An introduction to p-adic systems: A new kind of number system

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Let $T_C : \mathbb{N} \to \mathbb{N}$

$$n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

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Question for ultimate behaviour: Trivial!

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"Name" of 17 w.r.t. T_2: usual base 2 expansion
$(17, 8, 4, 2, 1, 0, \ldots) \,\%\, 2 = (1, 0, 0, 0, 1, 0, \ldots)$

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Notation: $S\,(T_C)[17] = (17, 26, 13, 20, \ldots)$: T_C-sequence of 17
$D(T_C)[17] = (1, 0, 1, 0, 0, 1, \ldots)$: T_C-(digit) expansion of 17

What do T_C and T_2 have in common?

Tables of sequences:

1	1	0	0	0	$\cdots$
2	2	1	0	0	$\cdots$
3	3	1	0	0	$\cdots$
4	4	2	1	0	$\cdots$
5	5	2	1	0	$\cdots$
6	6	3	1	0	$\cdots$
7	7	3	1	0	$\cdots$
8	8	4	2	1	$\cdots$
9	9	4	2	1	$\cdots$
10	10	5	2	1	$\cdots$
11	11	5	2	1	$\cdots$
12	12	6	3	1	$\cdots$
13	13	6	3	1	$\cdots$
14	14	7	3	1	$\cdots$
15	15	7	3	1	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T_2)$	0	1	2	3	$\cdots$

1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T_C)$	0	1	2	3	$\cdots$

Motivation

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1	1	0	0	0	$\cdots$
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11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	1	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(T_2)$	0	1	2	3	$\cdots$

1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	1	1	$\cdots$
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15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(T_C)$	0	1	2	3	$\cdots$

What do T_C and T_2 have in common?

Tables of expansions:

$D(T_2)$	0	1	2	3	$\cdots$
1	1	0	0	0	$\cdots$
2	0	1	0	0	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	1	0	$\cdots$
6	0	1	1	0	$\cdots$
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16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(T_C)$	0	1	2	3	$\cdots$
1	1	0	1	0	$\cdots$
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3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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First k digits of expansions of m and n coincide $\Leftrightarrow$ $m \equiv n \mod 2^k$

(Block property)

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Both generalize to the 2-adic integers $\mathbb{Z}_2$:

$T_C : \mathbb{Z}_2 \to \mathbb{Z}_2$

$$n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

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Subtract LSD before division by p if necessary: $(x, x, x)(5) = \frac{5-2}{3} = 1$

Basic properties of p-adic systems

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  then $D = D(T)$ for a "unique" p-adic system T

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$$\longrightarrow \quad T(6) = 3 \quad \longrightarrow \quad T[0](6) \in \underbrace{\{6, 7\}}$$

remember: $(x, x-1) = (x, x)$

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Consequence: Yes, many exist!

More examples of p-adic system

- $T_n = (x, \ldots, x)$, $T_C = (x, 3x+1)$

$S(T_C)$	0	1	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	$\cdots$
2	2	1	2	$\cdots$
3	3	5	8	$\cdots$
4	4	2	1	$\cdots$
5	5	8	4	$\cdots$
6	6	3	5	$\cdots$
7	7	11	17	$\cdots$
8	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
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3	1	1	0	$\cdots$
4	0	0	1	$\cdots$
5	1	0	0	$\cdots$
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⋮	⋮	⋮	⋮	⋰
1	1	34/11	64805/2541	⋯
2	2	227/21	2053/231	⋯
3	3	47/11	608/121	⋯
4	4	880/21	12621386/3087	⋯
5	5	60/11	64376/847	⋯
6	6	639/7	4346/77	⋯
7	7	73/11	777/121	⋯
8	8	3338/21	59723107/1029	⋯
⋮	⋮	⋮	⋮	⋱
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2	0	1	1	⋯
3	1	1	0	⋯
4	0	0	0	⋯
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  Example: $P(x) = 10x^2 - 3x + 4$ is a 2-permutation polynomial

⋮	⋮
1	11
2	38
3	85
4	152
5	239
6	346
7	473
8	620
⋮	⋮
P	

⋮	⋮	⋮	⋮	⋰
1	1	1	0	⋯
2	0	1	1	⋯
3	1	0	1	⋯
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What about the Collatz conjecture?

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$1, 0$	$1, 2$
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  Remember: $S(F)[n]$ ultimately periodic $\Leftrightarrow$ $D(F)[n]$ ultimately periodic

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