The level of distribution of the Zeckendorf sum of digits

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The Zeckendorf expansion

Every nonnegative integer n is the sum of different, non-adjacent Fibonacci numbers F_i and such a representation is unique $\rightsquigarrow$ Zeckendorf expansion.

0	0	0	8	10000	1	16	100100	2
1	1	1	9	10001	2	17	100101	3
2	10	1	10	10010	2	18	101000	2
3	100	1	11	10100	2	19	101001	3
4	101	2	12	10101	3	20	101010	3
5	1000	1	13	100000	1	21	1000000	1
6	1001	2	14	100001	2	22	1000001	2
7	1010	2	15	100010	2	23	1000010	2

- The number of 1s needed is the *Zeckendorf sum of digits* $Z(n)$ of n.
- The Zeckendorf expansion is a generalization of the Fibonacci word, which is given by the lowest digit.

A morphic sequence

We are specifically interested in the function $Z(n)$ modulo 2; the resulting sequence is morphic and can be obtained by the following substitution σ followed by the projection π.

$$\sigma : \left\{ \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & c \\ c & \mapsto & cd \\ d & \mapsto & a \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{ccc} a & \mapsto & 0 \\ b & \mapsto & 1 \\ c & \mapsto & 1 \\ d & \mapsto & 0 \end{array} \right\}.$$

We are interested in the fixed point starting with a. Therefore,

$$(Z(n) \bmod 2)_{n \geq 0} = (01110100100011000101110 0\ldots).$$

Möbius orthogonality

Recall that the Möbius function μ is defined by

$$\mu(n) = \begin{cases} 0, & n \text{ is divisible by a square;} \\ (-1)^k, & n \text{ is the product of } k \text{ different primes.} \end{cases}$$

Theorem (Drmota, Müllner, S. 2017)

$$\sum_{0 \le n < N} \mu(n)(-1)^{Z(n)} = o(N).$$

- This is one of the first examples of a morphic sequence satisfying a *Möbius randomness principle* (cf. Sarnak conjecture).
- One goal in this context is to prove a *prime number theorem* for $Z(n)$ mod 2: half of all prime numbers should have an even Zeckendorf sum of digits.

An easy start: the parity of $Z(n)$

Is the Zeckendorf sum-of-digits of n even for “one half” of the integers? In other words, we want to have

$$\sum_{0 \le n < N} (-1)^{Z(n)} = o(N).$$

- We set
$$S_k = \sum_{0 \le n < F_k} (-1)^{Z(n)}.$$
- These numbers satisfy the simple recurrence relation
$$S_{k+1} = S_k + \sum_{F_k \le n < F_{k+1}} (-1)^{Z(n)} = S_k - S_{k-1},$$
which leads to $S_{k+3} = -S_k$. From this we get the statement after decomposing $[0, N)$ into “Fibonacci intervals” analogous to dyadic intervals.

$Z(n)$ mod 2 on arithmetic progressions

- With a bit more work using trigonometric detection

$$\mathbf{1}_{a+d\mathbb{Z}}(n) = \frac{1}{d} \sum_{0 \le h < d} \exp(2\pi i h (n-a)/d)$$

  we can also obtain

$$\sum_{0 \le n < N} (-1)^{Z(nd+a)} = o(N)$$

  for each given $d \ge 1$ and $a \ge 0$.
- However, the error cannot be uniform in d and a! (van der Waerden's theorem.) $\rightsquigarrow$ we are interested in certain averages — *level of distribution*.

The level of distribution

The following result is work in progress.

Theorem (Drmota, Müllner, S. 2019+)

The Zeckendorf sum-of-digits function modulo 2 *has level of distribution* 1*. More precisely, for all* $\varepsilon > 0$ *we have*

$$\sum_{D \le d < 2D} \max_{\substack{y,z \ge 0 \\ z-y \le x}} \max_{0 \le a < d} \left| \sum_{\substack{y \le n < z \\ n \equiv a \bmod d}} (-1)^{Z(n)} \right| = \mathcal{O}(x^{1-\eta})$$

for some $\eta > 0$ *depending on* ε*, where* $D = x^{1-\varepsilon}$.

This is analogous to the main theorem proved in [S. 2018]: The Thue–Morse sequence (the sum-of-digits function in base 2, modulo 2) has level of distribution 1.

The van der Corput trick

In order to prove this, we use the *van der Corput inequality* as in the work of Mauduit and Rivat on the sum of digits of primes and squares.

Lemma

Let I be a finite interval in $\mathbb{Z}$ containing N integers and let $a_n \in \mathbb{C}$ for $n \in I$. Assume that $R \geq 1$ is an integer. Then

$$\left|\sum_{n\in I} a_n\right|^2 \leq \frac{N+R-1}{R} \sum_{-R<r<R} \left(1-\frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+r\in I}} a_{n+r}\overline{a_n}.$$

$\Rightarrow$ A sum can be estimated using certain correlations.

The reduced problem

- It follows that we have to estimate the correlations

$$S_1 = \sum_{0 \le n < N} (-1)^{Z(nd+a+rd)-Z(nd+a)}.$$

- Addition of *rd* will usually not change the digits above λ, where $F_\lambda \gg RD$.
- Since we take differences (thanks to van der Corput's inequality!) we can replace Z by the *truncated Zeckendorf sum-of-digits function* Z_λ:

$$Z_\lambda(\varepsilon_2 F_2 + \cdots + \varepsilon_\nu F_\nu) = \varepsilon_2 + \cdots + \varepsilon_{\min(\nu,\lambda-1)}$$

for all $\varepsilon_i \in \{0,1\}$ such that $\varepsilon_{i+1} = 1 \Rightarrow \varepsilon_i = 0$.
- It remains to handle

$$S_2 = \sum_{0 \le n < N} (-1)^{Z_\lambda(nd+a+rd)-Z_\lambda(nd+a)}.$$

Reducing the number of significant digits

- The function Z_λ takes λ many digits into account $\rightsquigarrow$ F_λ many combinations.
- In general, the length N of the sum will be much smaller than F_λ (a small power F_λ^ε).
- $\rightsquigarrow$ we want to further reduce the number of significant digits. For this, we use a variant of van der Corput's inequality repeatedly.
- In the i-th application, we eliminate the digits between $\lambda - i\mu$ and $\lambda - (i-1)\mu$; in the process, the number of terms in the exponent of (-1) doubles each time.
- Only a small window $[0, \rho)$ of digits remains.

The Zeckendorf Gowers norm

We obtain the expression

$$S_3 = \sum_{0 \le n < N} \prod_{\varepsilon_0,\ldots,\varepsilon_m \in \{0,1\}} (-1)^{Z_\lambda(nd+a+\varepsilon_0 rd+\varepsilon_1 k_1 d+\cdots+\varepsilon_m k_m d)}$$

This process leads us (after a lot of technicalities) to so-called *Gowers norms* for the Zeckendorf sum-of-digits function: we need to find a nontrivial upper bound for

$$\sum_{\substack{0 \le n < F_\rho \\ 0 \le r_1,\ldots,r_m < F_\rho}} \prod_{\varepsilon \in \{0,1\}^m} (-1)^{Z_\rho(n+\varepsilon\cdot r)}.$$

Such an estimate was provided by Clemens Müllner.

Detection of digits

The key of transforming S_3 to a Gowers norm lies in the detection of Zeckendorf digits in an "analytical" way. Let $k \geq 2$ and $n \geq 0$. For $k \geq 2$ and $n = \sum_{i \geq 2} \varepsilon_i F_i$, we define

$$v_k(n) = \sum_{2 \leq i < k} \varepsilon_i F_i.$$

- This function cuts off the digits with indices $\geq k$.
- In contrast to the base-q analogue, this function is not periodic. Note that v_3 is the Fibonacci word $0100101001001\ldots$.
- We are interested in integers n such that $v_k(n) = u$.

One-dimensional detection of digits

We define ($\varphi = (\sqrt{5}+1)/2$ is the golden mean)

$$A_k = \left[-\frac{1}{\varphi^{k-1}}, \frac{1}{\varphi^k}\right) \quad \text{and} \quad B_k = \left[-\frac{1}{\varphi^{k+1}}, \frac{1}{\varphi^k}\right).$$

Let $p_k(n) = (-1)^k n\varphi$. Moreover, we set

$$R_k(u) = p_k(u) + \begin{cases} A_k, & 0 \le u < F_{k-1} \\ B_k, & F_{k-1} \le u < F_k. \end{cases}$$

The sets $R_k(u) + \mathbb{Z}$ form a partition of $\mathbb{R}$. Moreover,

$$p_k(n) \in R_k\bigl(v_k(n)\bigr) + \mathbb{Z}.$$

In particular, the Zeckendorf digits of n up to k are fixed if and only if $n\varphi \in I + \mathbb{Z}$ for a certain interval I.
For $k = 3$ we rediscover the lowest digit, which is a Sturmian sequence.

One-dimensional detection is not sufficient

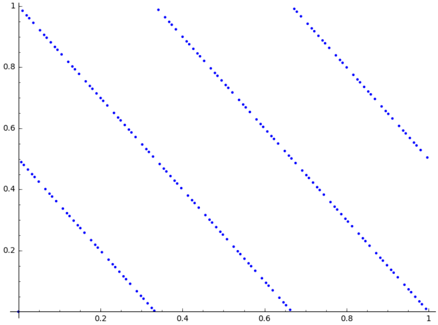
- However, the intervals corresponding to consecutive digit combinations are separated from each other: We have $R_5(0) + \mathbb{Z} \approx [-0.145, 0.090) + \mathbb{Z}$ (corresponding to 000), while $R_5(1) + \mathbb{Z} \approx [0.236, 0.472) + \mathbb{Z}$ (corresponding to 001). In between lies the interval corresponding to 100.
- In order to detect digits between indices $\lambda - i\mu$ and $\lambda - (i-1)\mu$, we would need a union of many small intervals I; $\rightsquigarrow$ too many!
- We therefore work with a two-dimensional detection of Zeckendorf digits.

Two-dimensional detection

We introduce the function $\tilde{p} : \mathbb{N} \to \mathbb{R}^2$ by

$$\tilde{p}_k(n) = \left(\frac{n}{\varphi^k}, \frac{n}{\varphi^{k+1}} \right).$$

The closure of the set of points $\tilde{p}_k(n) \bmod (1,1)$ is a finite set of lines with slope $-F_{k+1}/F_k$. Example for $k = 3$: slope $-3/2$.



Two-dimensional detection of digits

We define parallelograms A_k and B_k:

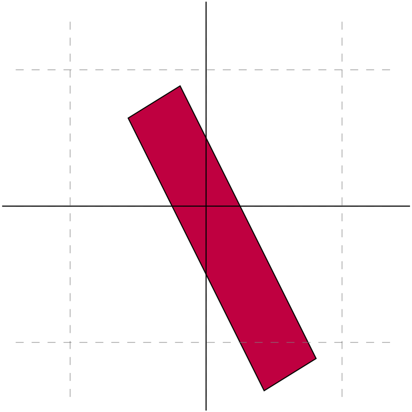
$$\begin{array}{c|c} A_k & B_k \\ -\frac{1}{2} \leq F_{k+1}x + F_k y < \frac{1}{2} & -\frac{1}{2} \leq F_{k+1}x + F_k y < \frac{1}{2} \\ -\varphi < -\frac{1}{\varphi}x + y < 1 & -\frac{1}{\varphi} \leq -\frac{1}{\varphi}x + y < 1. \end{array}$$

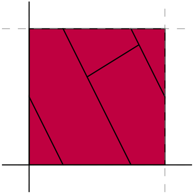
With their help we define regions R_u:

$$R_k(u) = \tilde{p}_k(u) + \begin{cases} A_k, & 0 \leq u < F_{k-1} \\ B_k, & F_{k-1} \leq u < F_k. \end{cases}$$

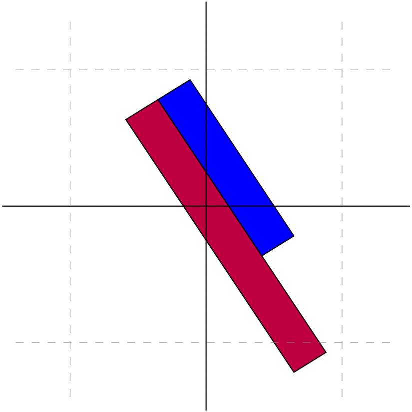
The sets $R_k(u) + \mathbb{Z}^2$ form a partition of $\mathbb{R}^2$ and we have $v_k(n) = u$ if and only if $\tilde{p}_k(n) \in R_k(u) + \mathbb{Z}^2$.

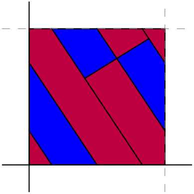
The regions R_u: 0 significant digits





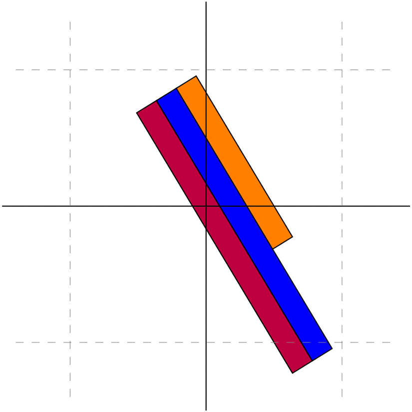
The regions R_u: 1 significant digit

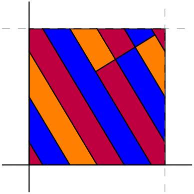




$0, 1$

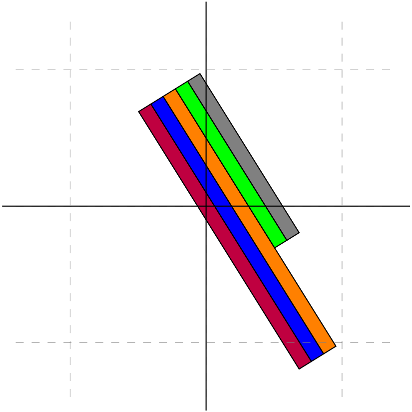
The regions R_u: 2 significant digits

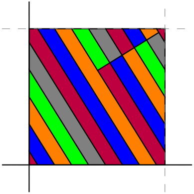




$\color{purple}{00}, \color{blue}{01}, \color{orange}{10}$

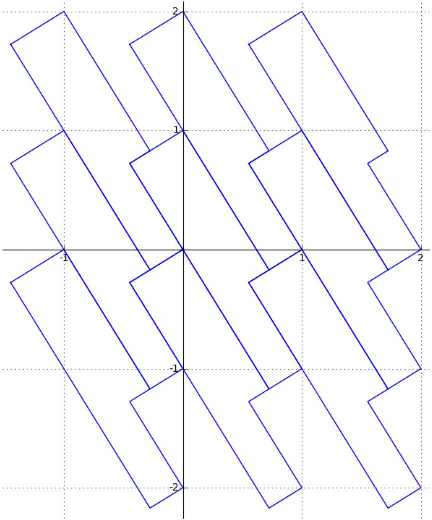
The regions R_u: 3 significant digits





000, 001, 010, 100, 101

The limiting fundamental domain



Applying the detection procedure

- In order to prove our result, we wish to prove uniform distribution of the Zeckendorf digits of multiples of d in $[0,\rho)$ and $[\lambda - i\mu, \lambda - (i+1)\mu)$.
- Combining the two types of digit detection and Fourier approximation of the corresponding regions, we see that it is sufficient to estimate

$$\left\| \frac{h_1}{\varphi^a} + \frac{h_2}{\varphi^{a+1}} + h_3\varphi \right\|$$

  from below, where $a = \lambda - i\mu$.
- There are nontrivial cases where the argument is an integer — powers of φ are not linearly independent over $\mathbb{Q}$.

Applying Binet's formula

- We have the identity

$$F_a\varphi + \frac{(-1)^a}{\varphi^a} = F_{a+1}$$

  for $a \geq 0$, directly from Binet's formula.
- If we have a solution of $\frac{h_1}{\varphi^a} + \frac{h_2}{\varphi^{a+1}} + h_3\varphi = k \in \mathbb{Z}$, this identity yields $(h_1 F_a - h_2 F_{a+1} + h_3)\varphi \in \mathbb{Z}$ and therefore

$$h_1 F_a - h_2 F_{a+1} + h_3 = 0.$$

- We have to exclude these exceptional cases.

The last slide: basic algebraic number theory

- The ring of integers in $\mathbb{Q}(\sqrt{5})$ is given by $\mathbb{Z}[\varphi]$; the norm of $k+\ell\varphi$ is therefore a nonzero integer for $(k,\ell)\neq(0,0)$.
  If $x=\frac{h_1}{\varphi^a}+\frac{h_2}{\varphi^{a+1}}+h_3\varphi$ is not an integer, we obtain, if L is the integer closest to x,

$$\begin{aligned}1&\le\left|\mathcal{N}\left(h_1\varphi+h_2+h_3\varphi^{a+2}-L\varphi^{a+1}\right)\right|\\&=\left|h_1\varphi+h_2+h_3\varphi^{a+2}-L\varphi^{a+1}\right|\underbrace{\left|h_1\overline{\varphi}+h_2+h_3\overline{\varphi}^{a+2}-L\overline{\varphi}^{a+1}\right|}_{\ll\max\{h_1,h_2,h_3\}=M}\end{aligned}$$

- It follows that

$$\|x\|\gg\frac{1}{\varphi^{a+1}M}.$$

Thank you!