, $\mathcal{O}([2, \pi(1, -1, 1), 1])$ consists of three elements: one smaller copy of $\mathcal{T}(1)$ and two smaller copies $\mathcal{T}(2)$. Note that the sets $\mathcal{O}([i, \gamma, j])$ are not disjoint. Indeed, the only element of $\mathcal{O}([1, \pi(1, -1, 1), 1])$ (a smaller copy of $\mathcal{T}(2)$) already appears in $\mathcal{O}([2, \pi(1, -1, 1), 1])$.

We will show that all of this tube is covered by 6 neighbours of the central tile in the lattice tiling. For our example this can be verified easily by looking at Figure 3 (right).

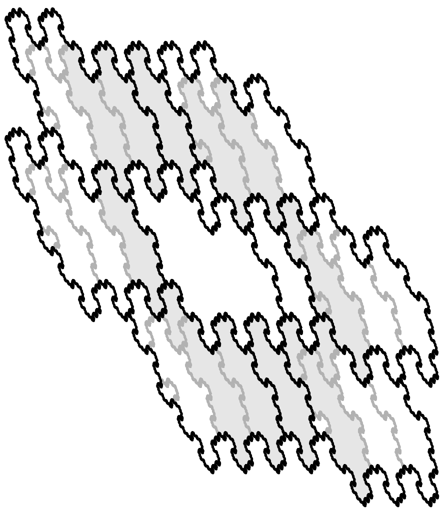


Figure 5: The idea of the proof of Theorem 3.5 on behalf of $\sigma_{3,2}$.

Lemma 6.2. *Let $[i, \gamma, j]$ a vertex of the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. Then the elements of $\mathcal{O}([i, \gamma, j])$ are given by the outgoing edges of $[i, \gamma, j]$. In particular,*

$$\begin{aligned}\mathcal{O}([i, \gamma, j]) = &\{\eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) | \, [i, \gamma, j] \xrightarrow{\eta} [i', \gamma', j'] \text{ is an edge of Type 1}\} \cup \\ &\{\eta' + \mathbf{h}(\mathcal{T}(i')) | \, [i, \gamma, j] \xrightarrow{\eta} [i', \gamma', j'] \text{ is an edge of Type 2}\}\end{aligned}$$

Proof. At first we show that each element of $\mathcal{O}([i,\gamma,j])$ is contained in the set on the right hand side of this equality. Let $C = \gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \in \mathcal{O}([i,\gamma,j])$. Thus $\mathcal{T}(i) \cap C \neq \emptyset$. The use of (6.1) yields

$$\bigcup_{B\in\mathcal{B}(i)} B \cap (\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset.$$

Now, there must be at least one $B = \pi(l(p_1)) + \mathbf{h}(\mathcal{T}(i')) \in \mathcal{B}(i)$ that satisfies the equation.

$$\pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(i')) \cap \gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \neq \emptyset.$$

Suppose that $\gamma = \pi(\mathbf{x})$. Hence,

$$\mathcal{T}(i') \cap (\mathbf{h}^{-1}(\pi(\mathbf{x}) + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))) + \mathcal{T}(j')) \neq \emptyset. \tag{6.3}$$

By (2.2) we have

$$\mathcal{T}(i') \cap (\pi(\mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1))) + \mathcal{T}(j')) \neq \emptyset.$$

Now note that $\langle \mathbf{x}, \mathbf{v}_\beta\rangle < v_j = \langle \mathbf{l}(j), \mathbf{v}_\beta\rangle$. Since

$$\beta \langle \mathbf{l}(j'), \mathbf{v}_\beta\rangle = \langle \mathbf{M}\mathbf{l}(j'), \mathbf{v}_\beta\rangle = \langle \mathbf{l}(\sigma(j')), \mathbf{v}_\beta\rangle \geq \langle \mathbf{l}(j) + \mathbf{l}(p_2), \mathbf{v}_\beta\rangle$$

we immediately see that $\langle \mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1)), \mathbf{v}_\beta\rangle < v_{j'}$. Analogously, we can show that $\langle \mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1)), \mathbf{v}_\beta\rangle > -v_{i'}$.

However, we either have $[i', \mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), j'] \in \mathcal{D}$ and $[\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), j'] \in \Gamma_{srs}$ or $[j', -\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), i'] \in \mathcal{D}$ and $[-\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), i'] \in \Gamma_{srs}$. Remember that for a vertex $[i,\gamma,j]$ of the boundary graph, we necessarily have $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma) \neq \emptyset$. Together with (6.3), this leads to the conclusion that one of the triples occurs as vertices in $\mathcal{G}^{(B)}_{srs}$ and we see that it has an incoming edge from $[i,\gamma,j]$. In the first case it is of Type 1 and labelled by $\pi(\mathbf{l}(p_1))$, in the second case it is of Type 2 and labelled by $\pi(\mathbf{l}(p_2)) + \gamma$. However, we see that

$$\begin{aligned} C \in &\{\eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) \,|\, [i,\gamma,j] \xrightarrow{\eta} [i',\gamma',j'] \text{ is an edge of Type 1}\} \cup \\ &\{\eta' + \mathbf{h}(\mathcal{T}(i)') \,|\, [i,\gamma,j] \xrightarrow{\eta} [i',\gamma',j'] \text{ is an edge of Type 2}\}. \end{aligned}$$

To prove the reverse inclusion, consider an edge $[i,\gamma,j] \xrightarrow{\eta} [i',\gamma',j']$. We have $\mathcal{T}(i') \cap (\gamma' + \mathcal{T}(j')) \neq \emptyset$. Suppose the edge is of Type 1. Then

$$\mathbf{h}(\mathcal{T}(i')) \cap (\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset$$

by the definition of the self replicating boundary graph $\mathcal{G}^{(B)}_{srs}$. Furthermore, we have $\eta = \pi(\mathbf{l}(p_1))$, $\pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(i')) \in \mathcal{B}(i)$ and $\pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \in \mathcal{B}(j)$. Thus

$$\mathcal{T}(i) \cap (\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset$$

and, hence,

$$\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) = \eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) \in \mathcal{O}([i,\gamma,j]).$$

For edges of Type 2 the proof runs analogously. □

Now, for each vertex $[i,\gamma,j]$ we consider the set $\mathcal{O}([i,\gamma,j])$. This set consists of the γ-translates of all subsets of $\mathcal{T}(j)$ induced by the decomposition (6.2) that intersect with $\mathcal{T}(i)$. The union of the elements of $\mathcal{O}([i,\gamma,j])$ for all vertices of $\mathcal{G}^{(B)}_{srs}$ with $\gamma \neq \mathbf{0}$ gives the mentioned tube. In Lemma 6.4 we show that in the lattice tiling the neighbours of the Rauzy fractal induced by the elements of $\{\pm\pi(0,1,-1), \pm\pi(1,0,-1), \pm\pi(1,-1,0)\}$ cover all of this tube. Lemma 6.3 is a preparation to Lemma 6.4.

Lemma 6.3. *For all $a \ge b \ge 1$ we have*

$$\pi(0,1,0) + \mathcal{T}(2) \subset \mathcal{T}(1).$$

Proof. By (6.1) for $i = 2$ we have $\mathcal{T}(2) = \pi(a,0,0) + \mathbf{h}(\mathcal{T}(1))$. Hence, by (2.2) and the shape of $\mathbf{M}$,

$$\pi(0,1,0) + \mathcal{T}(2) = \pi(\mathbf{M}(1,0,0)) + \mathbf{h}(\mathcal{T}(1)) = \mathbf{h}(\pi(1,0,0) + \mathcal{T}(1)). \tag{6.4}$$

Set

$$R := \bigcup_{k=0}^{a-1} (\pi(k,0,0) + \mathbf{h}(\mathcal{T}(1))) \cup \bigcup_{k=0}^{b-1} (\pi(k,0,0) + \mathbf{h}(\mathcal{T}(2))).$$

Now we use again (6.1) to obtain

$$\begin{aligned}
\pi(0,1,0) + \mathcal{T}(2) =& \mathbf{h}(\pi(1,0,0) + \mathcal{T}(1)) \\
=& \mathbf{h}(\pi(1,0,0) + R) \cup \mathbf{h}(\pi(1,0,0) + \mathbf{h}(\mathcal{T}(3))) \\
=& \mathbf{h}(\pi(1,0,0) + R) \cup \left[\mathbf{h}\left(\pi(1,0,0) + \mathbf{h}(\pi(b,0,0)) + \mathbf{h}^2(\pi(a,0,0)) + \mathbf{h}^3(\mathcal{T}(1))\right)\right].
\end{aligned}$$

Now observe that

$$\pi(1,0,0) + \mathbf{h}(\pi(b,0,0)) + \mathbf{h}^2(\pi(a,0,0)) = \pi((\mathbf{I}_3 + b\mathbf{M} + a\mathbf{M}^2)(1,0,0)) = \mathbf{h}^3(\pi(1,0,0)), \tag{6.5}$$

(where $\mathbf{I}_3$ denotes the 3×3 identity matrix) since $x^3 - ax^2 - bx - 1$ is the characteristic (and minimal) polynomial of $\mathbf{M}$. Hence,

$$\pi(0,1,0) + \mathcal{T}(2) = \mathbf{h}(\pi(1,0,0) + R) \cup \mathbf{h}^4(\pi(1,0,0) + \mathcal{T}(1)).$$

Iterating this procedure, we obtain

$$\pi(0,1,0) + \mathcal{T}(2) = \overline{\bigcup_{n=0}^{\infty} \mathbf{h}^{3n+1}(\pi(1,0,0) + R)} \tag{6.6}$$

since $\mathbf{h}$ is a contraction. Now we claim that $\mathbf{h}(\pi(1,0,0) + R) \subset \mathcal{T}(1)$. By definition of R we have

$$\mathbf{h}(\pi(1,0,0) + R) = \bigcup_{k=1}^{a} (\mathbf{h}(\pi(k,0,0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup \bigcup_{k=1}^{b} (\mathbf{h}(\pi(k,0,0)) + \mathbf{h}^2(\mathcal{T}(2))). \tag{6.7}$$

On the other hand, by (6.1), we have

$$\begin{aligned}
\mathcal{T}(1) \supset& \mathbf{h}(\mathcal{T}(1)) \cup \mathbf{h}(\mathcal{T}(2)) \cup \mathbf{h}(\mathcal{T}(3)) \\
=& \bigcup_{k=0}^{a-1} (\mathbf{h}(\pi(k,0,0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup \bigcup_{k=0}^{b-1} (\mathbf{h}(\pi(k,0,0)) + \mathbf{h}^2(\mathcal{T}(2))) \cup (\mathbf{h}(\pi(0,0,0)) + \mathbf{h}^2(\mathcal{T}(3))) \\
&\cup (\mathbf{h}(\pi(a,0,0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup (\mathbf{h}(\pi(b,0,0)) + \mathbf{h}^2(\mathcal{T}(2)))
\end{aligned}$$

which contains the set (6.7) and thus yields the claim. Observing that $\mathcal{T}(1) \supset \mathbf{h}^{3i}(\mathcal{T}(1))$ and that by the claim $\mathbf{h}^{3i+1}(\pi(1,0,0) + R) \subset \mathbf{h}^{3i}(\mathcal{T}(1))$ for all $i \in \mathbb{N}$ we obtain the assertion from (6.6). □

Lemma 6.4. *Let $m(a,b) = 1$. For each $O \subset \mathcal{O}([i,\gamma,j])$ of each vertex $[i,\gamma,j]$ of $\mathcal{G}_{srs}^{(B)}$ with $\gamma \neq \mathbf{0}$ there exists a vertex $[i',\gamma',j']$ of $\mathcal{L}(a,b)$ with $[\gamma',j'] \in \Gamma_{lat} \setminus (\{\mathbf{0}\} \times \mathcal{A})$ such that one of the following conditions hold:*

1. *$i = i'$ and $O \subset \gamma' + \mathcal{T}(j')$;*
2. *$i = j'$ and $O \subset -\gamma' + \mathcal{T}(i')$.*

Proof. We prove the lemma by analysing the vertices of $\mathcal{G}_{srs}^{(B)}$ one by one. Since for every $O \in \mathcal{O}([i,\gamma,j])$ we have $O \subset \gamma + \mathcal{T}(j)$ the lemma holds for all vertices $[i,\gamma,j]$ with $\gamma \in \{\pi(0,1,-1),\pi(1,0,-1),\pi(1,-1,0)\}$. Indeed, these vertices appear in $\mathcal{L}(a,b)$ too. Thus, we have 6 more vertices to investigate.

$[1,\pi(0,0,1),1]$**:** For convenience, define the 3 sets

$$\begin{aligned} C :=& \{\pi(k,0,1)+\mathbf{h}(\mathcal{T}(1)) | \, b-1 \le k \le a-2\},\\ D :=& \{\pi(a-1,0,1)+\mathbf{h}(\mathcal{T}(1))\},\\ E :=& \{\pi(b-1,0,1)+\mathbf{h}(\mathcal{T}(2))\}. \end{aligned}$$

Using Lemma 6.2 we easily obtain that

$$\mathcal{O}([1,\pi(0,0,1),1]) = C \cup D \cup E.$$

We claim that the vertices $[i',\pi(1,0,-1),1]$ for $i' \in \{1,2,3\}$ cover all elements of C, D and E according to (2). Indeed, the triples are vertices of $\mathcal{L}(a,b)$, $[\pi(1,0,-1),1] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ and at the third position we find 1. Furthermore, by (6.1), we have that

$$\begin{aligned} -\pi(1,0,-1)+\mathcal{B}(1) &\supset \{\pi(k,0,1)+\mathbf{h}(\mathcal{T}(1)) | \, b-1 \le k \le a-1\} = C\\ -\pi(1,0,-1)+\mathcal{B}(2) &= \{\pi(a-1,0,1)+\mathbf{h}(\mathcal{T}(1))\} = D\\ -\pi(1,0,-1)+\mathcal{B}(3) &= \{\pi(b-1,0,1)+\mathbf{h}(\mathcal{T}(2))\} = E, \end{aligned}$$

which proves the claim.

$[i,\pi(0,0,1),2]$**:** The vertex with $i=1$ always occurs while $i=2$ only exists for $a=b$. However, we have

$$\mathcal{O}([1,\pi(0,0,1),2]) = \mathcal{O}([2,\pi(0,0,1),2]) = \{\pi(a,0,1)+\mathbf{h}(\mathcal{T}(1))\}.$$

Now (6.1) and Lemma 6.3 yield

$$\pi(a,0,1)+\mathbf{h}(\mathcal{T}(1)) = \pi(0,0,1)+\mathcal{T}(2) \subset \pi(0,-1,1)+\mathcal{T}(1).$$

Since $[1,\pi(0,1,-1),1]$ as well as $[1,\pi(0,1,-1),2]$ occur in $\mathcal{L}(a,b)$ the case is accomplished.

$[1,\pi(0,1,0),1]$**:** Note that $[2,(1,-1,0),1]$ is a vertex of $\mathcal{L}(a,b)$. Also,

$$\mathcal{O}([1,\pi(0,1,0),1]) = \{\pi(a-1,1,0)+\mathbf{h}(\mathcal{T}(1))\} = \{\pi(-1,1,0)+\mathcal{T}(2)\},$$

where we used (6.1). This proves the lemma in this case.

$[1,\pi(1,-1,1),1]$**:** We have for $b \ge 2$

$$\begin{aligned} \mathcal{O}([1,\pi(1,-1,1),1]) =& \{\pi(k,-1,1)+\mathbf{h}(\mathcal{T}(1)) | \, 1 \le k \le b-2\}\\ &\cup \{\pi(k,-1,1)+\mathbf{h}(\mathcal{T}(2)) | \, 1 \le k \le b-1\}. \end{aligned}$$

(there is nothing to prove for $b=1$). Using (6.1) we easily obtain that $\pi(0,-1,1)+\mathcal{T}(1)$ covers all elements of $\mathcal{O}([1,\pi(1,-1,1),1])$. This finishes the case since $[1,\pi(0,1,-1),1]$ is a vertex of $\mathcal{L}(a,b)$.

$[2,\pi(1,-1,1),1]$**:** We have

$$\begin{aligned} \mathcal{O}([2,\pi(1,-1,1),1]) = \{&\pi(b-1,-1,1)+\mathbf{h}(\mathcal{T}(1)),\pi(b-1,-1,1)+\mathbf{h}(\mathcal{T}(2)),\\ &\pi(b,-1,1)+\mathbf{h}(\mathcal{T}(2))\}. \end{aligned}$$

Now observe that $\{\pi(b-1,0,0)+\mathbf{h}(\mathcal{T}(1)),\pi(b-1,0,0)+\mathbf{h}(\mathcal{T}(2))\} \subset \mathcal{B}(1)$ and $\mathcal{B}(3) = \{(\pi(b,0,0)+\mathbf{h}(\mathcal{T}(2))\}$. Hence, the lemma is proved since $[3,\pi(0,1,-1),2]$ and $[1,\pi(0,1,-1),2]$ are vertices of $\mathcal{L}(a,b)$.

□

We are finally able to prove Theorem 3.4. However, we will exactly go through the proof to show which conditions we need.

Proof of Theorem 3.5. We already know from Theorem 3.4 that $\mathcal{L}(a,b)$ is a subgraph of $\mathcal{G}_{lat}^{(B)}$.

Thus, we just have to prove that $\mathcal{L}(a,b)$ contains $\mathcal{G}_{lat}^{(B)}$. At first we show that each vertex $[\tilde{i},\tilde{\gamma},\tilde{j}]$ of $\mathcal{G}_{lat}^{(B)}$ with $[\tilde{\gamma},\tilde{j}] \in \Gamma_{lat} \setminus \{0\} \times \mathcal{A}$ is also a vertex of $\mathcal{L}(a,b)$. Let $\xi \in \tilde{\gamma} + \mathcal{T}(\tilde{j}) \cap \mathcal{T}(\tilde{i})$. The tiles are the closure of their interiors, hence, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of interior points of $\tilde{\gamma} + \mathcal{T}(\tilde{j})$ that converges to ξ. For each $n \in \mathbb{N}$ we can find an $\varepsilon_n > 0$ such that the open ball $K(\xi_n, \varepsilon_n)$ is completely contained in the interior of $\tilde{\gamma} + \mathcal{T}(\tilde{j})$. Since σ has the tiling property, we conclude that none of the ξ_n is contained in $\mathcal{T}(\tilde{i})$.

Now consider the aperiodic tiling induced by σ. We use the covering property to deduce that each of the ξ_n is contained in some translate of the self-replicating tiling. By the local finiteness, there are only finitely many possibilities. Thus, suppose that $\gamma + \mathcal{T}(j)$ (with $[\gamma, j] \in \Gamma_{srs}$) contains ξ_n for infinitely many $n \in \mathbb{N}$. By the above considerations and, again, by the tiling property we conclude that $\gamma \neq \mathbf{0}$. Since $\gamma + \mathcal{T}(j)$ is compact and contains an infinite subsequence of ξ_n, it necessarily includes the limit point ξ, too. Thus, $\gamma + \mathcal{T}(j) \cap \mathcal{T}(\tilde{i}) \neq \emptyset$ which makes $[\hat{i}, \gamma, j]$ a vertex of $\mathcal{G}_{srs}^{(B)}$. Furthermore, as $\gamma + \mathcal{T}(j)$ includes points of the sequence $(\xi_n)_{n\in\mathbb{N}}$, it necessarily intersects with the respective neighbourhoods $K(\xi_n, \epsilon_n)$. This shows that $\operatorname{int}(\gamma + \mathcal{T}(j)) \cap \operatorname{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \neq \emptyset$.

Now divide the subtile $\mathcal{T}(j)$ with respect to (6.1). Then there must be at least one $B \in \mathcal{B}(j)$ such that $\gamma + B$ includes ξ_n for infinitely many $n \in \mathbb{N}$. Similarly as before we have $\operatorname{int}(\gamma + B) \cap \operatorname{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \neq \emptyset$ and $\xi \in \gamma + B$. The latter relation yields $\gamma + B \in \mathcal{O}([\tilde{i}, \gamma, j])$. By Lemma 6.4 there exists a vertex $[i', \gamma', j'] \in \mathcal{L}(a,b)$ with $[\gamma, j'] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ such that $i' = \tilde{i}$ and $\gamma + B \subset \gamma' + \mathcal{T}(j')$ or $j' = \tilde{i}$ and $\gamma + B \subset -\gamma' + \mathcal{T}(i')$. We claim that, in fact, the first relation holds. Indeed, suppose the second relation would hold. Then

$$\operatorname{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \cap \operatorname{int}(-\gamma' + \mathcal{T}(i')) \neq \emptyset$$

and, by the tiling property of σ, $\tilde{\gamma} + \mathcal{T}(\tilde{j}) = -\gamma' + \mathcal{T}(i')$. Hence, $[\tilde{j}, -\tilde{\gamma}, \tilde{i}]$ would be a vertex of $\mathcal{L}(a,b)$. But $[\tilde{i}, \tilde{\gamma}, \tilde{j}]$ is a vertex of that $\mathcal{G}_{lat}^{(B)}$ with $\tilde{\gamma} \neq \{\mathbf{0}\}$. Thus, by definition, $\langle \mathbf{x}, \mathbf{v}_\beta \rangle > 0$ where $\tilde{\gamma} = \pi(\mathbf{x})$. The same consideration apply for $\mathcal{L}(a,b)$ which shows that $[\tilde{j}, -\tilde{\gamma}, \tilde{i}]$ impossibly can be a vertex of $\mathcal{L}(a,b)$. Therefore, the first relation must hold necessarily. Now, the same considerations yield that $[\tilde{i}, \tilde{\gamma}, \tilde{j}]$ is a vertex of $\mathcal{L}(a,b)$.

From the first part of the proof we can deduce that, whenever $[i', \gamma', j']$ with $[\gamma', j'] \in \Gamma_{lat} \setminus \{0\} \times \mathcal{A}$ is a vertex of $\mathcal{G}_{lat}^{(B)}$, it is also a vertex of $\mathcal{L}(a,b)$. Note that these 7 vertices (or only 6 vertices if $a = b < 4$) are also vertices of $\mathcal{G}_{srs}^{(B)}$. By Definition 2.1, $\mathcal{G}_{srs}^{(B)}$ contains all infinite paths starting from one of these vertices. Since $\mathcal{L}(a,b)$ also contains all of these paths, we conclude that $\mathcal{L}(a,b)$ contains $\mathcal{G}_{lat}^{(B)}$. □

7. Comments

We want to say a few words on possible proofs of Conjecture 3.6. For a, b that satisfy $m(a,b) \leq k$ for a given constant k the same strategy seems to work. But it requires additional assertions in the spirit of Lemma 6.3. The following considerations may yield another strategy. The sets $\mathcal{O}([i,\gamma,j])$ induce a neighbourhood of the central tile. By Lemma 6.2 it corresponds to the paths of length 1 of $\mathcal{G}_{srs}^{(B)}$. One may obtain a smaller neighbourhood by considering a refinement of $\mathcal{O}([i,\gamma,j])$. This would lead us to investigate longer paths of $\mathcal{G}_{srs}^{(B)}$. This will involve very lengthy hand calculations. We should rather use a computational implementation.

ANNEX - Execution of the algorithmic proofs

Here we carefully execute the algorithmic parts of our proofs.

7.1. Algorithmic parts of Lemma 5.2.

Table 9 and Table 11, respectively, represent edges between several vertices. We will determine the strongly connected components algorithmically by successively deleting vertices that have no incoming nor outgoing edges.

We start with Table 9. It induces 6 vertices shown in Table 12.

$[1, \pi(t, -t, t), 1]$	$[1, \pi(t, -t, t), 2]$	$[2, \pi(t, -t, t), 1]$	$[2, \pi(t, -t, t), 2]$	$[3, \pi(t, -t, t), 1]$	$[3, \pi(t, -t, t), 2]$

Table 12

We will now repeat the following two steps:

a) find vertices that have no incoming (outgoing, respectively) edge and delete the respective cells in Table 12;
b) delete the cells in Table 9 that correspond to edges that end (start, respectively) in vertices that we deleted in a);

until we cannot find suitable vertices in a) any more.

The application reads as follows[4].

Step 1a) remove $[1, \pi(t, -t, t), 2]$, $[2, \pi(t, -t, t), 2]$ and $[3, \pi(t, -t, t), 2]$ since none of these vertices has an outgoing edge;
Step 1b) remove $(1, 1) \to (1, 2)$, $(3, 1) \to (1, 2)$, $(1, 1) \to (2, 2)$, $(3, 1) \to (2, 2)$, $(1, 1) \to (3, 2)$ and $(3, 1) \to (3, 2)$;
Step 2a) remove $[3, \pi(t, -t, t), 1]$ since the vertex has no outgoing edge;
Step 2b) remove $(1, 1) \to (3, 1)$ and $(2, 1) \to (3, 1)$;

What remains are the vertices and edges that are highlighted in grey in Table 12 and Table 9, respectively.

The edges denoted in Table 11 induce the 36 vertices given in Table 13.

$[1, \pi(0, 1, 0), 1]$	$[2, \pi(0, 1, 0), 1]$	$[3, \pi(0, 1, 0), 1]$			
$[1, \pi(0, 1, -1), 1]$	$[1, \pi(0, 1, -1), 2]$	$[1, \pi(0, 1, -1), 3]$	$[2, \pi(0, 1, -1), 1]$	$[2, \pi(0, 1, -1), 2]$	$[2, \pi(0, 1, -1), 3]$
$[3, \pi(0, 1, -1), 1]$	$[3, \pi(0, 1, -1), 2]$	$[3, \pi(0, 1, -1), 3]$			
$[1, \pi(1, 0, -1), 1]$	$[1, \pi(1, 0, -1), 2]$	$[1, \pi(1, 0, -1), 3]$	$[2, \pi(1, 0, -1), 1]$	$[2, \pi(1, 0, -1), 2]$	$[2, \pi(1, 0, -1), 3]$
$[3, \pi(1, 0, -1), 1]$	$[3, \pi(1, 0, -1), 2]$	$[3, \pi(1, 0, -1), 3]$			
$[1, \pi(1, -1, 0), 1]$	$[1, \pi(1, -1, 0), 2]$	$[1, \pi(1, -1, 0), 3]$	$[2, \pi(1, -1, 0), 1]$	$[2, \pi(1, -1, 0), 2]$	$[2, \pi(1, -1, 0), 3]$
$[3, \pi(1, -1, 0), 1]$	$[3, \pi(1, -1, 0), 2]$	$[3, \pi(1, -1, 0), 3]$			
$[1, \pi(1, -1, 1), 1]$	$[1, \pi(1, -1, 1), 2]$	$[2, \pi(1, -1, 1), 1]$	$[2, \pi(1, -1, 1), 2]$	$[3, \pi(1, -1, 1), 1]$	$[3, \pi(1, -1, 1), 2]$

Table 13

We use the same procedure as before:

a) find vertices that have no incoming (outgoing, respectively) edge and delete the respective cells in Table 13;
b) delete the cells in the respective blocks in Table 11 that correspond to edges that end (start, respectively) in vertices that we deleted in a);

[4] Of course, the order for the removal of vertices is not specified. We present an order that requires a few steps only. However, the algorithm always terminates and necessarily yields the same result but more steps might be necessary.

until we cannot find suitable vertices in a) any more.

Application yields:

- Step 1a) remove $[2, \pi(0,1,0), 1]$ and $[3, \pi(0,1,0), 1]$ since none of these vertices has an outgoing edge;
- Step 1b) remove $(1,1) \to (2,1)$, $(1,3) \to (2,1)$, $(2,1) \to (2,1)$, $(2,3) \to (2,1)$, $(1,1) \to (3,1)$, $(2,1) \to (3,1)$ from Block 4
- Step 2a) remove $[1, \pi(1,-1,0), 1]$, $[1, \pi(1,-1,0), 2]$, $[1, \pi(1,-1,0), 3]$, $[2, \pi(1,-1,0), 2]$, $[2, \pi(1,-1,0), 3]$, $[3, \pi(1,-1,0), 2]$, $[3, \pi(1,-1,0), 3]$ since none of these vertices has an outgoing edge;
- Step 2b) remove $(1,1) \to (1,1)$, $(1,2) \to (1,1)$, $(2,2) \to (1,1)$, $(1,1) \to (1,2)$, $(1,3) \to (1,2)$, $(2,3) \to (1,2)$, $(1,1) \to (1,3)$, $(1,1) \to (2,2)$, $(1,3) \to (2,2)$, $(3,3) \to (2,2)$, $(1,1) \to (2,3)$, $(1,1) \to (3,2)$, $(1,3) \to (3,2)$, $(1,1) \to (3,3)$ from Block 2 and $(1,1) \to (1,1)$, $(2,1) \to (1,1)$, $(1,1) \to (1,2)$, $(3,1) \to (1,2)$, $(1,1) \to (2,2)$, $(3,1) \to (2,2)$, $(1,1) \to (3,2)$, $(3,1) \to (3,2)$ from Block 7;
- Step 3a) remove $[1, \pi(0,1,-1), 3]$ since the vertices has no incoming edge;
- Step 3b) remove $(1,3) \to (1,2)$, $(1,3) \to (2,2)$, $(1,3) \to (3,2)$ from Block 3;
- Step 4a) remove $[2, \pi(0,1,-1), 1]$, $[2, \pi(0,1,-1), 2]$, $[2, \pi(0,1,-1), 3]$, $[3, \pi(0,1,-1), 1]$, $[3, \pi(0,1,-1), 3]$ since none of these vertices has an outgoing edge;
- Step 4b) remove $(1,1) \to (2,1)$, $(1,3) \to (2,1)$, $(2,1) \to (2,1)$, $(2,3) \to (2,1)$, $(3,1) \to (2,2)$, $(1,1) \to (3,1)$, $(2,1) \to (3,1)$ from Block 5;
- Step 5a) remove $[1, \pi(1,0,-1), 2]$, $[1, \pi(1,0,-1), 3]$, $[2, \pi(1,0,-1), 2]$, $[2, \pi(1,0,-1), 3]$, $[3, \pi(1,0,-1), 2]$, $[3, \pi(1,0,-1), 3]$ since none of these vertices has an outgoing edge;
- Step 5b) remove $(1,1) \to (1,2)$, $(1,1) \to (1,3)$, $(1,1) \to (2,2)$, $(1,1) \to (2,3)$, $(1,1) \to (3,2)$, $(1,1) \to (3,3)$ from Block 1 and $(3,1) \to (1,2)$, $(3,1) \to (2,2)$, $(3,1) \to (3,2)$ from Block 6;
- Step 6a) remove $[3, \pi(1,-1,0), 1]$ since the vertices has no outgoing edge;
- Step 6b) remove $(1,1) \to (3,1)$ from Block 2 and $(1,1) \to (3,1)$, $(2,1) \to (3,1)$ from Block 7;
- Step 7a) remove $[1, \pi(1,-1,1), 2]$, $[2, \pi(1,-1,1), 2]$, $[3, \pi(1,-1,1), 2]$ since none of these vertices has an outgoing edge;
- Step 7b) $(1,1) \to (1,2)$, $(1,1) \to (2,2)$, $(1,1) \to (3,2)$ from Block 3 and $(1,1) \to (1,2)$, $(3,1) \to (1,2)$, $(1,1) \to (2,2)$, $(3,1) \to (2,2)$, $(1,1) \to (3,2)$, $(3,1) \to (3,2)$ from Block 8;
- Step 8a) remove $[3, \pi(1,-1,1), 1]$ since the vertices has no outgoing edge;
- Step 8b) remove $(1,1) \to (3,1)$ and $(1,2) \to (3,1)$ from Block 3 and $(1,1) \to (3,1)$, $(2,1) \to (3,1)$ from Block 8;

The resulting vertices and edges are highlighted in grey in Table 13 and Table 11, respectively.

7.2. Algorithmic part of Proof of Theorem 3.2.

Remember that we wish to prove the following claim.

Claim. *No more edges (nor vertices) satisfying Items 1. and 2. of Theorem 4.1 can be added to $\mathcal{S}(a,b)$.*

We carry out the following algorithm. We go through all types of vertices of $\mathcal{S}(a,b)$ and investigate the possible incoming edges:

- we use Lemma 4.2 to show that a predecessor $[i, \gamma, j]$ of a vertex $[i', \gamma', j']$ can obtain at most two different values for γ;
- we use Lemma 4.5 to determine i and j.

The #-letters refer to the names of the vertices we used in Ajacency Table 1 to 3.

Vertices of the form $[i', \pi(0,0,0), j']$: By Lemma 4.2 the only possible incoming edges have an initial vertex of the shape $[i, \pi(0,0,0), j]$. Moreover, $i < j$ by the definition of the self-replicating boundary graph. We have to investigate the following pairs.

$(i', j') = (1, 2)$ **(#J_1):** We use Table 8 to find out the possible edges. In column $(1, 2)$ there occur three lists that may include 0. The list in line $(1, 1)$ is not relevant since $i = j = 1$ is not allowed. Hence, we only consider $(i, j) = (1, 3)$ or $(i, j) = (2, 3)$. For $a = b$ row $(1, 3)$ contains strictly positive values only, hence, this edge cannot occur in this case. For $a \neq b$ we consult Γ_σ and see that there is only one edge from 3 to 2 labelled by $(1^b, 3, \varepsilon)$.

Hence, we have only one possible edge $[1, \pi(0, 0, 0), 3] \xrightarrow{\pi(b,0,0)} [1, \pi(0, 0, 0), 2]$ (#$K_1 \to$ #J_1, Type 1). Analogously, $(i, j) = (2, 3)$ gives one edge from $[2, \pi(0, 0, 0), 3]$ (#L_1) labelled by $\pi(a, 0, 0)$ provided that $a = b$. In both cases the edges can already be found in Adjacency Table 1.

$(i', j') = (1, 3)$ **(#K_1) or** $(2, 3)$ **(#L_1):** These vertices cannot have any incoming edge since in columns $(1, 3)$ and $(2, 3)$ of Table 8 the only list including 0 is the one in row $(1, 1)$, which is not relevant here.

Vertices of the form $[i', \pi(0, 0, 1), j']$**:** Similar as above, $[i, \pi(0, 0, 0), j]$ with $i < j$ is the only type of predecessor. The significant difference equals 1 and, thus, we are looking for the entry 1 in Table 8. In particular,

$(i', j') = (1, 1)$ **(#A_1):** Since $i < j$ we see by Table 8 that $(i, j) = (1, 2)$ (#J_1) is the only possibility and gives only one edge (which is already included in $\mathcal{S}(a, b)$).

$(i', j') = (1, 2)$ **(#B_1):** In column $(1, 2)$ of Table 8 the only row that contains 1 is the row $(1, 3)$. The corresponding edge starts in $[1, \pi(0, 0, 0), 3]$ (#K_1) and is contained in $\mathcal{S}(a, b)$.

$(i', j') = (2, 2)$ **(#M_1):** Similarly as before, row $(1, 3)$ is the only row that contains 1 in column $(2, 2)$ of Table 8. The corresponding edge (#$K_1 \to$ #M_1) is contained in $\mathcal{S}(a, b)$ provided that $[2, \pi(0, 0, 1), 2]$ is contained in $\mathcal{S}(a, b)$, *i.e.*, $b = 1$.

The vertex $[1, \pi(0, 1, 0), 1]$ **(#N_1):** By Lemma 4.2 the incoming edges of Type 1 start in vertices of the form $[i, \pi(0, 0, 1), j]$ with the significant difference $b > 0$. In the corresponding column of Table 8 we find that row $(1, 2)$ includes b. The associated edge (#$B_1 \to$ #N_1) is already included in $\mathcal{S}(a, b)$. Row $(1, 1)$ includes b provided that $a \neq b$, and in Γ_σ there are $a - b$ possibilities to choose edges $(p_1, 1, s_1)$ and $(p_2, 1, s_2)$ from 1 to 1 such that $\mathbf{l}(p_2) - \mathbf{l}(p_1) = (b, 0, 0)$. All the edges (#$A_1 \to$ #N_1) appear in $\mathcal{S}(a, b)$. All incoming edges of Type 2 start from vertices of the type $[i, \pi(1, 0, -1), j]$. Now observe that we already collected all possibilities in Block 4 in Table 11 in Lemma 5.2. There are two cells whose right entry equals $(1, 1)$. Their left entries give the possible pairs (i, j). The first one is $(1, 1)$ (#G_1) and yields $a - b - 1$ edges (hence, edges only if $a \geq b + 2$), the other one is $(2, 1)$ (#E_1) and yields one edge provided that $a \neq b$. The edges are included in $\mathcal{S}(a, b)$.

Vertices of the form $[i', \pi(0, 1, -1), j']$**:** The incoming edges of Type 1 have initial vertices of the shape $[i, \pi(0, 0, 1), j]$ with significant difference $b - 1 \geq 0$. As $[\pi(0, 0, 1), 3] \notin \Gamma_{srs}$ we conclude that $j \neq 3$.

$(i', j') = (1, 1)$ **(#C_1):** $\mathcal{S}(a, b)$ includes $a - b - 1$ edges that start in $[1, \pi(0, 0, 1), 1]$ (#A_1). There is also another edge that starts in $[1, \pi(0, 0, 1), 2]$ (#B_1) if $b > 1$, and in $[2, \pi(0, 0, 1), 2]$ (#M_1) if $b = 1$. By Table 8 and Γ_σ there is no other possibility.

$(i', j') = (1, 2)$ **(#D_1) or** $(3, 2)$ **(#O_1):** By Table 8 the only possibility is $(i, j) = (1, 1)$ (#A_1) since $j = 3$ is not allowed.

We already investigated the incoming edges of Type 2 in the proof of Lemma 5.2. They are of Shape 5, start in vertices of the form $[i, (1, 0, -1), j]$ and Block 5 gives the possible pairs. All 4 cells that we find there correspond to edges that are included in $\mathcal{S}(a, b)$ (#$E_1 \to$ #C_1, #$F_1 \to$ #D_1, #$F_1 \to$ #O_1, #$G_1 \to$ #C_1).

The vertex $[2, \pi(1, -1, 0), 1]$ ($\#P_1$) and vertices of the form $[i', \pi(1, 0, -1), j']$ ($\#E_1$, $\#F_1$, $\#G_1$): In the blocks 1, 2, 6 and 7 of Table 11 in Lemma 5.2 we can check that actually all possible incoming edges are already included in $\mathcal{S}(a, b)$.

Vertices of the form $[i', \pi(t, -t, t), j']$: The incoming edges of Type 1 have initial vertices of the shape $[i, \pi(1-t, t, -t), j]$ with significant difference $t(a-b+2)-1 = \delta_t - 1 > a - b$, those of Type 2 start in vertices of the form $[i, \pi(t, -t, t), j]$ with significant difference $t(a-b+2) = \delta_t > a - b + 1 > 0$. We already studied the latter ones detailed in Lemma 5.2. We see that there cannot be other incoming edges than those that are included in $\mathcal{S}(a, b)$. For examining the possible edges of Type 1 we investigate the three vertices of the present form.

- **$(i', j') = (1, 1)$ ($\#I_t$ for $t = 1, \ldots, m(a, b)$):** Note that column $(1, 1)$ of Table 8 shows two lists that include strictly positive entries: the rows $(1, 1)$ and $(1, 2)$. The first one gives $a - \delta_t + 1$ edges starting from $[1, \pi(1-t, t, -t), 1]$ ($\#C_t$). The other one gives an edge starting from $[1, \pi(1-t, t, -t), 2]$ ($\#D_t$).
- **$(i', j') = (2, 1)$ ($\#H_t$ for $t = 1, \ldots, m(a, b)$):** Again, we consult Table 8 and find two lists with suitable entries, $(1, 1)$ and $(1, 2)$. They give the edges that are included in $\mathcal{S}(a, b)$.
- **$(i', j') = (3, 1)$ ($\#Q_{t+1}$ for $t = 1, \ldots, m(a, b) - 1$):** Note that $t \leq m(a, b) - 1$ induces that $\delta_t < a$. Thus, in Table 8 the only row of interest is $(1, 1)$. It gives the edge starting from $[1, \pi(1-t, t, -t), 1]$ ($\#C_t$).

Vertices of the form $[i', \pi(2-t, t-1, -t), j']$ (for $t = 2, \ldots, m(a, b)$): Incoming edges of Type 1 can only start at vertices of the form $[i, \pi(t-2, 2-t, t-1), j]$ with significant difference $a - (t-1)(a-b+2) = a - \delta_{t-1} > 0$, those of Type 2 start at vertices of the form $[i, \pi(3-t, t-2, 1-t), j]$ with significant difference $a - (t-1)(a-b+2) + 1 = a - \delta_{t-1} + 1 > 0$.

$(i', j') = (1, 1)$ ($\#G_t$): The two rows which have positive entries in column $(1, 1)$ of Table 8 are $(1, 1)$ and $(1, 2)$. They induce the $a - \delta_{t-1} + 1$ edges of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 1]$ ($\#A_{t-1}$) and the single edge of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 2]$ ($\#B_{t-1}$) that are included in $\mathcal{S}(a, b)$. On the other hand, the edges of Type 2 start in $[1, \pi(3-t, t-2, 1-t), 1]$ ($\#G_{t-1}$) and $[2, \pi(3-t, t-2, 1-t), 1]$ ($\#E_{t-1}$) and are included in $\mathcal{S}(a, b)$, too.

$(i', j') = (2, 1)$ ($\#E_t$): In Table 8 we find in column $(2, 1)$ that the rows $(1, 1)$ and $(1, 2)$ yield the edges of Type 1 (starting in $\#A_{t-1}$ and $\#B_{t-1}$) that are included in $\mathcal{S}(a, b)$. Note that by the definition of $m(a, b)$ we have $a - (t-1)(a-b+2) > a - b + 2 > a - b$ and, hence, neither $(i, j) = (3, 1)$ nor $(i, j) = (3, 2)$ come into question. For edges of Type 2 we only have the possibilities $(i, j) = (1, 1)$ ($\#G_{t-1}$) and $(i, j) = (2, 1)$ ($\#E_{t-1}$) and obtain edges that already are contained in $\mathcal{S}(a, b)$. For the same reason as before, (i, j) cannot be $(1, 3)$ or $(2, 3)$.

$(i', j') = (3, 1)$ ($\#F_t$): Similar as before, we see that there is always one incoming edge of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 1]$ ($\#A_{t-1}$) and of Type 2 starting at $[1, \pi(3-t, t-2, 1-t), 1]$ ($\#G_{t-1}$). Since $a - \delta_{t-1} < a - (t-1)(a-b+2) + 1 \leq a - 1 < a$ we cannot have $(i, j) = (1, 2)$ (Type 1) or $(i, j) = (2, 1)$ (Type 2), respectively.

Vertices of the form $[i', \pi(t-1, 1-t, t), j']$ (for $t = 2, \ldots, m(a, b)$): The incoming edges originate in $[i, \pi(2-t, t-1, 1-t), j]$ (Type 1 with significant difference $(t-1)(a-b+2) = \delta_{t-1}$) and $[i, \pi(t-1, 1-t, t-1), j]$ (Type 2 with significant difference $(t-1)(a-b+2)+1 = \delta_{t-1}+1 > 1$). Note that $\delta_{t-1} \leq b - 2$.

$(i', j') = (1, 1)$ ($\#A_t$): Analogously as before we easily find that the only possible incoming edges of Type 1 are $a - \delta_{t-1}$ edges starting at $[1, \pi(2-t, t-1, 1-t), 1]$ ($\#C_{t-1}$) and one edge starting at $[1, \pi(2-t, t-1, 1-t), 2]$ ($\#D_{t-1}$). The incoming edges of Type 2 start at $[1, \pi(t-1, 1-t, t-1), 1]$ ($\#I_{t-1}$, $a - \delta_{t-1} - 1$ edges) and $[2, \pi(t-1, 1-t, t-1), 1]$ ($\#H_{t-1}$, one edge).

$(i',j')=(1,2)$ **($\#B_t$):** In Table 8 we find that the lists in row $(1,1)$ and $(1,3)$ include suitable values. Indeed, $[1,\pi(2-t,t-1,1-t),1]$ ($\#C_{t-1}$) is the origin of $b-\delta_{t-1}$ edges of Type 1. By (5.1) we see that $[\pi(2-t,t-1,1-t),3]\not\in\Gamma_{srs}$ and, hence, $j=3$ is no option. On the other hand, for the the edges of Type 2 we have one edge starting at $[3,\pi(t-1,1-t,t-1),1]$ ($\#Q_t$) besides the $b-\delta_{t-1}-1$ incoming edges that have their origin in $[1,\pi(t-1,1-t,t-1),1]$ ($\#I_{t-1}$).

Vertices of the form $[i',\pi(1-t,t,-t),j']$ **(for** $t=2,\dots,m(a,b)$**):** The incoming edges of Type 1 have initial vertices of the form $[i,\pi(t-1,1-t,t),j]$ with significant difference $a-t(a-b+2)+1=a-\delta_t+1>0$, those of Type 2 originate in vertices of the shape $[i,\pi(2-t,t-1,-t),j]$ with significant difference $a-t(a-b+2)+2=a-\delta_t+2>2>0$.

$(i',j')=(1,1)$ **($\#C_t$):** We see that for an edge of Type 1 we must have $(i,j)\in\{(1,1),(1,2)\}$ and, respectively, $(i,j)\in\{(1,1),(2,1)\}$ for an edge of Type 2. All possible edges (from $\#A_t$, $\#B_t$, $\#G_t$, $\#E_t$) are included in $\mathcal{S}(a,b)$.

$(i',j')=(1,2)$ **($\#D_t$):** $(i,j)=(1,1)$ ($\#A_t$) is the only option for edges of Type 1. $(i,j)=(1,3)$ is not possible since $[\pi(t-1,1-t,t),3]\not\in\Gamma_{srs}$ (see (5.1)). $(i,j)=(1,1)$ ($\#G_t$) and $(i,j)=(3,1)$ ($\#F_t$) give the only possibilities for edges of Type 2. Since all these edges are contained in $\mathcal{S}(a,b)$ we finally showed the claim.

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