Subsequences of digitally defined functions

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Outline

1. Digital expansions
2. Sparse subsequences
3. Long arithmetic subsequences

Section 1

Digital expansions

In the simplest case, a *digital expansion* Φ assigns to each natural number a finite string of *digits* in an injective manner.

n	$\Phi(n)$	n	$\Phi(n)$	n	$\Phi(n)$
0	0	8	1000	16	10000
1	1	9	1001	17	10001
2	10	10	1010	18	10010
3	11	11	1011	19	10011
4	100	12	1100	20	10100
5	101	13	1101	21	10101
6	110	14	1110	22	10110
7	111	15	1111	23	10111

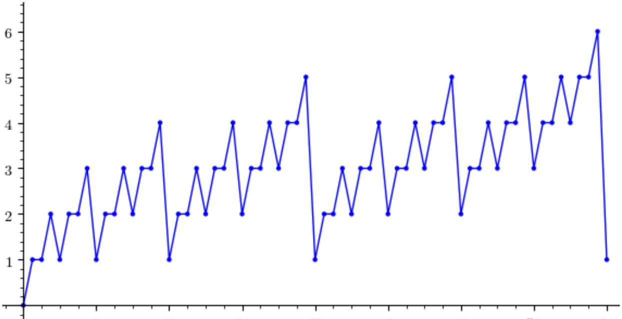
Figure: The binary expansion

Let $\mathsf{s}_2(n)$ be the number of 1s in the binary expansion of n.
The sequence s_2 is a fixed point of the substitution defined by

$$n \mapsto (n, n+1), \quad n \geq 0.$$

$$\mathsf{s}_2 = \quad \texttt{01121223122323341223233423343445}\ldots$$

$$\mathsf{s}_2 = \quad \begin{array}{l} \color{green}{\texttt{01121223122323341223233423343445}} \\ \color{red}{\texttt{12232334233434452334344534454556}\ldots} \end{array}$$



The Thue–Morse sequence

Let $\mathsf{t}(n) = \mathsf{s}_2(n) \bmod 2$.

$$\mathsf{t} = 01101001 \cdots .$$

This sequence is an *automatic sequence*, and a fixed point of

$$0 \mapsto 01, \quad 1 \mapsto 10.$$

Automatic sequences are given by deterministic finite automata with output (DFAO), where the input is the base-q expansion of integers.

2-automatic sequences

Each subsequence $n \mapsto \mathrm{t}(An + B)$ is an automatic sequence.

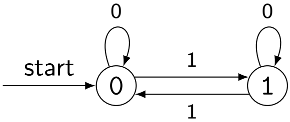


Figure: An automaton for $\mathrm{t}(n)$

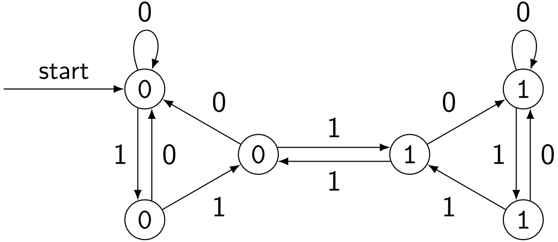


Figure: An automaton for $\mathrm{t}(3n)$

Section 2

Sparse subsequences

Few big steps

General idea: along sparse subsequences, t behaves "randomly". For example, every finite sequence on $\{0,1\}$ appears as an arithmetic subsequence of t (Avgustinovich–Fon-Der-Flaass–Frid 2003, Müllner–Spiegelhofer 2017 ("correct" rate of appearance), Konieczny–Müllner 2023+ (general automatic sequences)).

Very sparse arithmetic subsequences of t

The Thue–Morse sequence has level of distribution 1.

Theorem (S. 2020, Compos. Math.)

For all $\varepsilon > 0$ we have

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} (-1)^{\mathsf{s}_2(n)} \right| \le C x^{1-\eta}$$

for some C and $\eta > 0$ depending on ε, where $D = x^{1-\varepsilon}$.

In more relaxed language: let $R > 0$. As $N \to \infty$, the following holds.

Most $d \asymp N^R$ have the property that for all a, the number

$$\#\{0 \le n < N : \mathrm{t}(nd + a) = 0\}$$

is close to $N/2$.

Drmota–Müllner–S. 2023+

Together with Drmota and Müllner, we proved a theorem on the subsequence indexed by the sequence of primes. Let $\mathrm{z}(n)$ be the smallest k such that n is the number of k Fibonacci numbers.

Theorem (Drmota–Müllner–S., to appear in Mem. Amer. Math. Soc.)

- *The sequence* $n \mapsto \exp(2\pi i \vartheta \mathrm{z}(n))$ *has level of distribution* 1.
- *For* $m \geq 1$ *and* $a \in \mathbb{Z}$, *we have*

$$\{p < x : p \textit{ prime}, \mathrm{z}(p) \equiv a \bmod m\} \sim \frac{\pi(x)}{m}$$

  as $x \to \infty$.
- *For* k *large enough, there exists a prime number* p *that is the sum of exactly* k *different, non-consecutive Fibonacci numbers.*

(Informal) open questions

- Let $R > 0$. As $N \to \infty$, most $d \in \left[N^R, 2N^R\right)$ should have the property that
$$m \mapsto \#\left\{0 \le n < N : \mathrm{s}_2(nd) = m\right\}$$
closely follows a Gaussian.

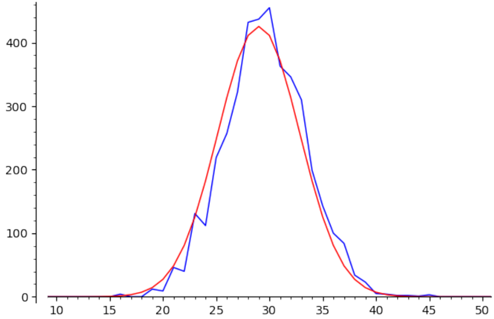


Figure: $N = 2^{12}$, $d = 3^{30}$

More open questions

- Prove prime number theorems for more general *morphic sequences*, such as

$$\sigma : \left\{ \begin{array}{llllllllll} \mathtt{a} & \mapsto & \mathtt{ae}, & \mathtt{b} & \mapsto & \mathtt{af}, & \mathtt{c} & \mapsto & \mathtt{a} \\ \mathtt{d} & \mapsto & \mathtt{db}, & \mathtt{e} & \mapsto & \mathtt{dc}, & \mathtt{f} & \mapsto & \mathtt{d} \end{array} \right\},$$

$$\pi(\mathtt{a}) = \pi(\mathtt{b}) = \pi(\mathtt{c}) = 0,$$
$$\pi(\mathtt{d}) = \pi(\mathtt{e}) = \pi(\mathtt{f}) = 1$$

  The projection under π of the fixed point starting with $\mathtt{a}$ is

$$\mathtt{tr} = 011010010010110010110110100101101101001101 0 \cdots .$$

- Prove a level-of-distribution result for arbitrary automatic sequences.

Section 3

Long arithmetic subsequences

Digital expansions and addition

how does the sum of digits of an integer change when a constant d is added repeatedly?

Differences along an arithmetic progression $d\mathbb{N}$:

$$\delta(d,a,j) \coloneqq \lim_{N\to\infty} \frac{1}{N} \#\bigl\{n < N : \mathsf{s}_2((n+1)d) - \mathsf{s}_2(nd) = j\bigr\}.$$

This value is in fact identical to

$$\boxed{\delta(d,j) \coloneqq \lim_{N\to\infty} \frac{1}{N} \#\bigl\{n < N : \mathsf{s}_2(n+d) - \mathsf{s}_2(n) = j\bigr\}.}$$

Cusick's conjecture

When traversing an infinite arithmetic subsequence of s_2, how often does the value weakly increase? This is the subject of *Cusick's conjecture*.

Conjecture (Cusick)

For all $d \ge 0$, we have

$$c_d > 1/2,$$

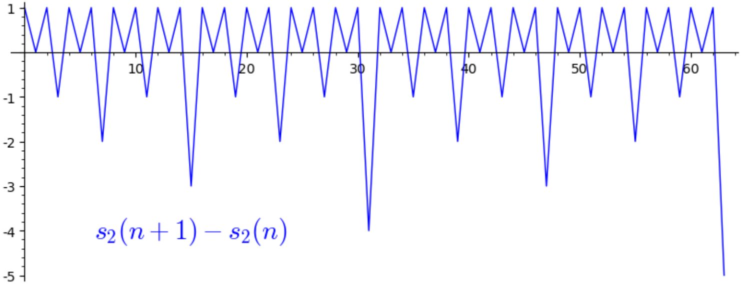
where

$$\begin{aligned} c_d &= \lim_{N\to\infty} \frac{1}{N} \#\{n < N : \mathsf{s}_2(n+d) \ge \mathsf{s}_2(n)\} \\ &= \sum_{j\ge 0} \delta(d,j). \end{aligned}$$

First example: $d = 1$

$$\begin{cases} s_2(n+1) - s_2(n) = 1 & \text{if and only if } n \equiv 0 \bmod 2, \\ s_2(n+1) - s_2(n) = 0 & \text{if and only if } n \equiv 1 \bmod 4, \end{cases}$$

and $s_2(n+1) - s_2(n) < 0$ for $n \equiv 3 \bmod 4$. Therefore $c_1 = 3/4$.

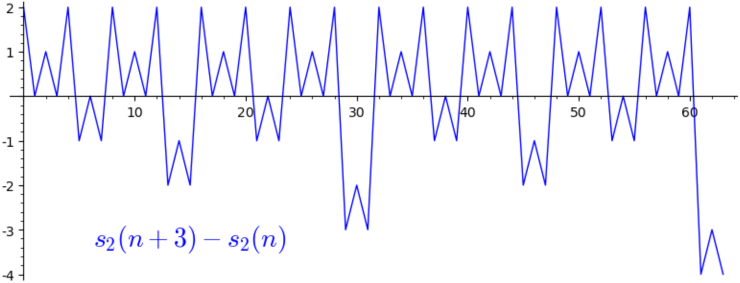


$\rightsquigarrow$ "ruler sequence".

Second example: $d = 3$

$$\begin{cases} s_2(n+3) - s_2(n) = 2 & \text{if and only if } n \equiv 0 \bmod 4, \\ s_2(n+3) - s_2(n) = 1 & \text{if and only if } n \equiv 2 \bmod 8, \\ s_2(n+3) - s_2(n) = 0 & \text{if and only if } \begin{cases} n \equiv 1 \bmod 8 & \text{or} \\ n \equiv 3 \bmod 8 & \text{or} \\ n \equiv 6 \bmod 16, \end{cases} \end{cases}$$

and $s_2(n+3) - s_2(n) < 0$ otherwise, therefore $c_3 = 11/16$.



SW2023

Let $M = M(d)$ be the number of maximal blocks of 1s in the binary expansion of d.

Theorem (S.–Wallner 2023, Ann. Sc. norm. super. Pisa - Cl. sci.)

Let $d \geq 1$. If $M(d)$ is larger than some absolute, effective constant M_1, then $c_d > 1/2$.

SW2023, part II

Again, let $M = M(d)$ be the number of blocks of 1s in d.

Theorem (S.–Wallner 2023)

Set $\kappa(1) = 2$, *and for* $d \ge 1$ *let* $\kappa(2d) = \kappa(d)$, *and*

$$\kappa(2d+1) = \frac{\kappa(d) + \kappa(d+1)}{2} + 1.$$

Then

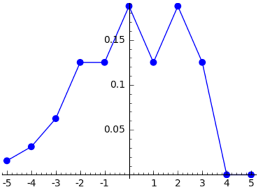
$$\delta(j, d) = \frac{1}{\sqrt{2\pi\kappa(d)}} \exp\left(-\frac{j^2}{2\kappa(d)}\right) + \mathcal{O}\left(\frac{(\log M)^4}{M}\right)$$

for all integers j. *The implied constant is absolute.*

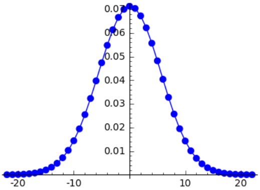
"The sum of digits along arithmetic progressions varies according to a normal distribution."

"Your paper reduces my conjecture to what I will call the 'hard cases' [...]" (T. W. Cusick, 2021)

Consider graphs of $j \mapsto \delta(d, j)$:



hard



easier

More structures to be discovered!

THANK YOU!

M. Drmota, C. Müllner, and L. Spiegelhofer, *Primes as sums of Fibonacci numbers*, 2021.
Accepted for publication in Mem. Amer. Math. Soc. (2022).

L. Spiegelhofer, *The level of distribution of the Thue–Morse sequence*, Compos. Math., 156 (2020), pp. 2560–2587.

L. Spiegelhofer and M. Wallner, *The binary digits of $n + t$*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5), 24 (2023), pp. 1–31.

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