Primes as sums of Fibonacci numbers, II

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Joint work with Michael Drmota and Clemens Müllner (TU Wien)

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We want to sketch the proof of the following theorem.

Proposition 1 (Drmota, Müllner, S. 2022+)

There exist constants $c > 0$ and C such that

$$\sum_{p \le x} \mathrm{e}\big(\vartheta \mathrm{z}(p)\big) \le C(\log x)^4 x^{1-c\|\vartheta\|^2}$$

for all real ϑ and $x \ge 2$.

- p is a prime,
- $\mathrm{e}(x) = \exp(2\pi i\, x)$,
- $\mathrm{z}(n)$ is the minimal number of Fibonacci numbers needed to write n as their sum,
- $\|\vartheta\|$ is the distance of ϑ to $\mathbb{Z}$.

The sum over primes can be rewritten, using summation by parts (e.g. Mauduit–Rivat 2010, Ann. of Math.):

$$\sum_{p\le N} \mathrm{e}\bigl(\vartheta \mathrm{z}(p)\bigr) \le \frac{2}{\log N} \max_{t\le N} \left| \sum_{n\le t} \mathrm{e}\bigl(\vartheta \mathrm{z}(n)\bigr)\Lambda(n) \right| + O\bigl(\sqrt{N}\bigr),$$

Von Mangoldt function:

$$\Lambda(n) = \begin{cases} \log p, & n = p^k \text{ for some } k \ge 1 \text{ and some prime } p, \\ 0, & \text{otherwise.} \end{cases}$$

Note. The prime number theorem asserts that $\sum_{n<x} \Lambda(n) \sim x$.

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Vaughan's identity leads to the starting point of our method.

Lemma (cf. Davenport, Multiplicative number theory)

Let $f:\mathbb{N}\to\mathbb{C}$ such that $|f(n)|\le 1$ for all $n\ge 1$. For all $N,U,V\ge 2$ such that $UV\le N$ we have

$$\sum_{n\le N} f(n)\Lambda(n) \ll U + (\log N)\sum_{t\le UV}\max_{w}\left|\sum_{w\le r\le N/t} f(rt)\right|$$

$$+\sqrt{N}(\log N)^3 \max_{\substack{U\le M\le N/V\\ V\le q\le N/M}}\left(\sum_{V<p\le N/M}\left|\sum_{\substack{M<m\le 2M\\ m\le\min(N/p,N/q)}} f(mp)\overline{f(mq)}\right|\right)^{1/2},$$

with an absolute implied constant.

Let us write

$$S_{\mathrm{I}}(N,U,V) := \sum_{t\le UV} \max_{w} \left| \sum_{w\le r\le N/t} f(rt) \right|,$$

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Good control over S_{I} will simplify the treatment of S_{II}.

Choose $U = N^{3/4}$ (and V small).

- S_{I}, inner sum: the difference t is usually *large* ($\gg N^{3/4-\varepsilon}$) compared to the length of the sum ($\ll N^{1/4+\varepsilon}$).
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We establish a strong estimate for type-I sums —

"the Zeckendorf sum-of-digits function has level of distribution 1".

Theorem (DMS 2022+)

Let $\varepsilon > 0$. There exist $c_1 = c_1(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ such that for all $\vartheta \in \mathbb{R}$ and all real $x \geq 2$,

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} \mathrm{e}\bigl(\vartheta \mathrm{z}(n)\bigr) \right| \le C\,(\log x)^{11/4}\, x^{1-c_1\|\vartheta\|^2},$$

where $D = x^{1-\varepsilon}$.

This is a statement on z along *very sparse finite arithmetic progressions*

$$A + \bigl(0, d, 2d, 3d, \ldots, (N-1)d\bigr),$$

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Sketch of proof of S2020.

- Apply the van der Corput inequality repeatedly in order to obtain *higher order correlations*.
- Estimate

$$\sum_{\substack{0\le n<2^\rho\\ 0\le r_1,\ldots,r_m<2^\rho}} \mathrm{e}\left(\frac{1}{2}\sum_{\varepsilon\in\{0,1\}^m} \mathrm{s}_2^{(\rho)}(n+\varepsilon\cdot r)\right),$$

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Eliminating digits

Consider a digital expansion along a sparse arithmetic progression:

in each step, *many digits* change!

Each application of van der Corput's inequality "eliminates" digits with indices in a certain interval $[A, B)$.

It is easy to detect base-q digits $\delta_j(n)$ with indices in $[A, B)$: we have

$$\big(\delta_A(n), \ldots, \delta_{B-1}(n)\big) = (\nu_A, \ldots, \nu_{B-1}) \quad \text{if and only if} \quad \left\{\frac{n}{q^B}\right\} \in J,$$

where

$$J = \left[\frac{m}{q^{B-A}}, \frac{m+1}{q^{B-A}}\right) \quad \text{and} \quad m = \sum_{A \le j < B} \nu_j q^{j-A}.$$

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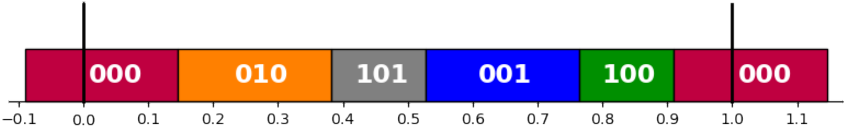
The Zeckendorf case: digit detection

Ostrowski expansion, $\varphi = \frac{\sqrt{5}+1}{2}$: The Zeckendorf digits of n with indices below B are equal to prescribed values if and only if

$$n\varphi$$

is contained in a certain interval modulo 1.



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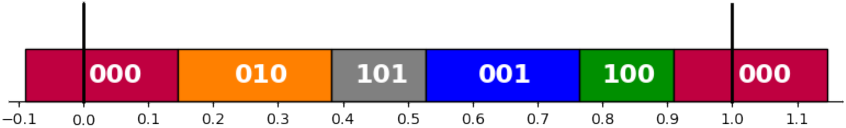

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Wrap

Wrap this around the two-dimensional torus in such a way that adjacent digit combinations (w.r.t. the lexicographical ordering) lie "parallel to each other" (illustration in a moment): set

$$p(n) = \left(\frac{n}{\varphi^B}, \frac{n}{\varphi^{B+1}} \right).$$

The closure of the set of points $p(n) \bmod 1 \times 1$ is a union of finitely many line segments, since

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The closure of the set of points $p(n)$ mod 1×1 is a union of finitely many line segments, since

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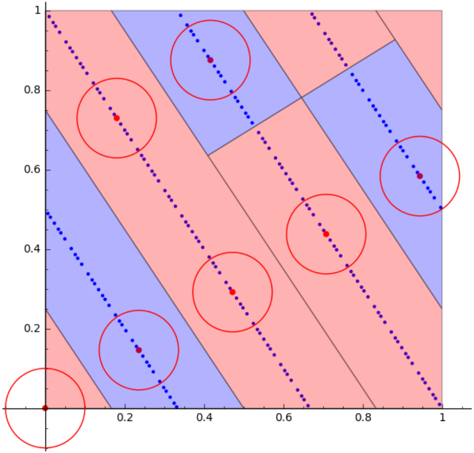
Detecting the lowest Zeckendorf digit: $B = 3$



The least significant digit is given by the Fibonacci word

$$F = (0100101001001010010100100101001001\cdots),$$

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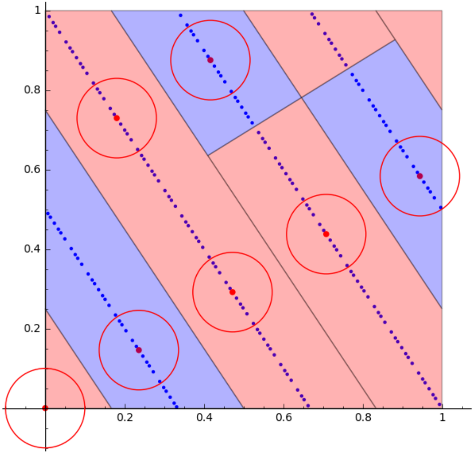

0
1
10
100
101
1000
1001

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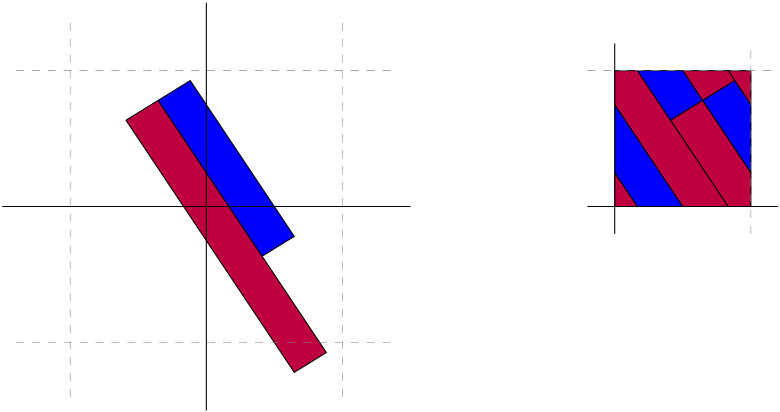
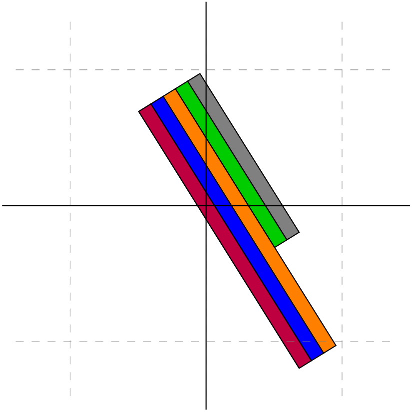
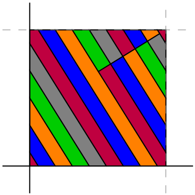


0, 1

Three significant digits





000, 001, 010, 100, 101

no longer separated!

The Zeckendorf digits of n with indices in $[A, B)$ are equal to prescribed values $\omega_A, \ldots, \omega_{B-1}$ if and only if

$$\left(\frac{n}{\varphi^B}, \frac{n}{\varphi^{B+1}}\right)$$

is contained in $Q + \mathbb{Z}^2$, where Q is a certain parallelogram depending on the values ω_j. $\rightsquigarrow$ trigonometric approximation of Q!

Together with the following lemma, we may eliminate Zeckendorf digits in our sum $\sum_n \mathrm{e}(\vartheta \mathrm{z}(nd + a))$.

Lemma (generalized vdC inequality)

Let I be a finite interval in $\mathbb{Z}$ containing M integers and $x_m \in \mathbb{C}$ for $m \in I$. Assume that $K \subset \mathbb{N}$ is a finite nonempty set. Then

$$\left|\sum_{m \in I} x_m\right|^2 \leq \frac{M + \max K - \min K}{|K|^2} \sum_{(k,k') \in K^2} \sum_{m \in I \cap (I-k+k')} x_m \overline{x_{m+k-k'}}.$$

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. . . . . .

Generalizations

For the (Ostrowski) α-sum-of-digits function s_α, we expect that the method generalizes, at least for the case that α is an algebraic number. Algebraicity is probably needed for multi-dimensional detection.

More interestingly, we are interested in *morphic sequences*: take a fixed point of a substitution over a finite alphabet, and possibly rename the letters afterwards. Examples: the Thue–Morse sequence defined by

$$\sigma : 0 \mapsto 01, \quad 1 \mapsto 10,$$

or the Zeckendorf sum-of-digits function modulo 2,

$$\sigma : \left\{ \begin{array}{lll} \mathtt{a} & \mapsto & \mathtt{ad} \\ \mathtt{b} & \mapsto & \mathtt{a} \\ \mathtt{c} & \mapsto & \mathtt{cb} \\ \mathtt{d} & \mapsto & \mathtt{c} \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{lll} \mathtt{a} & \mapsto & \mathtt{0} \\ \mathtt{b} & \mapsto & \mathtt{0} \\ \mathtt{c} & \mapsto & \mathtt{1} \\ \mathtt{d} & \mapsto & \mathtt{1} \end{array} \right\}.$$

Automatic sequences were treated by Müllner (Duke Math. J. 2017), but the general case (morphic sequences) is wide open.

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Automatic sequences were treated by Müllner (Duke Math. J. 2017), but the general case (morphic sequences) is wide open.

Generalizations

For the (Ostrowski) α-sum-of-digits function s_α, we expect that the method generalizes, at least for the case that α is an algebraic number. Algebraicity is probably needed for multi-dimensional detection.

More interestingly, we are interested in *morphic sequences*: take a fixed point of a substitution over a finite alphabet, and possibly rename the letters afterwards. Examples: the Thue–Morse sequence defined by

$$\sigma : \mathtt{0} \mapsto \mathtt{01}, \quad \mathtt{1} \mapsto \mathtt{10},$$

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Linear recurrent number systems

The *Tribonacci* numeration system is based on the Tribonacci numbers

$$\begin{gathered} a_n = a_{n-1} + a_{n-2} + a_{n-3}, \\ a_0 = a_1 = 0, a_2 = 1, \quad \text{that is,} \\ (a_n)_{n \ge 3} = (1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots). \end{gathered}$$

Every natural number is the unique sum of pairwise different a_n, $n \ge 3$, where taking three consecutive numbers is forbidden.
The lowest two Tribonacci digits exhibit a close connection to the Tribonacci word

$$\sigma : \left\{ \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 02 \\ 2 & \mapsto & 0 \end{array} \right\}.$$

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Tribonacci sum of digits

Meanwhile, the Tribonacci sum-of-digits function modulo 2 appears to be given by

$$\sigma : \left\{ \begin{array}{lll} \mathtt{a} & \mapsto & \mathtt{ae} \\ \mathtt{b} & \mapsto & \mathtt{af} \\ \mathtt{c} & \mapsto & \mathtt{a} \\ \mathtt{d} & \mapsto & \mathtt{db} \\ \mathtt{e} & \mapsto & \mathtt{dc} \\ \mathtt{f} & \mapsto & \mathtt{d} \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{lll} \mathtt{a} & \mapsto & \mathtt{0} \\ \mathtt{b} & \mapsto & \mathtt{0} \\ \mathtt{c} & \mapsto & \mathtt{0} \\ \mathtt{d} & \mapsto & \mathtt{1} \\ \mathtt{e} & \mapsto & \mathtt{1} \\ \mathtt{f} & \mapsto & \mathtt{1} \end{array} \right\}.$$

$$\mathtt{tr} = \mathtt{0110100100101100101101101001011011010011010}\cdots$$

Detecting a given letter in the Tribonacci word leads to the (classical) two-dimensional *Rauzy fractal*.

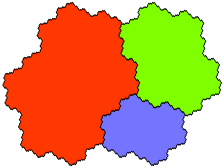

Figure: Jolivet, Loridant, Luo 2014

For detecting Tribonacci digits with indices in $[A, B)$, we will have to consider three-dimensional *cylinders* with the Rauzy fractal as base!

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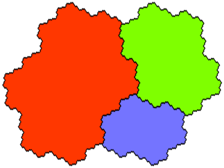

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Thank you!