A comparison of topologies on the fundamental group

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Salzburg, September 12, 2017

Setting

(X, x_0) based path connected space.

1. path space $PX = \{\alpha\colon (\mathbb{I}, 0) \to (X, x_0)\}$
2. loop space $LX = \{\alpha\colon (\mathbb{I}, \partial\mathbb{I}) \to (X, x_0)\}$

Quotient via homotopy $\sim$:

1. universal covering space $\widetilde{X} = PX/\sim$
2. fundamental group $\pi_1(X, x_0) = LX/\sim$

How to topologize them?

Whisker and Lasso topology on π_1

Definition

Whisker basis neighborhood for $\alpha \in LX, x_0 \in U$ open:

$$N^w([\alpha], U) = \{[\alpha * \delta] \mid \delta \subset U\}$$

Used for classical covering spaces (Spanier, ...).

Definition

For open cover $\mathcal{U}$ of X (a normal) subgroup $\pi_1(\mathcal{U}, x_0) \leq \pi_1(X, x_0)$ is generated by $\beta * \delta * \beta^-$ with $\delta \subset U$ for some $U \in \mathcal{U}$.
Lasso basis neighborhood for $\alpha \in LX$ and an open cover $\mathcal{U}$ of X:

$$N^\ell([\alpha], \mathcal{U}) = [\alpha] * \pi_1(\mathcal{U}, x_0) = \pi_1(\mathcal{U}, x_0) * [\alpha]$$

Defined by Brodsky, Dydak, Labuz, Mitra. $\pi_1^\ell(X, x_0)$ is always a topological group.

CO and τ topologies

Definition

CO topology is the quotient of the compact open topology.
Fabel: $\pi_1^{CO}(HE, 0)$ is not a topological group.

Definition

τ is the strongest topology contained in CO, making $\pi_1^{\tau}(X, x_0)$ a topological group.
Defined by Brazas. $\pi_1^{\tau}(X, x_0)$ is always a topological group. τ weaker than CO.

Topologies agree for semi-locally simply connected spaces.

Comparison in the LPC case

$$\pi_1^w(X, x_0) \geq \pi_1^{CO}(X, x_0) \geq \pi_1^\tau(X, x_0) \geq \pi_1^\ell(X, x_0)$$

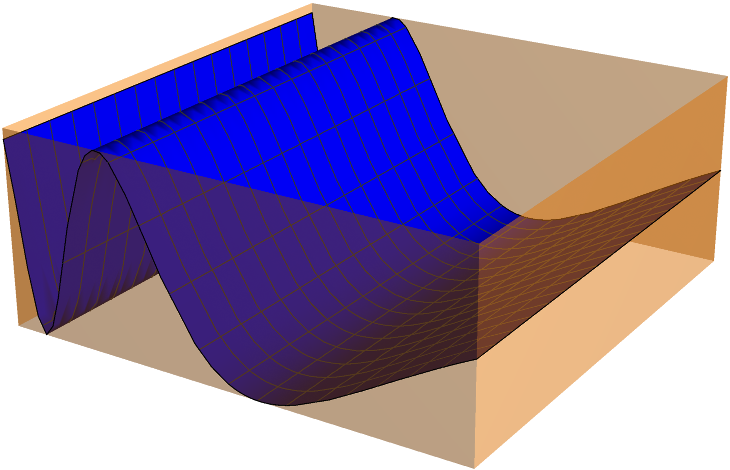
Strict inequalities obtained for HE.
Use:

- Fischer, Zastrow: there exists an open subgroup G of $\pi_1^{CO}(HE, x_0)$ which does not contain any nontrivial normal subgroup of $\pi_1^{CO}(HE, x_0)$
- Lasso topology is in a way uniform. CO is not

Comparison in general

$$\pi_1^{w}(X, x_0) \geq \pi_1^{CO}(X, x_0) \geq \pi_1^{\tau}(X, x_0)$$
$$\pi_1^{w}(X, x_0) \geq \pi_1^{\ell}(X, x_0)$$

No other relation holds.



Comparison on $\widetilde{X}$

- The comparison on $\widetilde{X}$ is the same.
- Topology $\tilde{\tau}$ on $\widetilde{X}$ is the strongest topology weaker than $\widetilde{CO}$ topology for which the action

$$\pi_1(X, x_0) \times \widetilde{X} \to \widetilde{X}$$

  is continuous.
- $\tilde{\tau}|_{\pi_1(X,x_0)} = \tau$

Fundamental grupoid?