Möbius orthogonality and the sum-of-digits function

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Joint work with Michael Drmota, Christian Mauduit and Joël Rivat

TU Vienna

July 10, 2019

Numeration 2019, ESI, Vienna

The Möbius function

- The Möbius μ-function is the inverse of the constant function 1 with respect to Dirichlet convolution:

$$\sum_{d|n} \mu(d) \cdot 1 = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

- More explicitly, if $n = \prod_p p^{\nu_p}$ is the prime factor decomposition of $n \geq 1$, then $\mu(n) = 0$ if $\nu_p > 1$ for some p, and $\mu(n) = (-1)^{\sum_p \nu_p}$ else.
- It is believed to exhibit random-like behaviour; the Riemann hypothesis is equivalent to the statement

$$\sum_{n \leq x} \mu(n) = O\left(x^{1/2+\varepsilon}\right)$$

for all $\varepsilon > 0$ (while no exponent < 1 is known).

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Möbius orthogonality

- Sarnak's conjecture states that a large class of functions f should be *orthogonal* to the Möbius function:

$$\sum_{1\le n\le N} \mu(n)f(n) = o(N)$$

  (f satisfies a *Möbius randomness principle*).
- Let $f : \mathbb{N} \to A$, where $A \subseteq \mathbb{C}$ is a finite set. Such a sequence f is *deterministic* if the number of *factors* (contiguous finite subsequences) of f of length k is bounded by $\exp(o(k))$.
- The term "deterministic" is in fact more general, but we don't go into the details.

Conjecture (Sarnak)

Let $f : \mathbb{N} \to \mathbb{C}$ be a deterministic sequence. Then

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Möbius orthogonality and digitally defined sequences

- It follows from Dartyge–Tenenbaum (2005) that

$$\sum_{1\le n\le N} (-1)^{s_2(pn)-s_2(qn)} = o(N),$$

where s_2 is the binary sum of digits of n and p, q are different odd positive integers.

- Applying the (Bourgain–Sarnak–Ziegler–)Daboussi–Kátai criterion (which we state later), we obtain

$$\sum_{1\le n\le N} \mu(n)\mathbf{t}(n) = o(N),$$

where $\mathbf{t}$ is the *Thue–Morse sequence* defined by $\mathbf{t}(n) = (-1)^{s_2(n)}$.

- C. Müllner (2017) generalized this to all *automatic sequences*, thus verifying Sarnak's conjecture for this class of functions.

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Möbius orthogonality and more digitally defined sequences

- Drmota, Müllner and S. proved that
$$\sum_{n<N} \mu(n)(-1)^{Z(n)} = o(N),$$
where Z is the *Zeckendorf sum-of-digits function*: $Z(n)$ is the minimal number of Fibonacci numbers needed to represent n as their sum.
- We note that the factor complexity p_k of automatic sequences satisfies $p_k \le Ck$ for some C, while $p_k \le C_2k^2$ for *morphic sequences* such as $(-1)^{Z(n)}$. Therefore they are deterministic.
- Possible generalization: *Zeckendorf-automatic sequences*!

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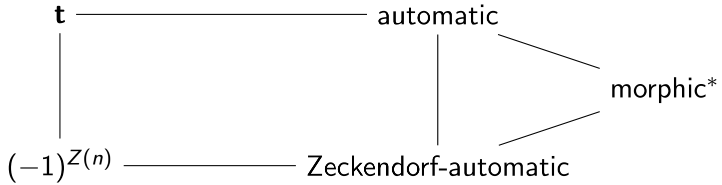

Möbius orthogonality for a non-deterministic sequence

We want to prove the following theorem.

Theorem (Drmota, Mauduit, Rivat, S. 2019+)

$$\sum_{n<N} \mu(n)\mathbf{t}(n^2) = o(N).$$

- The analogous statement for Λ instead of μ is open; this would prove a result on the sum of digits of *squares of primes*.
- This result shows Möbius orthogonality for a non-deterministic sequence: the sequence $n \mapsto \mathbf{t}(n^2)$ has full factor complexity $p_k = 2^k$ (Moshe 2007), in fact it is a *normal sequence* (Drmota, Mauduit, Rivat 2019) and even looks *random* (open).

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First ingredient of the proof

We will use the (Bourgain–Sarnak–Ziegler–)Daboussi–Kátai criterion.

Proposition ((BSZ)DK)

Let $f : \mathbb{N} \to \mathbb{C}$ *be bounded and*

$$\sum_{n \le x} f(pn)\overline{f(qn)} = o(x)$$

for all pairs (p, q) *of distinct primes such that* $p, q > M$. *Then*

$$\sum_{n \le x} \mu(n) f(n) = o(x).$$

- We have to verify this for the function $f(n) = \mathbf{t}(n^2)$, therefore we need to show that

$$\sum_{n \le x} \mathbf{t}(p^2 n^2)\mathbf{t}(q^2 n^2) = o(x). \qquad (1)$$

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Second ingredient of the proof

We are concerned with $g(n) = \mathbf{t}(p^2 n)\mathbf{t}(q^2 n)$ and need to show that $\sum_{n \le x} g(n^2) = o(x)$.

For this, we use Mauduit and Rivat (2019).

Theorem (Corollary of MR2019)

Assume that $h : \mathbb{N} \to \{z \in \mathbb{C} : |z| = 1\}$ satisfies a certain carry property and has uniformly small Fourier coefficients,

$$\frac{1}{2^\lambda} \sum_{0 \le u < 2^\lambda} h(2^\kappa u)\,\mathrm{e}(-ut) \ll 2^{-\eta\lambda}$$

for some $\eta > 0$, uniformly for $t \in \mathbb{R}$ and $\kappa \le c\lambda$. Then

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The carry property for $f(n) = \mathbf{t}(p^2 n)\mathbf{t}(q^2 n)$ is straightforward to verify; it remains to estimate

$$\sup_{t\in\mathbb{R}} \frac{1}{2^\lambda} \sum_{0\le n<2^\lambda} \mathbf{t}(p^2 n)\mathbf{t}(q^2 n)\,\mathrm{e}(-nt).$$

For this, we use a result by Dartyge and Tenenbaum:

Proposition (Corollary of Dartyge–Tenenbaum 2005)

Let p' and q' be different odd positive integers. Then

$$\sum_{x\le n<x+y} \mathbf{t}(p' n)\mathbf{t}(q' n)\,\mathrm{e}(-nt) = O(y^{1-\eta})$$

for some $\eta > 0$, uniformly in t and x.

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Summary

Summarizing:

- Dartyge–Tenenbaum implies that uniformly in t and x,

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- Mauduit–Rivat implies that

$$\sum_{0\le n<N} \mathbf{t}(p^2 n^2)\mathbf{t}(q^2 n^2) = o(N).$$

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A generalization: b-multiplicative sequences

We also want to prove Möbius orthogonality for the sequence $g(n^2)$, where g is *strictly b-multiplicative*. Such a function g is of the form

$$g(0) = 1 \quad \text{and} \quad g(\varepsilon_0 b^0 + \cdots + \varepsilon_\nu b^\nu) = g(\varepsilon_0) \cdots g(\varepsilon_\nu). \qquad (2)$$

That is, each digit $\neq 0$ gets assigned a *weight*, and these weights are multiplied. The Thue–Morse sequence is the function g satisfying (2) for $b = 2$ and $g(1) = -1$.

Theorem (DMRS 2019+)

Let $b \geq 2$ be an integer and g a strictly b-multiplicative function. Then

$$\sum_{n \leq N} \mu(n) g(n^2) = o(N).$$

Note that $g(n) = 1$ is strictly b-multiplicative, and the statement degenerates into $\sum_{n \leq N} \mu(n) = o(N)$, which is the prime number theorem.

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Generalizing Dartyge–Tenenbaum

- In fact, the "trivial" part of the theorem concerns b-multiplicative functions g satisfying $g(n) = \mathrm{e}(\alpha s_q(n))$, where $\alpha(b-1) \in \mathbb{Z}$ (we write $\mathrm{e}(x) = \exp(2\pi i x)$). This is the periodic case, since $s_b(n) \equiv n \bmod b-1$ ("preuve par neuf") $\rightsquigarrow$ prime number theorem in arithmetic progressions.
- We need to generalize Dartyge–Tenenbaum: for distinct positive integers p', q' not divisible by b (in fact squares of large different primes are sufficient), we have to show

$$\sum_{n \le y} g(p'n) g(q'n)\, \mathrm{e}(-nt) = O(y^{1-\eta})$$

for some $\eta > 0$, uniformly in t.

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Different bases

Drmota, Mauduit and Rivat (submitted) proved in particular the following result on the sum-of-digits function s_b in two different bases.

Theorem (Drmota, Mauduit, Rivat 2019+)

Assume that $b_1, b_2 \geq 2$ *are coprime, and* $\alpha_1, \alpha_2 \in \mathbb{R}$ *such that* $\alpha_1(b_1 - 1) \notin \mathbb{Z}$ *and* $\alpha_2(b_2 - 1) \notin \mathbb{Z}$*. Then*

$$\sum_{n<N} \mu(n)\,\mathrm{e}(\alpha_1 s_{b_1}(n) + \alpha_2 s_{b_2}(n)) = o(N)$$

and

$$\sum_{n<N} \Lambda(n)\,\mathrm{e}(\alpha_1 s_{b_1}(n) + \alpha_2 s_{b_2}(n)) = o(N).$$

Here Λ is the von Mangoldt function, defined by $\Lambda(p^k) = \log p$ for primes p and integers $k \geq 0$, and $\Lambda(n) = 0$ if n contains two different primes in its prime factor decomposition.

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deterministic for rational α_1, α_2

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Sums of type I and II

- The method of proof of Drmota–Mauduit–Rivat uses sums of type I and II: In order to bound the sum $\sum_n \mu(n)F(n)$, it is sufficient to estimate certain sums

$$\sum_m \max_I \left| \sum_{n\in I} F(mn) \right| \quad \text{(type I)}$$

and

$$\sum_m \sum_n a_m b_n F(mn) \quad \text{(type II)}.$$

- DMR's proof is not sufficient to handle three or more bases.

Excursus: the level of distribution

Theorem (S. 2019+)

The Thue–Morse sequence has level of distribution 1*. More precisely, for all* $\varepsilon > 0$ *we have*

$$\sum_{M \le m < 2M} \max_{\substack{y,z \ge 0 \\ z-y \le x}} \max_{0 \le a < d} \left| \sum_{\substack{y \le n < z \\ n \equiv a \bmod m}} (-1)^{s_2(n)} \right| = O(x^{1-\eta})$$

for some $\eta > 0$ *depending on* ε*, where* $M = x^{1-\varepsilon}$*.*

- This is similar to a sum of type I, allowing m to be a large power of n.
- This improvement on sums of type I simplifies the treatment of sums of type II! (cf. e.g. Heath–Brown's identity)

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An approach to the problem announced in the abstract

We plan to extend the theorem prove a result on the level of distribution of

$$\mathrm{e}(\vartheta_1 s_{b_1}(n) + \cdots + \vartheta_k s_{b_k}(n)),$$

which is of intrinsic interest; via simplified sums of type II this might lead to a proof of the statements

$$\sum_{n \le N} \mu(n)\,\mathrm{e}(\vartheta_1 s_{b_1}(n) + \cdots + \vartheta_k s_{b_k}(n)) = o(N)$$

and

$$\sum_{n \le N} \Lambda(n)\,\mathrm{e}(\vartheta_1 s_{b_1}(n) + \cdots + \vartheta_k s_{b_k}(n)) = o(N).$$

Thank you!