Automorphisms of low complexity subshifts 3

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Recall for minimal subshift

Examples of minimal subshift (X, σ) (any orbit is dense), with $\mathrm{Aut}(X, \sigma)$ isomorphic to

- $\mathbb{Q}$, with 1 identified with σ (BLR)

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Pb: Is it possible to obtain "more complicated" groups ?

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Open pb: Is $\mathrm{Aut}(X, \sigma)$ always locally virtually abelian when (X, σ) is a minimal subshift ?

Basic notion for group: Growth rate of a group.

Let G be a group generated by a finite set $S \subset G$.

$$s(n) := \#\{s_1 \cdots s_k : \; s_i \in S \cup S^{-1} \cup \{1_G\} \text{ and } k \leq n\}$$

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- G has exponential growth if $\lim_n \log(s(n))/n > 0$

Example:

- The free group has an exponential growth.

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- G has polynomial growth of degree at most d if $\liminf_n \frac{\log(s(n))}{\log n} \leq d$.

Example:

- The free group has an exponential growth.
- $\mathbb{Z}^d$ has a polynomial growth rate of degree at most d.

Subquadratic complexity

Theorem (Cyr-Kra (14))

If (X, σ) is a transitive subshift such that

$$\liminf_n \frac{p_X(n)}{n^2} = 0,$$

then $Aut(X, \sigma)/\langle\sigma\rangle$ is a torsion group: i.e.,

$$\forall \phi \in Aut(X, \sigma), \exists n, p \in \mathbb{Z} \ s.t. \ \phi^p = \sigma^n.$$

Subquadratic complexity: Idea of proof

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Theorem (Curtis-Hedlund-Lyndon)

An automorphism ϕ *of* (X, σ) *is a sliding block code,*
i.e. *there exists a* block map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \text{ for any } n \in \mathbb{Z}.$$

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Theorem (Epifanios-Koskas-Mignosi (01), Quas-Zamboni (04), Cyr-Kra (13))

If $\eta\colon \mathbb{Z}^2 \to A$ *is a coloring and there exist* $k, n \in \mathbb{N}$ *s.t. the number of coloring of* $n \times k$ *rectangles in* η *satisfies*

$$P_\eta(n,k) \leq nk/\lambda,$$

where $\lambda = 144$ *(EKM),* $\lambda = 16$ *(QZ),* $\lambda = 2$ *(CK).*
Then η *has a period.*

Subpolynomial complexity

Theorem (Cyr-Kra (15))

Let (X, σ) be a minimal subshift s.t. there exists $d \geq 1$

$$\limsup_n \frac{p_X(n)}{n^d} = 0.$$

Then every finitely generated, torsion free subgroup of $\mathrm{Aut}(X, \sigma)$ has a polynomial growth rate at most $d - 1$.

In particular if $p_X(n) = \mathrm{o}(n^d)$, $\mathrm{Aut}(X, \sigma)$ does not contains $\mathbb{Z}^d$.

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Corollary (Cyr-Kra (15))

Let (X, σ) be a minimal subshift s.t. there exists $d \geq 1$

$$\limsup_n \frac{p_X(n)}{n^d} = 0.$$

Every finitely generated, torsion free subgroup of $\mathrm{Aut}(X, \sigma)$ is virtually nilpotent of degree at most $\lfloor (-1 + \sqrt{8d - 7})/2 \rfloor$.

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Corollary (Cyr-Kra (15))

Let (X, σ) be a minimal subshift s.t.

$$\limsup_n \frac{p_X(n)}{n^3} = 0.$$

Every finitely generated, torsion free subgroup of $\mathrm{Aut}(X, \sigma)$ is virtually abelian.

Main ideas to control the growth rate of $\mathrm{Aut}(X, \sigma)$

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Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a bloc map $\hat{\phi}$: $\mathcal{L}_{2r_{\hat{\phi}}+1}(X) \to A$ *s.t.*

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The range of $\phi \in \mathrm{Aut}(X, \sigma)$ is

$$\mathbf{r}(\phi) := \inf\{r_{\hat{\phi}};\ \hat{\phi} \text{ is a bloc map defining } \phi\} \geq 0.$$

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$$\mathbf{r}(\phi \circ \psi) \leq \mathbf{r}(\phi) + \mathbf{r}(\psi) \qquad \forall \phi, \psi \in \mathrm{Aut}(X, \sigma).$$

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$$\mathbf{r}(\phi_1 \circ \cdots \circ \phi_n) \leq \mathbf{r}(\phi_1) + \cdots + \mathbf{r}(\phi_n) \leq n \sup_i \mathbf{r}(\phi_i)$$

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Goal: estimate the cardinal of

$$\mathrm{Aut}(X,\sigma)_R := \{\phi \in \mathrm{Aut}(X,\sigma); \mathbf{r}(\phi) \leq R\}$$

Lemma

Let (X, σ) *be a subshift s.t.* $\limsup_n p_X(n)/n^d < +\infty$*. Then there exists* $C > 1$ *and infinitely many words* $w \in \mathcal{L}(X)$ *s.t.*

$$\#\{(a, b) \in \mathcal{L}(X)^2 ; awb \in \mathcal{L}(X), |a| = |b| = \lfloor \frac{|w|}{C} \rfloor\} = 1. \qquad (1)$$

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Proof. By contradiction. Assume for all $C > 1$ and sufficiently large $u \in \mathcal{L}(X)$, $n = |u| \geq n_0$, there are words a_1, b_1, a_2, b_2 with $|a_i| = |b_i| = \lfloor \frac{|w|}{C} \rfloor$ s.t. $a_1 u b_1 \neq a_2 u b_2 \in \mathcal{L}(X)$.

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Contradiction when $C >> 1$

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G has a polynomial growth.

Subexponential complexity

Theorem (Cyr-Kra (15))

Let (X, σ) be a minimal subshift s.t. there exists $\beta < 1/2$

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Corollary

Under the same hypothesis:

$\mathrm{Aut}(X, \sigma)$ *is amenable.*

Obstruction to embedding: distortion

Let G be a countable group and a finite set $S \subset G$.
For $g \in \langle S \rangle$, $\ell_S(g)$ denotes the length of the shortest presentation of g by elements of S:

$$\ell_S(g) = \inf \left\{ k \in \mathbb{N}; \exists s_1, \ldots, s_k \in S \cup S^{-1}; g = s_1 \cdots s_k \right\}$$

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For any $n \in \mathbb{Z}$,

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ϕ distorted $\Rightarrow \mathbf{r}(\phi^n) = \mathrm{o}(n)$.

E.g.: the shift $\mathbf{r}(\sigma^n) = n$ for infinite subshift X.
The shift map is not distorted in $\mathrm{Aut}(X, \sigma)$.

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- Baumslag-Solitar group $\mathrm{BS}(1, n) = \langle a, b : bab^{-1} = a^n \rangle$.

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$$a^{n(\cdots)+\alpha_0} = ba^{(\cdots)}b^{-1}a^{\alpha_0} = b^k a^{\alpha_k} b^{-1} a^{\alpha_{k-1}} b^{-1} \cdots b^{-1} a^{\alpha_0}.$$

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- $\mathrm{SL}(2, \mathbb{Z}[1/p])$, for any prime p.

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Theorem (Cyr, Franks, Kra & P.)

Let (X, σ) be a subshift with zero entropy. Suppose $\phi \in \mathrm{Aut}(X, \sigma)$ is s.t. $\mathbf{r}(\phi^n) = \mathrm{O}(\log n)$. Then ϕ has finite order.

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Hochman (11): example of an automorphism polynomially range distorted.
Is it (group) polynomially distorted ?

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For a nilpotent group G, the torsion subgroup T is the group generated by the elements of finite order.
$T \triangleleft G$ is finite when G is finitely generated.

Corollary

Let (X, σ) be a subshift with a f. g. nilpotent group $G < \mathrm{Aut}(X, \sigma)$. If G/T is a d-step nilpotent group, then

$$\liminf_n \frac{p_X(n)}{n^{d+1}} > 0.$$

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$$\mathrm{H} = \langle s, t, u; su = us, ts = st, utu^{-1}t^{-1} = s \rangle.$$

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Growth rate of $\langle \sigma \rangle \oplus \mathrm{H}$ is n^5.

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Question

For zero entropy multidimensional shift, can the automorphism group contain the Heisenberg or a group with a distorted element of infinite order ?

Cantor minimal system

(X, T) is minimal if any orbit is dense in X.

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E.g.: finite groups, $\mathbb{Z}^n$, free group, finitely generated linear groups, $\mathrm{Aut}(\{0, 1\}^{\mathbb{Z}}, \sigma)$ ...

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Generally the examples are not expansive.

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