The simplest 2-regular sequences

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Outline

1. k-regular sequences

2. The discrepancy of the van der Corput sequence

3. Odd binomial coefficients

Section 1

k-regular sequences

k-regular sequences

If there exists an integer r, sequences $S_0, \ldots, S_{r-1}$, and $r \times r$-matrices $B_0, \ldots, B_{k-1}$ such that

$$\begin{pmatrix} S_0(kn+a) \\ \vdots \\ S_{r-1}(kn+a) \end{pmatrix} = B_a \begin{pmatrix} S_0(n) \\ \vdots \\ S_{r-1}(n) \end{pmatrix}$$

for all $n \geq 0$ and $0 \leq a < k$, the sequence S_0 is called k-regular. [Allouche–Shallit 1992]

First example. The *Stern sequence*, defined by

$$\begin{aligned} z(0) = 0, \quad z(1) = 1,\\ z(2n) = z(n),\\ z(2n+1) = z(n) + z(n+1) \end{aligned}$$

is 2-regular. Corresponding matrices are given by

$$B_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and we have

$$\begin{pmatrix} z(2n) \\ z(2n+1) \end{pmatrix} = B_0 \begin{pmatrix} z(n) \\ z(n+1) \end{pmatrix},$$
$$\begin{pmatrix} z(2n+1) \\ z(2n+2) \end{pmatrix} = B_1 \begin{pmatrix} z(n) \\ z(n+1) \end{pmatrix}.$$

The values of the Stern sequence are distributed according to a log-normal distribution [Bettin, Drappeau, Spiegelhofer 2017]

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Second example

The binary sum-of-digits function satisfies

$$s(2n) = s(n),$$
$$s(2n+1) = s(n) + 1.$$

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Third example

The number A_n of odd entries in row n of Pascal's triangle satisfies

$$A_n = 2^{s(n)},$$

that is,

$$A_{2n} = A_n, \quad A_{2n+1} = 2A_n.$$

$$\begin{aligned} \left(A_{2n}\right) &= \left(1\right)\left(A_n\right), \\ \left(A_{2n+1}\right) &= \left(2\right)\left(A_n\right). \end{aligned}$$

Fourth example

The *discrepancy of the van der Corput sequence* satisfies

$$\begin{aligned} d(1) &= 1, \\ d(2n) &= d(n), \\ d(2n+1) &= \frac{d(n)+d(n+1)+1}{2}. \end{aligned}$$

$$\begin{pmatrix} d(2n) \\ d(2n+1) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d(n) \\ d(n+1) \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} d(2n+1) \\ d(2n+2) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d(n) \\ d(n+1) \\ 1 \end{pmatrix}.$$

Section 2

The discrepancy of the van der Corput sequence

It is known [Drmota, Larcher, Pillichshammer 2005] that $d(n)$ is normally distributed.

Theorem (DLP2005)

For every $y \in \mathbb{R}$ we have

$$\frac{1}{M}\#\left\{N < M : d(n) \le \frac{1}{4}\log_2 N + y\frac{1}{4\sqrt{3}}\sqrt{\log_2 N}\right\} = \Phi(y) + o(1),$$

where

$$\Phi(y) = \int_{-\infty}^{y} e^{-t^2/2}\,\mathrm{d}t.$$

We want to obtain a finer distribution result using moments of $d(n)$, for example, involving a polynomially perturbed Gaussian

$$C\int_{-\infty}^{y} e^{-t^2/2}p(t)\,\mathrm{d}t$$

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The expectation of $d(n)$

We have

$$\frac{1}{2^\lambda} \sum_{2^\lambda \le n < 2^{\lambda+1}} d(n) = 1 + \frac{\lambda}{4}.$$

For $\lambda \ge 0$ and $2^\lambda \le n < 2^{\lambda+1}$, we define

$$\begin{aligned} e_\lambda(n) &= d(n) - 1 - \frac{\lambda}{4}, \\ \Delta(n) &= d(n) - d(n+1) = e_\lambda(n) - e_\lambda(n+1). \end{aligned}$$

By construction,

$$\sum_{2^\lambda \le n < 2^{\lambda+1}} e_\lambda(n) = \sum_{2^\lambda \le n < 2^{\lambda+1}} \Delta(n) = 0.$$

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Some values

$$
\begin{aligned}
&d(1) = 1, && d(2) = 1, && d(3) = \tfrac{3}{2},\\
&e_0(1) = 0, && e_1(2) = -\tfrac{1}{4}, && e_1(3) = \tfrac{1}{4},\\
&\Delta(1) = 0, && \Delta(2) = -\tfrac{1}{2}, && \Delta(3) = \tfrac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
&d(4) = 1, && d(5) = \tfrac{7}{4}, && d(6) = \tfrac{3}{2}, && d(7) = \tfrac{7}{4},\\
&e_2(4) = -\tfrac{1}{2}, && e_2(5) = \tfrac{1}{4}, && e_2(6) = 0, && e_2(7) = \tfrac{1}{4},\\
&\Delta(4) = -\tfrac{3}{4}, && \Delta(5) = \tfrac{1}{4}, && \Delta(6) = -\tfrac{1}{4}, && \Delta(7) = \tfrac{3}{4}.
\end{aligned}
$$

Mixed moments

For integers $k, \ell, \lambda \ge 0$ we define

$$m_{k,\ell,\lambda} = \frac{1}{2^\lambda} \sum_{2^\lambda \le n < 2^{\lambda+1}} e_\lambda^k(n) \Delta^\ell(n).$$

After some rewriting we obtain

$$m_{k,\ell,\lambda+1} = \frac{1}{2^{\lambda+1}} \sum_{2^\lambda \le n < 2^{\lambda+1}} \left(e_\lambda(n) - \frac{1}{4} \right)^k \left(\frac{\Delta(n)}{2} - \frac{1}{2} \right)^\ell$$
$$+ \frac{1}{2^{\lambda+1}} \sum_{2^\lambda \le n < 2^{\lambda+1}} \left(e_\lambda(n) - \frac{\Delta(n)}{2} + \frac{1}{4} \right)^k \left(\frac{\Delta(n)}{2} + \frac{1}{2} \right)^\ell.$$

The contribution of the moments $m_{k,r,\lambda}$, where $r \le \ell$, is given by

$$\frac{1}{2^\ell} \sum_{\substack{0 \le r \le \ell \\ 2 \mid r}} \binom{\ell}{r} m_{k,\ell-r,\lambda}.$$

We see that matrix products such as

$$\begin{pmatrix} \frac{1}{2^6} & 0 & \frac{1}{2^6}\binom{6}{2} & 0 & \frac{1}{2^6}\binom{6}{4} & 0 & \frac{1}{2^6}\binom{6}{6} \\ 0 & \frac{1}{2^5} & 0 & \frac{1}{2^5}\binom{5}{2} & 0 & \frac{1}{2^5}\binom{5}{4} & 0 \\ 0 & 0 & \frac{1}{2^4} & 0 & \frac{1}{2^4}\binom{4}{2} & 0 & \frac{1}{2^4}\binom{4}{4} \\ 0 & 0 & 0 & \frac{1}{2^3} & 0 & \frac{1}{2^3}\binom{3}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2^2} & 0 & \frac{1}{2^2}\binom{2}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^M \begin{pmatrix} m_{k,6,\lambda} \\ m_{k,5,\lambda} \\ m_{k,4,\lambda} \\ m_{k,3,\lambda} \\ m_{k,2,\lambda} \\ m_{k,1,\lambda} \\ m_{k,0,\lambda} \end{pmatrix}$$

will play a role!

Example.

$$A = \begin{pmatrix} \frac{1}{16} & 0 & \frac{3}{16} & 0 & \frac{1}{16} \\ 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{then}$$

$$A^k = \begin{pmatrix} \frac{1}{16^k} & 0 & \frac{2}{4^k} - \frac{2}{16^k} & 0 & \frac{16^k-1}{5\cdot 16^k} - 2\frac{4^k-1}{3\cdot 16^k} \\ 0 & \frac{1}{8^k} & 0 & \frac{1}{2^k} - \frac{1}{8^k} & 0 \\ 0 & 0 & \frac{1}{4^k} & 0 & \frac{16^k-4^k}{3\cdot 16^k} \\ 0 & 0 & 0 & \frac{1}{2^k} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Idea: Formulas for powers of such matrices of size K will enable us to describe all moments of $e_\lambda(n)$ with precision $O\bigl(\lambda^{-K/2}\bigr)$.

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Section 3

Odd binomial coefficients

Let us define

$$S(N) = \#\left\{(n,k) : 0 \le k \le n < N : 2 \nmid \binom{n}{k}\right\} = \sum_{0 \le n < N} 2^{s(n)}.$$

It is easy to see that

$$S(2N) = 3S(N), \quad S(2N+1) = 2S(N) + S(N+1).$$

Many authors have studied this quantity (Flajolet, Grabner, Harborth, Kirschenhofer, Larcher, Prodinger, Stolarsky, Tichy,...).

In particular, $S(N)$ was compared to the function $f(x) = x^{\log 3/\log 2}$, which also has the property $f(2x) = 3f(x)$, and therefore satisfies $f(2^N) = S(2^N)$.

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The quantity

$$\min_{x\in[0,1]} \varphi(x) = \liminf_{n\to\infty} S(N) 2^{-\log 3/\log 2}$$
$$= 0.812556559016006387694882101649536712 4\cdots$$

is called Stolarsky–Harborth constant.

Larcher (1992) showed that the position y of each minimum is given by a Sturmian word with slope $\vartheta = \log 3/\log 2 - 1$: for some $d \in \mathbb{R}$, we have

$$y = \sum_{k\ge 1} 2^{-k} e_k,$$

where

$$e_k = \lfloor (k+1)\vartheta + d \rfloor - \lfloor k\vartheta + d \rfloor.$$

Admissible values of d, corresponding to minima, as well as their number, remain mysterious.

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Simplification of the problem

Let $\lambda \geq 0$ be an integer. On the interval $\left[2^\lambda, 2^{\lambda+1}\right]$ we compare S with the line segment connecting

$$\left(2^\lambda, 3^\lambda\right) \quad \text{and} \quad \left(2^{\lambda+1}, 3^{\lambda+1}\right).$$

Set

$$\Delta_\lambda(n) = S(N) - 3^\lambda - \left(n - 2^\lambda\right)\frac{2 \cdot 3^\lambda}{2^\lambda}.$$

Normalizing suitably, we obtain a continuous function

$$\Phi : [0,1] \to \mathbb{R}$$

such that

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The Straight Stolarsky–Harborth constant

Surprisingly,

1. this does not seem to be handled in the literature,
2. the minimum of Φ seems to be attained at

$$x = \sum_{k\ge 1} 2^{-k} e_k = 0.67741147741147752252682561330016\cdots,$$

where

$$e_k = \lfloor (k+1)\vartheta + \mathbf{0} \rfloor - \lfloor k\vartheta + \mathbf{0} \rfloor$$

and $\vartheta = \log 3/\log 2 - 1$ as before.

The value of Φ at this position is

$$-0.4782436127025850978521771039475291\cdots.$$

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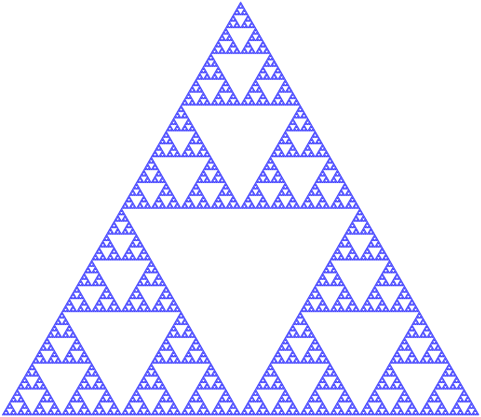
A remark

Note that the continuous function Φ satisfies, and is uniquely determined by,

$$\Phi(0) = \Phi(1) = 0,$$
$$\Phi\left(\frac{2k+1}{2^{\lambda+1}}\right) = \frac{2}{3}\Phi\left(\frac{k}{2^\lambda}\right) + \frac{1}{3}\Phi\left(\frac{k+1}{2^\lambda}\right) - \frac{1}{3\cdot 2^\lambda}$$

for all integers $\lambda \ge 0$ and $2^\lambda \le k < 2^{\lambda+1}$.

Remark. Consider the Sierpiński triangle T of height 1 and Hausdorff measure $H^{\log 3/\log 2}(T) = C$, and the red portion $A(h)$ of height h.



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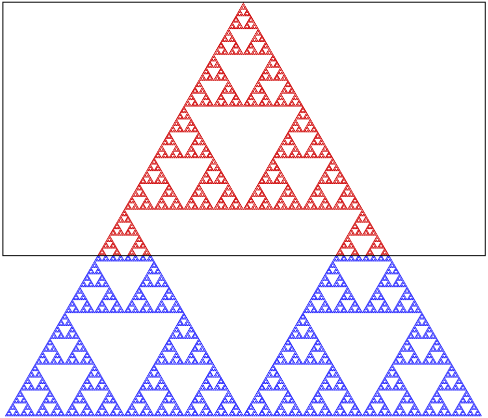


Prove that $H^{\log 3/\log 2}(A(h)) = C\left(\frac{1}{2}\Phi(h) + h\right)$.

Higher divisibility in Pascal's triangle

Barat and Grabner (2001) also considered the number of binomial coefficients exactly divisible by a power of a prime.

Theorem (Barat–Grabner 2001, Theorem 5)

Let p be a prime, and $j \ge 0$ an integer. Define

$$S_j(N) = \#\left\{(k,n) : 0 \le k \le n < N, p^j \| \binom{n}{k}\right\}.$$

There exist continuous periodic functions of period 1, $\psi_r^{(j)}$, for $0 \le r \le j$, such that

$$S_j(N) = N^\alpha \sum_{r=0}^{j} \bigl(\log_p N\bigr)^r \psi_r^{(j)}\bigl(\log_p N\bigr) + o\bigl(N^\varepsilon\bigr)$$

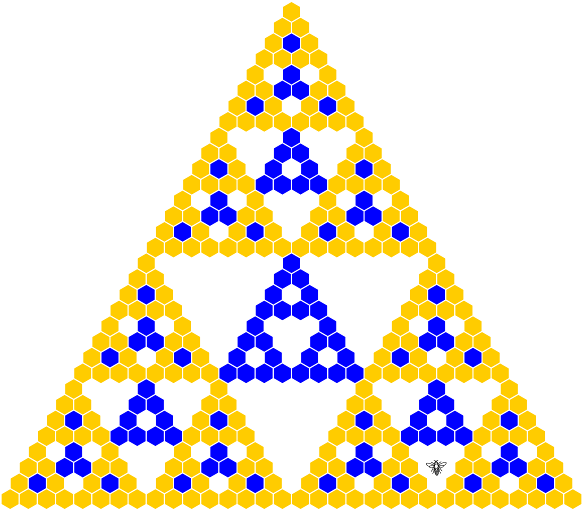
for $\alpha = \log_p \frac{p(p+1)}{2}$ and any $\varepsilon > 0$. Also, $\psi_j^{(j)} = \frac{1}{j!}\bigl(\frac{p-1}{p+1}\bigr)^{2j}\psi_0^{(0)}$.

More open questions

- (Grabner) Does there exist an exact representation of $S_j(N)$, where $j \geq 1$?
- What is the minimum of $\psi_r^{(j)}$ resp. of a *straightened* version?
- (Drmota, Mauduit, Rivat) Can any bounds for the sum

$$\sum_{p \leq x} 2^{s(p)}$$

be proved?



THANK YOU!

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