Thue–Morse along the sequence of cubes

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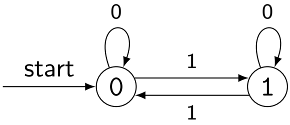
Section 1

Subsequences of the Thue–Morse [tʉː mɔːrs] sequence

The Thue–Morse sequence

We denote the Thue–Morse sequence on $\{0, 1\}$ by $\mathbf{t}$, and the Thue–Morse sequence on $\{1, -1\}$ by $\mathbf{u}$. It is given by the binary sum-of-digits function s, reduced modulo 2.



$$\mathbf{t} = 01101001100101101001011001101001 \cdots$$

Arithmetic subsequences of s

Each subsequence $n \mapsto \mathbf{t}(An + B)$ is an automatic sequence.

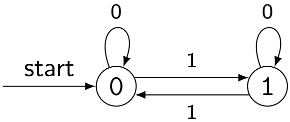


Figure: An automaton for $\mathbf{t}(n)$

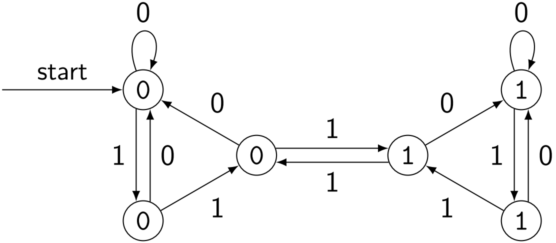
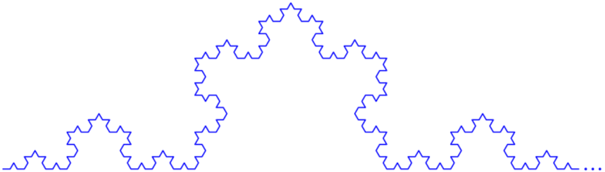


Figure: An automaton for $\mathbf{t}(3n)$

Thue–Morse $\rightleftharpoons$ Koch

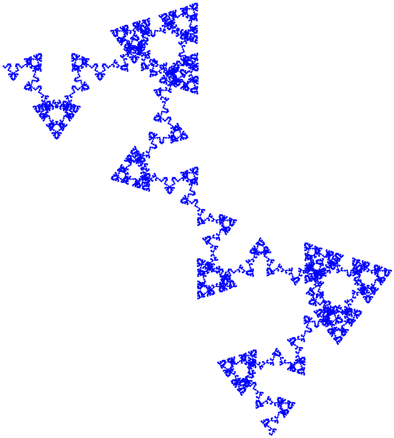
Let $\mathrm{e}(x) = e^{2\pi i x}$. The sequence $n \mapsto (-1)^{s(n)}\,\mathrm{e}(-n/3)$ describes the direction of the $(n+1)$th segment in the "unscaled Koch curve":



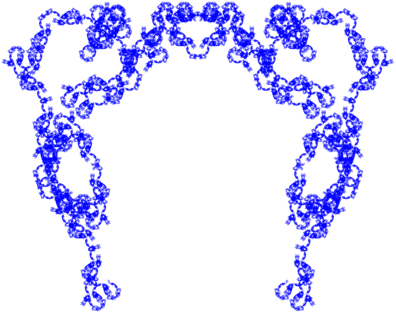
Partial sums of $(-1)^{s(n)}\,\mathrm{e}(-n/3)$

The sum of digits along arithmetic progressions

For all integers p, q, d, k such that $q \geq 1$ and $d \geq 0$, the sequence $n \mapsto \exp\left(\frac{ps(dn)+kn}{q}\right)$ is 2-automatic. Partial sums yield interesting pictures.

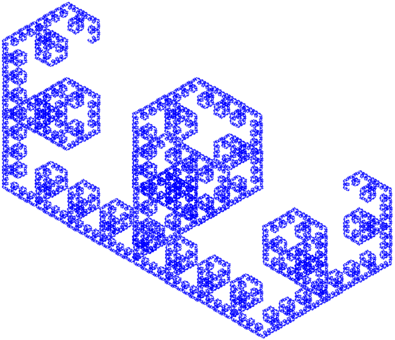


$\mathrm{e}\left(\frac{1}{2}s(3n) - n/5\right)$

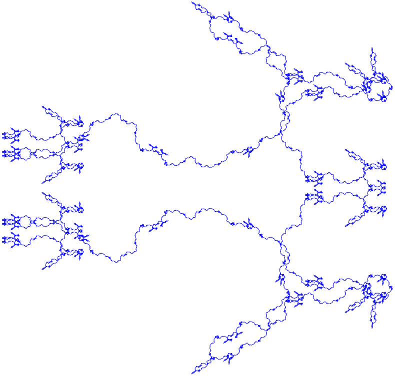


$\mathrm{e}\left(\frac{2}{5}s(3n) - n/5\right)$

The sum of digits along arithmetic progressions

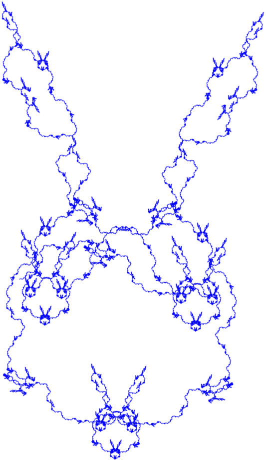


$\mathrm{e}(s(3n)/3)(-1)^n$



$\mathrm{e}(s(7n)/3)(-1)^n$

The sum of digits along arithmetic progressions



$\mathrm{e}(s(7n)/3)(-1)^n$, closeup

Source: Wikipedia, “Rabbit”

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Every finite sequence $\omega \in \{0,1\}^L$ appears as an arithmetic subsequence of $\mathbf{t}$: the Thue–Morse word has full *arithmetical complexity* (Avgustinovich–Fon-Der-Flaass–Frid 2003, Müllner–Spiegelhofer 2017, Konieczny–Müllner 2024+).

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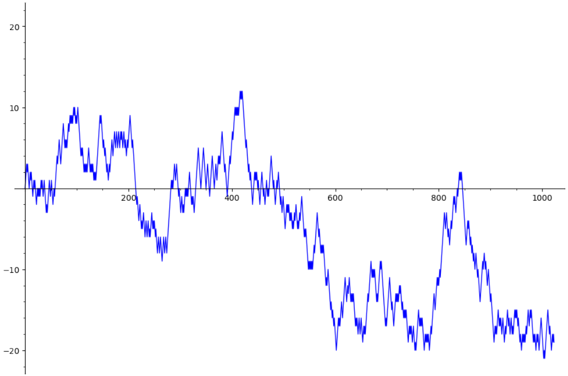
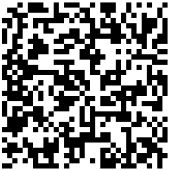
Short arithmetic subsequences of $\mathbf{t}$ even seem to behave randomly.





$N = 32 \times 32$ terms, common difference $d = 3^{21}$

The sum of digits along arithmetic progressions

The function s_q along arithmetic progressions is uniformly distributed in residue classes modulo m if $\gcd(q-1, m) = 1$. We state the following special case.

Theorem (Gelfond 1968)

Let $d \geq 1$ and a be integers. There is an absolute $\lambda < 1$ such that

$$\left|\{1 \leq n \leq x : \mathbf{t}(n) = 0, n \equiv a \bmod d\}\right| = \frac{x}{2d} + \mathcal{O}\left(x^{\lambda}\right).$$

Very sparse arithmetic subsequences of $\mathbf{t}$

The Thue–Morse sequence has mean value around $1/2$ along most very short arithmetic progressions — "$\mathbf{t}$ has level of distribution 1".

Theorem (S. 2020)

For all $\varepsilon > 0$ we have

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} (-1)^{s(n)} \right| \le Cx^{1-\eta}$$

for some C and $\eta > 0$ depending on ε, where $D = x^{1-\varepsilon}$.

In more relaxed language: let $R > 0$. As $N \to \infty$, the following holds.

Most $d \asymp N^R$ have the property that for all a, the number

$$\#\{0 \le n < N : \mathbf{t}(nd + a) = 0\}$$

is close to $N/2$.

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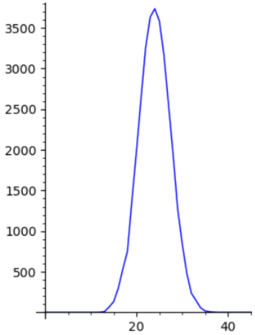
(Informal) Problem

Prove that most $d \asymp N^R$ have the property that for all a,

$$m \mapsto \#\{n < N : s(dn + a) = m\}$$

closely follows a Gaussian.



2^{15} terms of $s(3^{21}n)$

Gelfond's third problem

Let $S = s_q$ be the sum-of-digits function in base $q \geq 2$.

> *Finalement, signalons comme problème à résoudre l'estimation du nombre des valeurs du polynôme $P(t)$ ne prenant que des valeurs entières sur l'ensemble [...] des entiers rationels, pour lesquelles on a $S[P(n)] \equiv \ell \bmod m$.*
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That is, if P is a polynomial such that $P(\mathbb{N}) \subseteq \mathbb{N}$, we are interested in

$$A(q, P, m, \ell, x) := \#\{n < x : s_q(P(n)) \equiv \ell \bmod m\}.$$

Partial results

Write $s(n) = s_2(n)$. We have

$$\mathbf{t} = \big(s(n) \bmod 2\big)_{n \geq 0}$$

$$= \big(01101001100101101001011001101001100101100110100101 \cdots\big)$$

- Lower bounds for the numbers $A(q, P, m, \ell, x)$ are known (Dartyge–Tenenbaum 2006; Stoll 2012);
- The case $P(x) = x^2$ has been answered by Mauduit and Rivat (Acta Math., 2009). In particular, for some $c > 0$ and C,

$$\left| \#\big\{ n < x : \mathbf{t}\big(n^2\big) = 0 \big\} - \frac{x}{2} \right| \leq C x^{1-c}. \qquad (1)$$

- For "sufficiently large bases" q coprime to the leading coefficient of P, and $\gcd(q-1, m) = 1$, the equivalence $A(q, P, m, \ell, x) \sim x/m$ has been proved (Drmota–Mauduit–Rivat 2011).

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Block occurrences in $\mathbf{t}(n^2)$

The Thue–Morse sequence along n^2 is normal (Drmota–Mauduit–Rivat): each finite sequence over $\{0,1\}$ of length L appears with frequency 2^{-L} along $\mathbf{t}(n^2)$.

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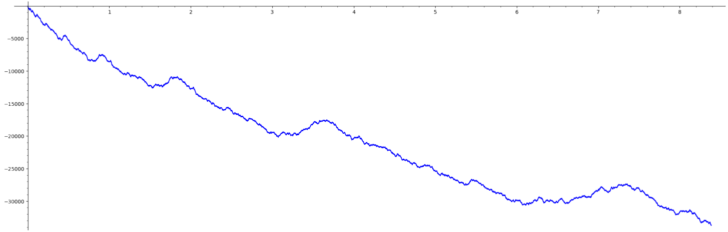
Partial sums of $(-1)^{s(n^2)}$ for $x < 2^{23}$:



Problem

A *drift* appears to be present. How is this related to the fact that n^2 avoids certain residue classes?

Problem, part 2 ☕

Prove that there exist real numbers $c \neq 0$ and $\eta \in (0,1)$, and a 1-periodic, continuous, nowhere differentiable function Φ, such that

$$\sum_{n<x} \mathbf{t}\bigl(n^2\bigr) \sim c x^{\eta} \Phi\bigl(\log x / \log 2\bigr).$$

The main result

Theorem (S. 2024+)

There exist real numbers $c > 0$ and C such that for all $x \geq 1$,

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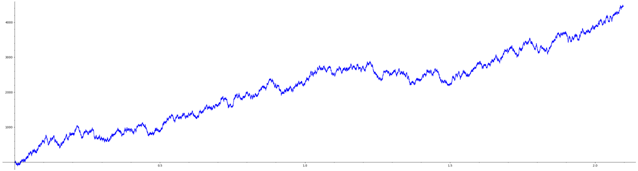

Partial sums of $(-1)^{s(n^3)}$

Section 2

Sketch of the proof

Carry Lemma (Mauduit–Rivat 2009, 2010)

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- The arguments $(n+r)^3$ and n^3 *usually* have the same digits with indices above

$$\lambda := \nu(2+\varepsilon),$$

  if r is *small* compared to 2^ν.

- These digits can therefore be discarded.

Too many significant digits

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Wikipedia: "Salami", by Aka (CC BY-SA 2.5)

Too many significant digits

Iterated van der Corput could so far not be used for removing sufficiently many digits of polynomial values $P(n)$, if $\deg P > 1$.

A trivial decomposition[1]

- Choose $\rho < \nu$ in such a way that $3\rho \geq \lambda$, and write

$$n = 2^\rho n_1 + n_0, \quad \text{where} \quad \begin{cases} 0 \leq n_1 < 2^{\nu-\rho}, \\ 0 \leq n_0 < 2^\rho \end{cases}$$

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- Expanding $n^3 \bmod 2^\lambda$, we see that the cubic term in n_1 disappears.
- On the *critical interval* $[2\rho, \lambda)$ of length $\kappa := \lambda - 2\rho$, the term n_1^2 is still relevant.
- We introduce an additional sum $\sum_{0 \leq j < 2^\kappa}$ that parametrizes the digit combinations in the critical interval.

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The critical interval of digits

For a subset $J \subseteq \mathbb{N}$, let s^J denote the *restricted binary sum-of-digits function*: only digits with indices in J are counted. We write

$$S_0 = \sum_{n<2^\nu} \mathrm{e}\Big(\tfrac{1}{2} s(n^3)\Big)$$

$$= \sum_{0\le j<2^\kappa} (-1)^{s(j)} \sum_{n<2^\nu} \mathrm{e}\Big(\tfrac{1}{2} s^{\mathbb{N}\setminus[2\rho,\lambda)}(n^3)\Big) \left[\!\!\left[\frac{n^3}{2^\lambda} \in \left[\frac{j}{2^\kappa}, \frac{j+1}{2^\kappa}\right) + \mathbb{Z} \right]\!\!\right].$$

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(3.) The rightmost factor is approximated by a trigonometric polynomial, evaluated at $(2^\rho n_1 + n_0)^3/2^\lambda$, which only depends on n_1 in a quadratic manner.

Even sketchier idea of the proof

- Applying van der Corput's inequality another time, the argument becomes linear in n_1 (cf. van der Corput difference theorem). At this point all squares and cubes have been eliminated, at the cost of a much longer summation.

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- The resulting trigonometric polynomial in n_1 is *decoupled* from the sum over n, using suitable arithmetic subsequences and summation by parts. (Note that "everything is linear"!)
- The trigonometric component yields a geometric sum

$$\sum_{0 \le h < H} \mathrm{e}(hx) \ll \min\Bigl(H, \|x\|^{-1}\Bigr),$$

where $\|x\|$ is the distance of x to the nearest integer.

THE AVERAGE IN x OF THIS EXPRESSION IS ONLY $\log H$ IN SIZE!

- Due to the small (logarithmic) contribution of the critical interval, we only have to obtain a small gain in the sum-of-digits component. This component is basically of the form

$$\sum \mathrm{e}\left(s^{[0,2\rho)}(dn+a) \right).$$

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  (In fact, four different slopes d_0, d_1, d_2, d_3 play a role, coming from two applications of van der Corput ...)
- This is amenable to an iterated digit-elimination procedure [S2020].
- As in that paper, we end up with a *Gowers norm* for the Thue–Morse sequence, which was estimated by Konieczny (2019) (and Byszewski–Konieczny–Müllner 2023 for general automatic sequences):

$$\frac{1}{2^{(m+1)\rho}} \sum_{n,r_1,\ldots,r_m<2^\rho} \prod_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} \mathbf{u}\Bigl(n+\sum_{1\le i\le m} \varepsilon_i r_i \bmod 2^\rho\Bigr) = O\bigl(\exp(-c\rho)\bigr).$$

Essence of the proof

The additional sum introduced for digit detection in the critical interval only contributes a logarithm. A linear digital problem remains, which can be handled by iterated digit block elimination.

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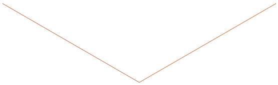
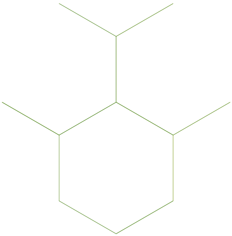
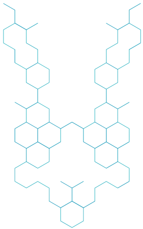
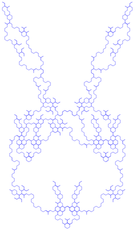
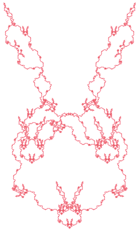
  Preprint, arXiv:2308.09498.

THANK YOU!

Supported by the FWF–ANR joint project ArithRand, and P36137 (FWF).

van der Corput's inequality

Lemma

Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left| \sum_{n \in I} a_n \right|^2 \leq \frac{N + K(R-1)}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R} \right) \sum_{\substack{n \in I \\ n+Kr \in I}} a_{n+Kr} \overline{a_n}.$$

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Instead of the original sum, we now have to estimate certain correlations (where KR will be small compared to N).

Higher degree polynomials

- Why not iterate the procedure of degree reduction?
- Note that

$$\int_0^1 \min\Bigl(H, \|x\|^{-1}\Bigr)\,\mathrm{d}x \asymp \log H,$$

while

$$\int_0^1 \Bigl|\min\Bigl(H, \|x\|^{-1}\Bigr)\Bigr|^2\,\mathrm{d}x \asymp H.$$