The level of distribution of the Thue–Morse sequence

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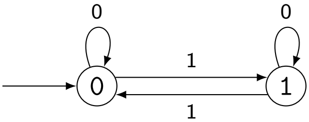
Section 1

Arithmetic subsequences of the Thue–Morse sequence

The Thue–Morse sequence $\mathbf{t}$

(1) is an automatic sequence:



(2) is the base-2 sum-of-digits function s, modulo 2.

(3) is the fixed point, starting with 0, of the substitution

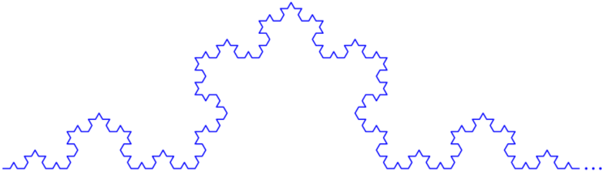
$$0 \mapsto 01$$
$$1 \mapsto 10.$$

(4) can be constructed starting with $\mathbf{t}^{(0)} = 0$ and setting $\mathbf{t}^{(k+1)} = \mathbf{t}^{(k)}\overline{\mathbf{t}^{(k)}}$ (Boolean complement).

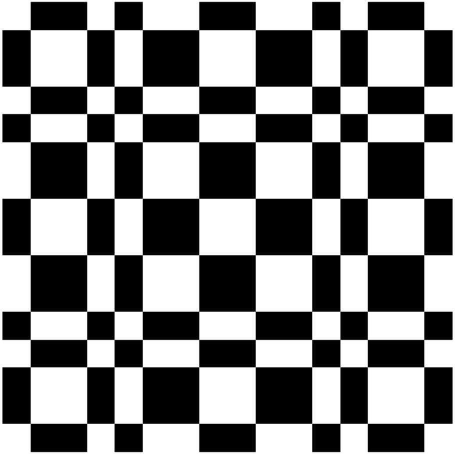
The Thue–Morse sequence, continued

A less well-known characterization uses the Koch snowflake curve.

(5) The sequence $n \mapsto (-1)^{s(n)}\,\mathrm{e}(-n/3)$ describes the orientation of the nth segment in the unscaled snowflake curve:



Thue–Morse, 16×16.



In particular, there are $\ll k$ many subwords of length k (as for any automatic sequence).

Thue–Morse along arithmetic progressions

011010011001011010010110011010011001011001101001011010

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More 0s than 1s (Newman's phenomenon), but not in the main term.

Theorem (Gelfond 1968)

Let $d \geq 1$ and a be integers. There is an absolute $\lambda < 1$ such that

$$\left|\{1 \leq n \leq x : \mathbf{t}(n) = 0, n \equiv a \bmod d\}\right| = \frac{x}{2d} + \mathcal{O}\left(x^{\lambda}\right).$$

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That is, for all $d \geq 1$ and $a < d$ we have

$$\sum_{1 \leq m \leq M} (-1)^{s(md+a)} \ll M^{\lambda}.$$

The error term for sums over APs

- We cannot get a uniform constant C in the estimate

$$\sum_{1 \le m \le M} (-1)^{s(md+a)} \le CM^{\lambda}.$$

- This follows since there are arbitrarily long arithmetic progressions on which $\mathbf{t}$ is constant.

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Theorem (Fouvry–Mauduit 1996)

$$\sum_{1\le d\le D} \max_{1\le y\le x} \max_{0\le a<d} \left| \sum_{\substack{0\le n<y\\ n\equiv a \bmod d}} (-1)^{s(n)} \right| \le Cx^{1-\eta}$$

for some $\eta > 0$ *and* C *and* $D = x^{0.5924}$.

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selects the "worst" AP contained in $[0, x)$

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Theorem (Spiegelhofer 2018+)

The Thue–Morse sequence has level of distribution 1. *More precisely, let* $0 < \varepsilon < 1$. *There exist* $\eta > 0$ *and* C *such that*

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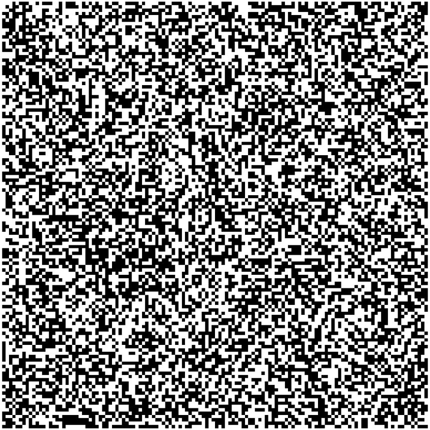
- This is a statement on *sparse* arithmetic progressions: the Thue–Morse sequence usually shows cancellation along N-term arithmetic progressions having common difference $\sim N^R$, where $R > 0$ is arbitrary.

Sparse arithmetic subsequences of TM

TM along short arithmetic subsequences (with respect to the common difference) even seems to behave randomly.



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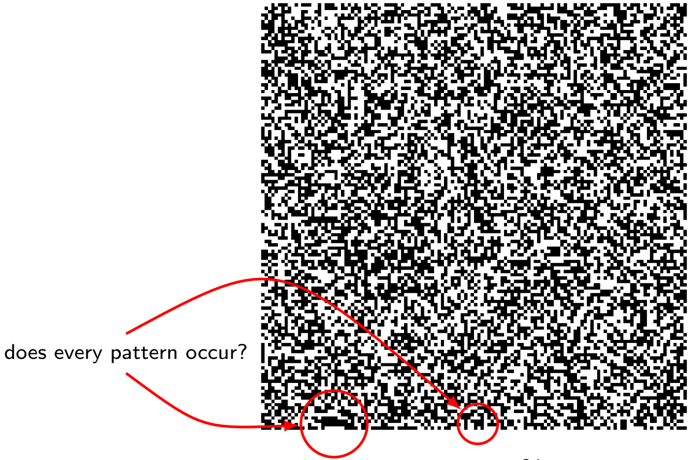

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Theorem (Müllner, Spiegelhofer 2017)

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?? Pseudorandom properties: is $\mathbf{t}(nd+a)$ a good PRNG?

Section 2

Sparse infinite subsequences of Thue–Morse

The Thue–Morse sequence along sparse subsequences

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Theorem (Corollary of Mauduit–Rivat 2005)

Let $1 < c < 1.4$*. There exists a* $\eta > 0$ *such that*

$$\sum_{1 \le n \le x} (-1)^{s(\lfloor n^c \rfloor)} \ll x^{1-\eta}.$$

In particular, for $1 < c < 1.4$, the sequence $n \mapsto \mathbf{t}(\lfloor n^c \rfloor)$ is *simply normal*.

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The Thue–Morse sequence along the sequence of squares is normal: every block $B \in \{0, 1\}^k$ *appears as a subword with asymptotic frequency* $1/2^k$.

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Theorem (Spiegelhofer 2014)

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Theorem (Müllner–Spiegelhofer 2017)

The Thue–Morse sequence along $\lfloor n^c \rfloor$ is normal for $1 < c < 1.5$.

Application: the Thue–Morse sequence along $\lfloor n^c \rfloor$

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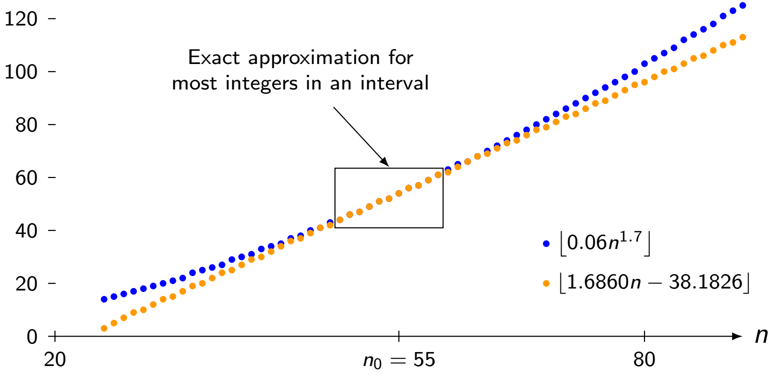
☕ The Thue–Morse sequence along $\lfloor n^c \rfloor$ is normal for $1 < c < 2$.

Connection to our main theorem: linear approximation.



Piatetski-Shapiro via Beatty sequences

Proposition (Spiegelhofer 2014)

We write $f(x) = x^c$, where $1 < c < 2$ is a real number. There exists a constant C such that for all $N \geq 2$ and $K > 0$ we have

$$\left| \frac{1}{N} \sum_{N < n \leq 2N} (-1)^{s(\lfloor n^c \rfloor)} \right| \leq C \left(f''(N) K^2 + \frac{(\log N)^2}{K} + \frac{J(f'(N), K)}{f'(N) K} \right),$$

where

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- The problem is reduced to a Beatty sequence version of our main theorem!
- The proof of this new statement is analogous to the main theorem.

Thank you! [1]

[1] Supported by SFB 55 and the FWF–ANR joint project MuDeRa