# Automorphisms of low complexity subshifts

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$(Y, S)$ is a (topological) factor of $(X, T)$, or $(X, T)$ is an extension of $(Y, S)$, if there exists a continuous <u>surjective</u> $\phi \colon X \to Y$ such that

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Q: How does $\mathrm{Aut}(X, T)$ acts on $X$? On $T$-invariant measures?

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The open sets are unions of cylinders:

$$[u.v] := \{(x_n)_n \in A^{\mathbb{Z}} : x_{-|u|} \dots x_{|v|-1} = uv\}; \qquad u, v \in A^*$$

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For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma|_X)$.

Similarly

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The language

$$\mathcal{L}(X) := \{u \in A^* : u = x_0 \cdots x_{|u|-1} \text{ for some } (x_n)_n \in X\}.$$

$$\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n.$$

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The system $(X, \sigma)$ is expansive: $\exists \epsilon > 0$, $x \neq y \in X$,

$$\sup_{n \in \mathbb{Z}} \mathrm{dist}(\sigma^n(x), \sigma^n(y)) > \epsilon.$$

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  $$\langle T\rangle < Z(\mathrm{Aut}(X, T))$$
- The centralizer of $S \subset G$: $C_G(S) := \{g \in G : gs = sg \ \forall s \in S\}$.
- $$C_{\mathrm{Homeo}(X)}(T) = \mathrm{Aut}(X, T).$$

# Algebraic motivations

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The (topological) full group of a subshift $(X, \sigma)$ is

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Giordano-Putnam-Skau (1999): If $(X, \sigma)$ is minimal (without proper subshift)

$$\mathrm{Out}([[\sigma]]) \simeq \{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}/\langle\sigma\rangle.$$

$$\{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}_{/\mathrm{Aut}(X,\sigma)} \subset \mathbb{Z}/2\mathbb{Z}.$$

**Theorem (Curtis-Hedlund-Lyndon)**

*An automorphism $\phi$ of $(X, \sigma)$ is a sliding block code,*
i.e. *there exists a* block map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

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| 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

$$x = \ \ldots 010011.1010101 \quad 000 \quad 0111\ldots$$

$$\downarrow$$

$$\phi(x) = \ \cdots\cdot .01010100 \quad 0$$

**Theorem (Curtis-Hedlund-Lyndon)**

*An automorphism $\phi$ of $(X, \sigma)$ is a sliding block code,* i.e. *there exists a* block map $\hat{\phi}$: $\mathcal{L}_{2r+1}(X) \to A$ *s.t.*

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**Theorem (Curtis-Hedlund-Lyndon)**

*Let $\phi$ be an automorphism of $(X,\sigma)$*
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**Corollary**

$Aut(X,\sigma)$ *is countable.*
$Aut(X,\sigma)$ *is a discrete subgroup of Homeo($X$) for the uniform convergence topology.*

# Complexity restrictions

The complexity $p_X : \mathbb{N} \to \mathbb{N}$,

$$p_X(n) = \#\mathcal{L}_n(X) = \# \text{ words of length } n \text{ in } X.$$

Q: How the growth of the complexity restricts $\mathrm{Aut}(X, \sigma)$?

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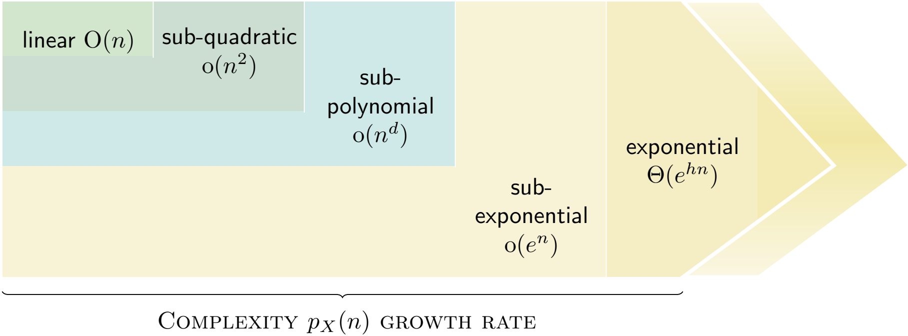

# Plan

1. Automorphism of SFT
2. Automorphism of classical minimal systems
   a) Linear complexity case
   b) Toeplitz subshifts case
3. Automorphism for sub-exponential complexity subshifts and restrictions on automorphisms groups.

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In this case:

$\mathrm{Aut}(X, \sigma)$ is not finitely generated, not amenable.

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Basic algebra shows it contains the free group on 2 generators, hence the free group with countably many generators.

# Automorphism of SFT

A group $G$ is residually finite if for any $g_1 \neq g_2 \in G$ there is a homomorphism $\pi\colon G \to G_0$ onto a finite group $G_0$ such that $\pi(g_1) \neq \pi(g_2)$.

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Since $\bigcup_n \mathrm{Per}_n$ is dense in $X$,

$$\pi_n(\phi_1) = \pi_n(\phi_2),\ \forall n \Rightarrow \phi_1 = \phi_2.$$

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**Corollary**

*For an irreducible SFT,* $\mathrm{Aut}(X, \sigma)$ *does not contains a divisible subgroup: For any* $\phi \in \mathrm{Aut}(X, \sigma) \setminus \{\mathrm{Id}\}$*, there exists* $n \in \mathbb{N}$ *s.t. the equation*

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**Ex:** $\mathrm{Aut}(X,\sigma)$ does not contains $\mathbb{Q}$.
**Open problem:** is $\mathbb{Z}[1/p]$ contained in $\mathrm{Aut}(X,\sigma)$ for any prime $p$?

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*Proof.* Given $\phi_1, \dots, \phi_\ell \in \mathrm{Aut}(X, \sigma)$, find a finite procedure to decide if

$$\psi = \phi_{i_1}^{\pm} \circ \cdots \circ \phi_{i_r}^{\pm} = \mathrm{Id}, \qquad i_1, \dots, i_r \in \{1, \dots, \ell\}.$$

By Curtys-Hedlund-Lyndon Theorem, it is enough to check if the block map of $\psi$ with range $r_\psi = \mathrm{O}(r)$ satisfies

$$\hat{\psi}(x_{-r_\psi} \cdots x_{r_\psi}) = x_0.$$

# Automorphism of $\mathbb{Z}^d$- SFT, $d \geq 2$

There exists a $\mathbb{Z}^d$-SFT whose automorphism group has undecidable word problem Guillon,Jeandel,Kari,Vanier (18).

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Let $(X, \sigma)$ be a $\mathbb{Z}^d$-SFT with $h(X, \sigma) > 0$ then

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- If moreover, minimal orbits are dense (e.g. periodic orbits), then $\mathrm{Aut}(X, \sigma)$ contains a copy of $\mathrm{Aut}(\{1, \ldots, n\}^{\mathbb{Z}}, \sigma)$ $\forall n$.

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**Open problem:** $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}^d}, \sigma) \simeq \mathrm{Aut}(\{0,1\}^{\mathbb{Z}^m}, \sigma)$ ?

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