Subsequences of digitally defined functions

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Outline

Section 1

Digital expansions

In the simplest case, a *digital expansion* Φ assigns to each natural number a finite string of *digits* in an injective manner.

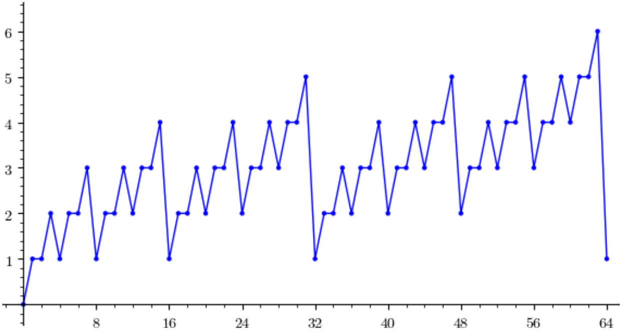
n	$\Phi(n)$	n	$\Phi(n)$	n	$\Phi(n)$
0	0	8	1000	16	10000
1	1	9	1001	17	10001
2	10	10	1010	18	10010
3	11	11	1011	19	10011
4	100	12	1100	20	10100
5	101	13	1101	21	10101
6	110	14	1110	22	10110
7	111	15	1111	23	10111

Figure: The binary expansion

Let $s_2(n)$ be the number of 1s in the binary expansion of n.
The sequence s_2 is a fixed point of the substitution defined by

$$n \mapsto (n, n+1), \quad n \geq 0.$$

$$s_2 = 011212231223233412232334233434 45\cdots$$



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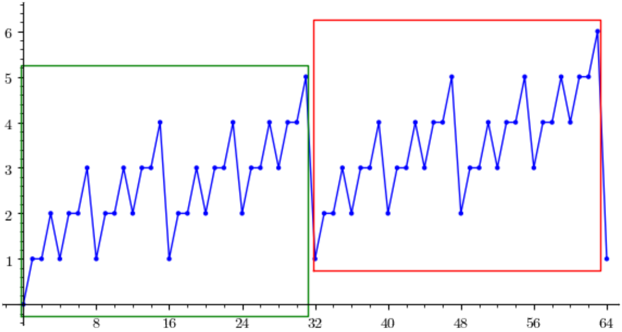

The Thue–Morse sequence

Let $\mathsf{t}(n) = \mathsf{s}_2(n) \bmod 2$.

$$\mathsf{t} = 01101001 \cdots .$$

This sequence is an *automatic sequence*, and a fixed point of

$$0 \mapsto 01, \quad 1 \mapsto 10.$$

Automatic sequences are given by deterministic finite automata with output (DFAO), where the input is the base-q expansion of integers.

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2-automatic sequences

Each subsequence $n \mapsto \mathrm{t}(An + B)$ is an automatic sequence.

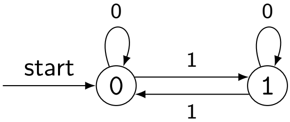

Figure: An automaton for $\mathrm{t}(n)$

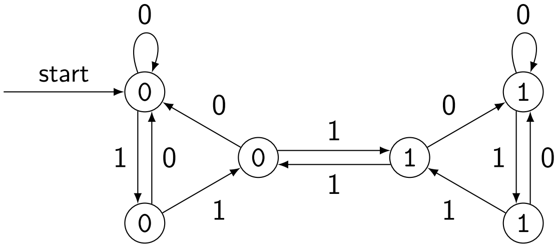

Figure: An automaton for $\mathrm{t}(3n)$

Section 2

Sparse subsequences

Few big steps

General idea: along sparse subsequences, t behaves “randomly”. For example, every finite sequence on $\{0, 1\}$ appears as an arithmetic subsequence of t (Avgustinovich–Fon-Der-Flaass–Frid 2003, Müllner–Spiegelhofer 2017 (“correct” rate of appearance), Konieczny–Müllner 2023+ (general automatic sequences)).

Very sparse arithmetic subsequences of t

The Thue–Morse sequence has level of distribution 1.

Theorem (S. 2020, Compos. Math.)

For all $\varepsilon > 0$ we have

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} (-1)^{\mathsf{s}_2(n)} \right| \le C x^{1-\eta}$$

for some C and $\eta > 0$ depending on ε, where $D = x^{1-\varepsilon}$.

In more relaxed language: let $R > 0$. As $N \to \infty$, the following holds.

> *Most $d \asymp N^R$ have the property that for all a, the number*
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> $$\#\{0 \le n < N : \mathrm{t}(nd + a) = 0\}$$
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Drmota–Müllner–S. 2023+

Together with Drmota and Müllner, we proved a theorem on the subsequence indexed by the sequence of primes. Let $\mathsf{z}(n)$ be the smallest k such that n is the number of k Fibonacci numbers.

Theorem (Drmota–Müllner–S., to appear in Mem. Amer. Math. Soc.)

- *The sequence* $n \mapsto \exp(2\pi i \vartheta \mathsf{z}(n))$ *has level of distribution* 1.
- *For* $m \geq 1$ *and* $a \in \mathbb{Z}$, *we have*

  $$\{p < x : p \textit{ prime}, \mathsf{z}(p) \equiv a \bmod m\} \sim \frac{\pi(x)}{m}$$

  as $x \to \infty$.
- *For* k *large enough, there exists a prime number* p *that is the sum of exactly* k *different, non-consecutive Fibonacci numbers.*

S. 2023+

The recent manuscript "Thue–Morse along the sequence of cubes" contains a proof the following theorem.

Theorem (S. 2023+)

As $N \to \infty$, we have

$$\frac{1}{N}\#\big\{n < N : \mathrm{t}\big(n^3\big) = 0\big\} \to \frac{1}{2}.$$

Informal open questions

- Prove that any finite word ω over $\{0,1\}$ appears in $\mathtt{t}$ about the expected number of times, for most very sparse arithmetic progressions.
- Prove prime number theorems for more general *morphic sequences*, such as

$$\sigma : \left\{ \begin{array}{lllllllll} \mathtt{a} & \mapsto & \mathtt{ae}, & \mathtt{b} & \mapsto & \mathtt{af}, & \mathtt{c} & \mapsto & \mathtt{a} \\ \mathtt{d} & \mapsto & \mathtt{db}, & \mathtt{e} & \mapsto & \mathtt{dc}, & \mathtt{f} & \mapsto & \mathtt{d} \end{array} \right\},$$

$$\pi(\mathtt{a}) = \pi(\mathtt{b}) = \pi(\mathtt{c}) = 0,$$
$$\pi(\mathtt{d}) = \pi(\mathtt{e}) = \pi(\mathtt{f}) = 1$$

The projection under π of the fixed point starting with $\mathtt{a}$ is

$$\mathtt{tr} = 01101001001011001011011010010110110100110 10\cdots.$$

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More open questions

- Complete the treatment of the *third Gelfond problem (1967/1968)*: consider the sum-of-digits function along arbitrary polynomials f such that $f(\mathbb{N}) \subseteq \mathbb{N}$.
- Prove a level-of-distribution result for arbitrary automatic sequences.

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Section 3

Long arithmetic subsequences

Digital expansions and addition

how does the sum of digits of an integer change when a constant d is added repeatedly?

Differences along an arithmetic progression $a + d\mathbb{N}$:

$$\delta(d,a,j) := \lim_{N\to\infty} \frac{1}{N} \#\{n < N : \mathsf{s}_2((n+1)d+a) - \mathsf{s}_2(nd+a) = j\}.$$

For all a, this value is in fact identical to

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Cusick's conjecture

When traversing an infinite arithmetic subsequence of s_2, how often does the value weakly increase? This is the subject of *Cusick's conjecture*.

Conjecture (Cusick)

For all $d \geq 0$, we have

$$c_d > 1/2,$$

where

$$\begin{aligned} c_d &= \lim_{N\to\infty} \frac{1}{N} \#\bigl\{n < N : \mathsf{s}_2(n+d) \geq \mathsf{s}_2(n)\bigr\} \\ &= \sum_{j\geq 0} \delta(d,j). \end{aligned}$$

SW2023

Let $M = M(d)$ be the number of maximal blocks of 1s in the binary expansion of d.

Theorem (S.–Wallner 2023, Ann. Sc. norm. super. Pisa - Cl. sci.)

Let $d \geq 1$. If $M(d)$ is larger than some absolute, effective constant M_1, then $c_d > 1/2$.

SW2023, part II

Again, let $M = M(d)$ be the number of blocks of 1s in d.

Theorem (S.–Wallner 2023)

Set $\kappa(1) = 2$, *and for* $d \ge 1$ *let* $\kappa(2d) = \kappa(d)$, *and*

$$\kappa(2d+1) = \frac{\kappa(d) + \kappa(d+1)}{2} + 1.$$

Then

$$\delta(j,d) = \frac{1}{\sqrt{2\pi\kappa(d)}} \exp\left(-\frac{j^2}{2\kappa(d)}\right) + \mathcal{O}\left(\frac{(\log M)^4}{M}\right)$$

for all integers j. *The implied constant is absolute.*

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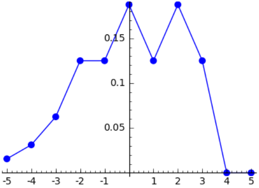

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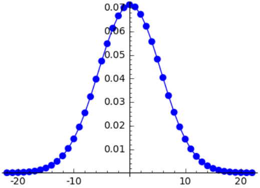

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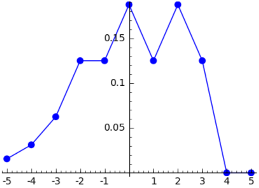

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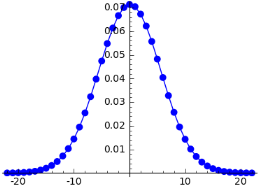

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More structures to be discovered!

Informal open questions

- Reduce the bound M_1 for the number of blocks needed for $c_t > 1/2$ to hold. Goal: $M_1 = 0$.
- Prove a *polynomially perturbed Gaussian law* for the numbers

$$\Theta(n,j) = \#\left\{0 \le k \le n : \nu_2\left(\binom{n}{k}\right) = j\right\},$$

where $\Theta(n,j)$ is compared to

$$p(j)\exp\left(-K(j-E)^2/2\right),$$

and $p = p_n$ is a certain polynomial. This was asked by R. Tichy (2020), and work in progress with B. Sobolewski (Kraków/Leoben);
- Study several differences $s_2(n+t_j) - s_2(n)$ simultaneously, and prove e.g. multidimensional Gaussian laws. For this, consider generalized correlations

$$\sum_n \exp\left(\vartheta s_2(n) + \sum_{j<J} \vartheta_j s_2(n+t_j)\right).$$

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Section 4

Digital expansions in different bases

“Collisions” of digit sums in different bases

A former folklore conjecture states that the equation

$$s_2(n) = s_3(n)$$

admits infinitely many solutions n in the positive integers.

Theorem (S. 2023, Israel J. Math.)

For all $\delta > 0$ we have

$$\#\{n < N : s_2(n) = s_3(n)\} \gg N^{\frac{\log 3}{\log 4} - \delta}, \qquad (1)$$

where the implied constant may depend on δ.
Note that $\log 3 / \log 4 = 0.792\ldots$.

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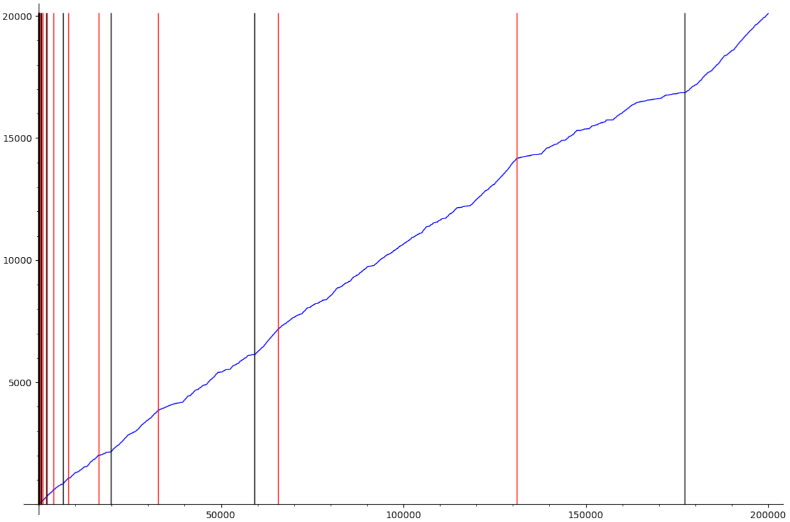

blue: number of collisions up to N; red: powers of 2; black: powers of 3

Even the arrangement of powers of 2 and 3 is somewhat cryptic.

$$(a_n)_{n\geq 0} = (1, 2, 3, 4, 8, 9, 16, 27, 32, 64, 81, 128, 243, 256, 512, 729, 1024, \ldots)$$

This amounts to understanding the *continued fraction expansion*

$$\frac{\log 3}{\log 2} = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4, 3, 1, 1, \ldots],$$

which is unknown!

This topic has connections to dynamical systems (*Furstenberg's conjectures* on joint digital expansions in different bases), Diophantine approximation (estimates for the irrationality exponent of $\log_2 3$), and Mahler's 3/2-problem (can we have $\{x(3/2)^n\} < 1/2$ for all $n \geq 0$?).

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A remark on the separation of sum-of-digits functions

The values of $\mathsf{s}_2(n)$ and $\mathsf{s}_3(n)$, as $n < N$ concentrate around $\log_4(N)$ and $\log_3(N)$ respectively. The standard deviations are small compared to the difference of expected values.

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Connection to the main topic of the talk

The central idea of the proof is a simple heuristic. We have

$$s_3\left(3^\zeta n\right) = s_3(n),$$

which has a value close to $\log(n)/\log(3)$ for most n. Meanwhile, the binary digits of $3^\zeta n$ should be "random", and therefore

$$s_2\left(3^\zeta n\right) \approx \log_4\left(3^\zeta n\right) = \frac{\log(n)}{\log 4} + \zeta\frac{\log 3}{\log 4}.$$

We can therefore adjust ζ so that $s_3\left(3^\zeta n\right)$ and $s_2\left(3^\zeta n\right)$ have many values close to each other.

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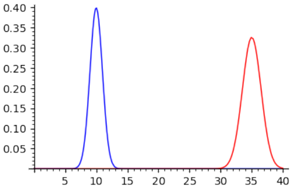

all n

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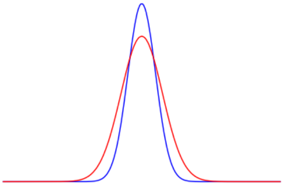

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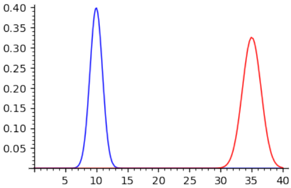
Transition to a *subsequence of a digitally defined function* enabled us to prove the folklore conjecture.



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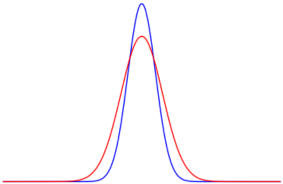

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Open questions

- Work in Progress with M. Drmota: new proof of the theorem, refinements. With J.-M. Deshouillers: digits of $n!$ in base 12.
- Conjecture: There exist constants c and η such that

$$\#\{n < N : \mathsf{s}_2(n) = \mathsf{s}_3(n)\} \sim cN^\eta.$$

- Prove that there are infinitely many prime numbers p such that $\mathsf{s}_2(p) = \mathsf{s}_3(p)$.

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THANK YOU!

- M. Drmota, C. Müllner, and L. Spiegelhofer, *Primes as sums of Fibonacci numbers*, 2021.
  Accepted for publication in Mem. Amer. Math. Soc. (2022).
- L. Spiegelhofer, *The level of distribution of the Thue–Morse sequence*, Compos. Math., 156 (2020), pp. 2560–2587.
- L. Spiegelhofer, *Collisions of digit sums in bases 2 and 3*, 2023.
  To appear in Israel J. Math.
- L. Spiegelhofer, *Thue–Morse along the sequence of cubes*, 2023.
  Preprint, http://arxiv.org/abs/2308.09498.
- L. Spiegelhofer and M. Wallner, *The binary digits of $n + t$*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5), 24 (2023), pp. 1–31.

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