Automorphisms of low complexity subshifts 2

Case of classical minimal subshifts

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Automorphism of classical minimal systems

A subshift $X \subset A^{\mathbb{Z}}$,(*i.e.* a closed shift invariant $\sigma(X) = X$ set) is minimal if it has no proper non empty subshift.

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Any subshift has a non empty minimal subshift.

Example:

- Periodic sequences.
- Sturmian subshift, substitutive,...
- Toeplitz subshift.

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To understand $\mathrm{Aut}(X, \sigma)$: first understand the minimal case!

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Lemma

Let (X, T) be a minimal dynamical system. The action of $Aut(X, T)$ on X

$$\begin{array}{rcl} Aut(X,T)\times X & \to & X \\ (\phi, x) & \mapsto & \phi(x), \end{array}$$

is free (the stabilizer of any point is trivial).

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Proof. For any automorphism ϕ, the set

$$\{x; \phi(x) = x\}$$

is closed and T invariant.
By minimality, it is either X ($\Rightarrow \phi = \mathrm{Id}$) or empty.

Automorphism of classical systems

Examples of minimal subshift (X, σ), with $\mathrm{Aut}(X, \sigma)$ isomorphic to

- $\mathbb{Q}$, with 1 identified with σ — Boyle-Lind-Rudolph (88)

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- $\mathbb{Q}$, with 1 identified with σ Boyle-Lind-Rudolph (88)
- $\langle \sigma \rangle \oplus G$ for an arbitrarily finite group G Host-Parreau - Lemańczyk-Mentzen, (89)
- $\langle \sigma \rangle \oplus G$ for an arbitrarily finitely generated abelian group G (eventually G trivial)

Non superlinear complexity

Complexity function $\quad p_X(n) = \#\mathcal{L}_n(X).$

Theorem (Donoso-Durand-Maass & P., Cyr & Kra (15), Coven-Quas & Yassawi (16))

Let (X, σ) be a minimal subshift such that

$$\liminf_n \frac{p_X(n)}{n} < +\infty.$$

Then any continuous $\phi \colon X \to X$ s.t. $\phi \circ \sigma = \sigma \circ \phi$, is bijective. Moreover $Aut(X, \sigma)/\langle \sigma \rangle$ is finite and

$$\# Aut(X, \sigma)/\langle \sigma \rangle \leq \liminf_n \frac{p_X(n)}{n}.$$

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Example.

- Sturmian subshifts: $p_X(n) = n + 1$ for all n (Olli 2013).
- Coding of minimal Interval Exchange Transformations.
- Pisot substitution (Salo-Törmä 2013)
- Linearly recurrent subshift (substitutive, ...).

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Example. This includes also

- Subshifts with subexponential complexity $p_X(n) \geq g(n)$ i.o. where $\lim_n g(n)/\alpha^n = 0$ for any $\alpha > 1$.

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idea of Proof. Let $G = \{g_0, \dots, g_\ell\}$, $g_0 = 1_G$, For $g \in G$.

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$\hat{\phi}_g : \mathcal{L}(X_\tau) \to \mathcal{L}(X_\tau) \qquad \phi_g\colon X_\tau \to X_\tau \qquad G < \mathrm{Aut}(X_\tau,\sigma).$

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Theorem (Donoso et al., Cyr et al. (15), Coven et al. (16))

Let (X, σ) *be a minimal subshift. If*

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $\# Aut(X, \sigma)/\langle \sigma \rangle \leq \liminf_n \frac{p_X(n)}{n}$.

Result is sharp. Salo (14), DDMP (16): $\forall \epsilon > 0$, there exists a Toeplitz subshift with complexity $O(n^{1+\epsilon})$ with a non finitely generated automorphism group.

Main Ideas

A word $w \in \mathcal{L}(X)$ is right special if there are two letters $a, b \in A$ s.t. wa and wb are words of X.

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Two sequences $x = (x_n)_{n\in\mathbb{Z}}$, $y = (y_n)_{n\in\mathbb{Z}} \in X$ are asymptotics if there is a $n_0 \in \mathbb{Z}$

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This defines an equivalence relation on σ-orbits.
Non trivial class are asymptotic pairs.

$$\lim_{n \to -\infty} \mathrm{dist}(\sigma^n(x), \sigma^n(y)) = 0.$$

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claim $\Rightarrow$ unif. bound on special words.

By contradiction: $\forall n \geq m$ large enough

$$\begin{aligned} p_X(n) - p_X(m) &= \sum_{i=m}^{n-1} p_X(i+1) - p_X(i) \geq (n-m)(K+1) \\ p_X(n) &\geq (n-m)(K+1) + p_X(m) \end{aligned}$$

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Let (X, σ) *be a subshift with* $\liminf_n p_X(n)/n = K < \infty$*.*
Then there is at most K *asymptotic pairs.*

Corollary

Let (X, σ) *be a minimal subshift s.t.* $\liminf_n p_X(n)/n < +\infty$*.*
Then any continuous $\phi\colon X \to X$ *s.t.* $\phi \circ \sigma = \sigma \circ \phi$*, is bijective.*
Moreover

$$\#Aut(X, \sigma)/\langle\sigma\rangle \leq \liminf_n \frac{p_X(n)}{n}.$$

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$p_4 = 60$	0	1	0	0	0	0	0	1	0	**1**	0	$*$	0	1	0	0	0	0

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A Toeplitz subshift is the subshift X generated by a Toeplitz sequence $x = (x_n)_n$.

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Dynamically:
For $x \in X$ a Toeplitz sequence, for any open set $U \subset X$, the return times of x in U:

$$\{n \in \mathbb{Z} : \sigma^n(x) \in U\},$$

contains a subgroup of $\mathbb{Z}$.

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- non uniquely ergodic minimal subshift — Williams (84)
- an arbitrary entropy $h \geq 0$ — Williams (84)
- complexity in $\Theta\left(n^{\alpha_0}(\log n)^{\alpha_1}(\log\log n)^{\alpha_2}\cdots(\log_{(k)} n)^{\alpha_k}\right)$, $\alpha_0 > 1, \alpha_1, \cdots, \alpha_k \in \mathbb{R}$.

  Goyon, Cassaigne

Adding machine or odometer

Given a sequence of periods $(p_n)_{n\geq 0}$, with $p_n | p_{n+1}$.
The odometer

$$\mathbb{Z}_{(p_n)} = \{(x_n)_{n\geq 0} \in \prod_{n=0}^{\infty} \mathbb{Z}/p_n\mathbb{Z} : x_{n+1} \equiv x_n \mod p_n \ \forall n\}.$$

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Any minimal equicontinuous system on a Cantor set is conjugated to an odometer.

Toeplitz subshift

Theorem (Williams)

Any Toeplitz subshift (X, σ) *is an extension of an odometer* $(\mathbb{Z}_{(p_n)}, +\mathbf{1})$.
Moreover the factor map $\pi \colon X \to \mathbb{Z}_{(p_n)}$ *is injective on a* G_δ *dense set.*

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Converse true — Downarowicz, Lacroix

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If $\pi\colon (X,\sigma) \to (\mathbb{Z}_{(p_n)}, +\mathbf{1})$ *is an almost one-to-one extension. Then*

$$\pi(x) = \pi(y) \Leftrightarrow \liminf_n d(\sigma^n(x), \sigma^n(y)) = 0.$$

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Corollary

For a Toeplitz subshift (X,σ)*.*

$$\mathrm{Aut}(X,\sigma)<\mathrm{Aut}(\mathbb{Z}_{(p_n)},+\mathbf{1})\simeq\mathbb{Z}_{(p_n)}.$$

In particular $\mathrm{Aut}(X,\sigma)$ *is abelian and residually finite.*

Toeplitz subshift

Corollary

For a Toeplitz subshift (X, σ).

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Consequences:

$\mathbb{Q} \not< \mathrm{Aut}(X, \sigma)$.

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A $g \in G$ is a torsion element if $g^n = 1_G$ for some integer n.

Lemma

The torsion group of $\mathbb{Z}_{(p_n)}$ *is isomorphic to* $\bigoplus_p \mathbb{Z}/p^k\mathbb{Z}$, *where the sum is taken over all the prime numbers p such that* $\lim_{n\to\infty} v_p(p_n) = k$ *is positive and finite.*

If X is a Toeplitz subshift, any f.g. torsion subgroup is cyclic.

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If X is a Toeplitz subshift, any f.g. torsion subgroup is cyclic.
If X has periods $(p_n) = (p^n)$ for some prime p.
Then $\mathrm{Aut}(X,\sigma)$ has no torsion element.

Toeplitz subshift

Corollary

For a Toeplitz subshift (X, σ).

$$\mathrm{Aut}(X, \sigma) \subset \mathrm{Aut}(\mathbb{Z}_{(p_n)}, +\mathbf{1}) \simeq \mathbb{Z}_{(p_n)}.$$

Corollary

If X *is a Toeplitz subshift* $\liminf_n p_X(n)/n < +\infty$, *then*

$$\mathrm{Aut}(X, \sigma) \simeq \mathbb{Z} \textit{ or } \mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z},$$

for some N.

See Coven, Quas,Yassawi (2016).

Toeplitz subshift

Examples of Toeplitz subshifts with:

- complexity $\mathrm{O}(n^{1+\epsilon})$ and $\mathrm{Aut}(X,\sigma)$ not f.g.

Salo, DDMP

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- positive entropy and $\mathrm{Aut}(X, \sigma) = \langle \sigma \rangle$.

  Bulatek, Kwiatkowski, Downarowicz, 90's

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- positive entropy and $\mathrm{Aut}(X, \sigma) = \langle \sigma \rangle$.

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- positive entropy and $\mathrm{Aut}(X, \sigma) = \langle \sigma \rangle \oplus G$ for an arbitrarily f.g. abelian group G.

  DDMP

Open pb: realize any countable subgroup of $\mathbb{Z}_{(p_n)}$ as $\mathrm{Aut}(X, \sigma)$?

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