Thue–Morse along the sequence of cubes

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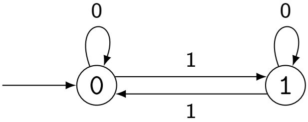
Section 1

Thue–Morse [tʉː mɔːrs]

The Thue–Morse sequence $\mathbf{t}$ is the fixed point of the substitution

$$0 \mapsto 01, \quad 1 \mapsto 10$$

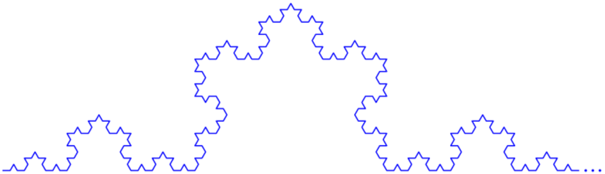
that starts with 0.
It is given by the binary sum-of-digits function s, reduced modulo 2.



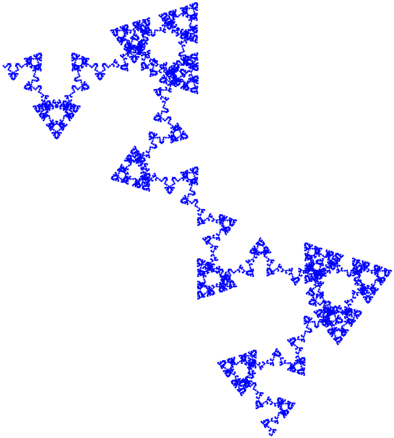
$$\mathbf{t} = 01101001100101101001011001101001 \cdots$$

Thue–Morse $\rightleftharpoons$ Koch

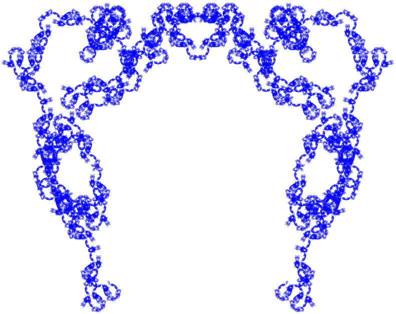
The sequence $n \mapsto (-1)^{s(n)}\, \mathrm{e}(-n/3)$ describes the orientation of the nth segment in the "unscaled Koch (snowflake) curve" (where $\mathrm{e}(x) = e^{2\pi i x}$):



The sum of digits along arithmetic progressions



$\mathrm{e}\left(\frac{1}{2}s(3n) - n/5\right)$



$\mathrm{e}\left(\frac{2}{5}s(3n) - n/5\right)$

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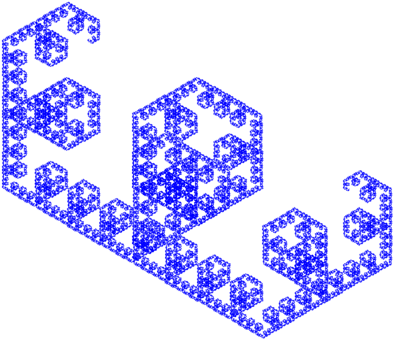
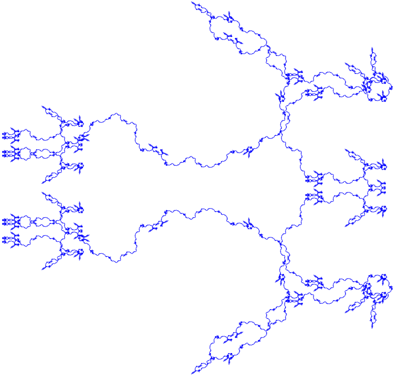


$\mathrm{e}(s(3n)/3)(-1)^n$



$\mathrm{e}(s(7n)/3)(-1)^n$

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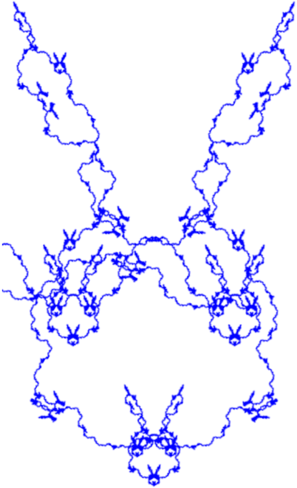

$\mathrm{e}(s(7n)/3)(-1)^n$, closeup

Every finite sequence $\omega \in \{0,1\}^L$ appears as an arithmetic subsequence of $\mathbf{t}$: the Thue–Morse word has full *arithmetical complexity* (Avgustinovich–Fon-Der-Flaass–Frid 2003, Müllner–Spiegelhofer 2017, Konieczny–Müllner 2023+).

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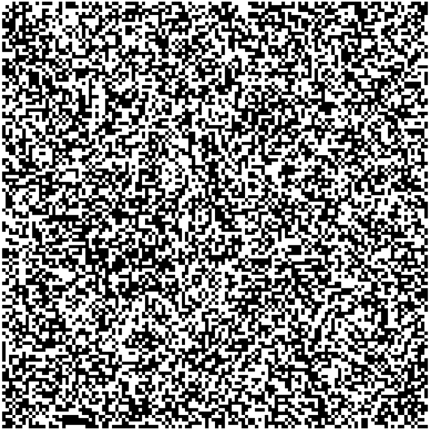

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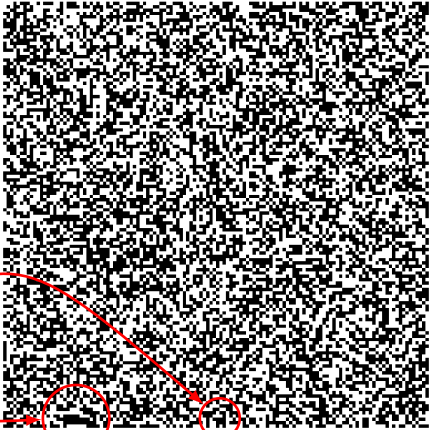
every pattern occurs



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Informal question

Let $A \gg N^R$, and assume that A contains many blocks of 1*s in binary. Is*

$$P : \{0, \ldots, N\} \rightarrow \{0, 1\}, \quad n \mapsto \mathbf{t}(nA + B)$$

a good pseudorandom number generator?

Gelfond's third problem

Let $S = s_q$ be the sum-of-digits function in base $q \geq 2$.

> *Finalement, signalons comme problème à résoudre l'estimation du nombre des valeurs du polynôme $P(t)$ ne prenant que des valeurs entières sur l'ensemble [...] des entiers rationels, pour lesquelles on a $S[P(n)] \equiv \ell \bmod m$.*

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That is, if P is a polynomial such that $P(\mathbb{N}) \subseteq \mathbb{N}$, we are interested in

$$A(q, P, m, \ell, x) := \#\{n < x : s_q(P(n)) \equiv \ell \bmod m\}.$$

Partial results

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- Lower bounds for the numbers $A(q, P, m, \ell, x)$ are known (Dartyge–Tenenbaum 2006; Stoll 2012);
- For "sufficiently large bases" q coprime to the leading coefficient of P, and $\gcd(q-1, m) = 1$, the equivalence $A(q, P, m, \ell, x) \sim x/m$ has been proved (Drmota–Mauduit–Rivat 2011);
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Generalizations

The Thue–Morse sequence along n^2 is normal (Drmota–Mauduit–Rivat): each finite sequence over $\{0, 1\}$ of length L appears with frequency 2^{-L} along $\mathbf{t}(n^2)$.

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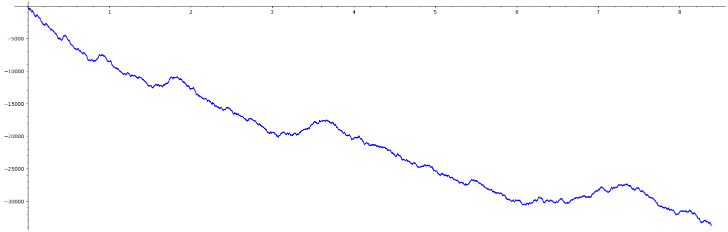
Partial sums of $\mathbf{t}\big(n^2\big)$ for $x < 2^{23}$:



A *drift* appears to be present. How is this related to the fact that n^2 avoids $2 + 3\mathbb{Z}$?

Conjecture ( )

There exist real numbers c and η, and a 1-periodic, continuous, nowhere differentiable function Φ, such that

$$\sum_{n<x} \mathbf{t}\big(n^2\big) \sim c x^{\eta} \Phi\big(\log x / \log 2\big).$$

The main result

Theorem (S. 2023+)

There exist real numbers $c > 0$ and C such that for all $x \geq 1$,

$$\left| \#\left\{ n < x : \mathbf{t}\left(n^3\right) = 0 \right\} - \frac{x}{2} \right| \leq C x^{1-c}. \tag{1}$$

Section 2

Sketch of the proof

- We are interested in the sum

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- This is standard.

S. 2020

In the paper (Compos. Math. 2020) we apply van der Corput's inequality repeatedly in order to eliminate blocks of digits, piece by piece.

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In this way, a statement on very sparse arithmetic subsequences of **t** could be derived. These progressions have length $\asymp N$, while their common difference is $\asymp N^R$, where $R > 0$ is arbitrary!

But: iterated van der Corput could so far not be used for removing sufficiently many digits of polynomial values, if $\deg P > 1$.

$$s_2\bigl(n^3\bigr) - s_2\bigl((n+r)^3\bigr) - s_2\bigl((n+s)^3\bigr) + s_2\bigl((n+r+s)^3\bigr)$$

A trivial decomposition

We write

$$n = 2^\rho n_1 + n_0,$$

where $3\rho \ge \lambda$ and $n_0 < 2^\rho$. The variable n_0 is treated as a parameter. Expanding $n^3 \bmod 2^\lambda$, we see that the cubic term in n_1 disappears.

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! In the actual proof, the elimination of the digits in the critical interval $[2\rho, \lambda)$ comes first.

The critical interval of digits

For a subset $J \subseteq \mathbb{N}$, let s^J denote the *restricted binary sum-of-digits function*: only digits with indices in J are counted. We write

$$S_0 = \sum_{0 \le j < 2^\kappa} (-1)^{s_2(j)} \sum_{n < 2^\nu} \mathrm{e}\Bigl(\tfrac{1}{2} s^{\mathbb{N} \setminus [2\rho, \lambda)}\bigl(n^3\bigr)\Bigr) \left[\!\left[\frac{n^3}{2^\lambda} \in \left[\frac{j}{2^\kappa}, \frac{j+1}{2^\kappa}\right) + \mathbb{Z} \right]\!\right].$$

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(2) The "prepared" set $\mathbb{N} \setminus [2\rho, \lambda)$ will lead to a linear sum-of-digits problem after cutting away the digits with indices $\ge \lambda$ (as above);

(3) The rightmost factor is approximated by a trigonometric polynomial, evaluated at $n^3/2^\lambda$.

Even sketchier idea of the proof

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- The linear trigonometric polynomial in n_1 is *decoupled* from the sum over n, using suitable arithmetic subsequences and summation by parts.
- The sum over h together with the decoupled exponential term yields a geometric sum

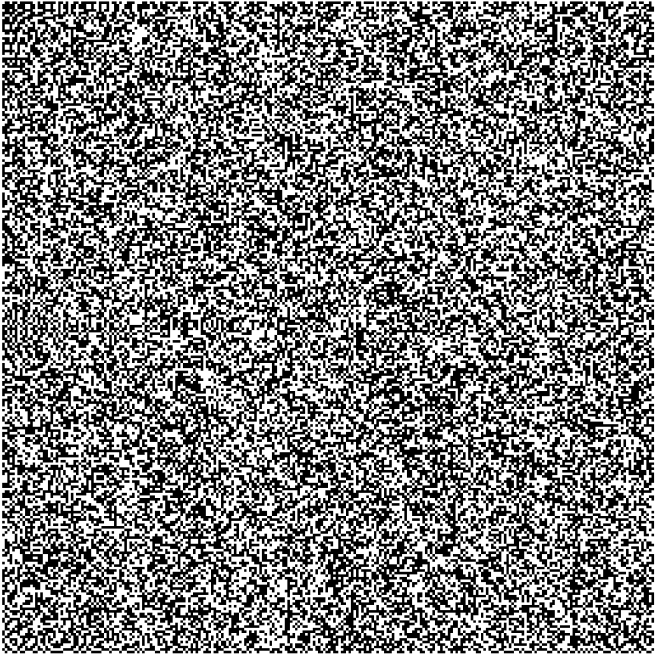
$$\sum_{0 \le h < H} \mathrm{e}(hx) \ll \min\Bigl(H, \|x\|^{-1}\Bigr),$$

where $\|x\|$ is the distance of x to the nearest integer.

This is only logarithmic in mean (over x)!

Essence of the proof

Summarizing, the additional sum introduced for digit detection in the critical interval only contributes a logarithm. A linear digital problem remains, for which there are methods available.



THANK YOU!

M. Drmota, C. Mauduit, and J. Rivat, *Normality along squares*, J. Eur. Math. Soc, 21 (2019), pp. 507–548.

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C. Mauduit and J. Rivat, *La somme des chiffres des carrés*, Acta Math., 203 (2009), pp. 107–148.

L. Spiegelhofer, *The level of distribution of the Thue–Morse sequence*, Compos. Math., 156 (2020), pp. 2560–2587.

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Preprint, http://arxiv.org/abs/2308.09498.

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van der Corput's inequality

Lemma

Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left|\sum_{n\in I} a_n\right|^2 \leq \frac{N+K(R-1)}{R} \sum_{|r|<R} \left(1-\frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+Kr\in I}} a_{n+Kr}\overline{a_n}.$$

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Instead of the original sum, we now have to estimate certain correlations (where KR will be small compared to N).