Thue–Morse along the sequence of cubes

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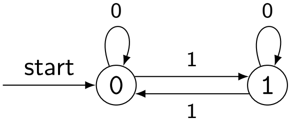
AAA7, Graz

Section 1

Subsequences of the Thue–Morse [tʉː mɔːrs] sequence

The Thue–Morse sequence

We denote the Thue–Morse sequence on $\{0,1\}$ by $\mathbf{t}$. It is given by the binary sum-of-digits function s, reduced modulo 2, or as the fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$ that starts with 0.

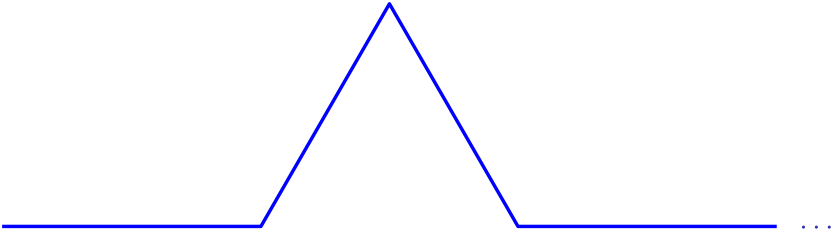


$$\mathbf{t} = 01101001100101101001011001101001\cdots$$

Also, denote the Thue–Morse sequence on $\{1,-1\}$ by $\mathbf{u}$.

Thue–Morse ⇌ Koch

Let $\mathrm{e}(x) = e^{2\pi i x}$. The sequence $n \mapsto (-1)^{s(n)}\,\mathrm{e}(-n/3)$ describes the direction of the $(n+1)$th segment in the "unscaled Koch curve":



Partial sums of $(-1)^{s(n)}\,\mathrm{e}(-n/3)$, reverse zoom

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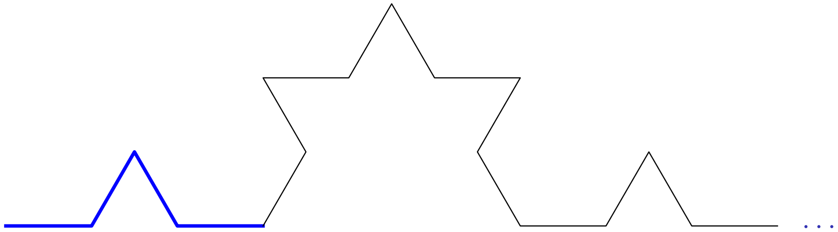

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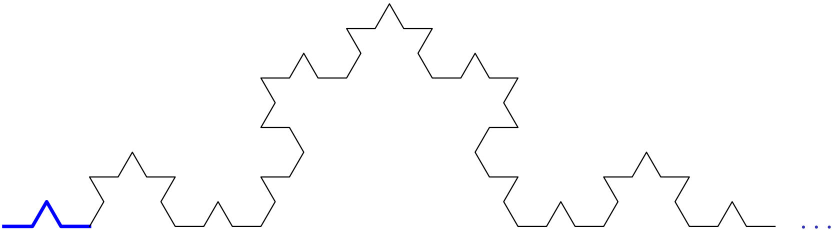

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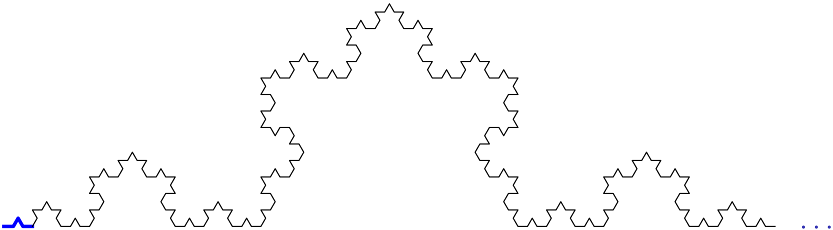

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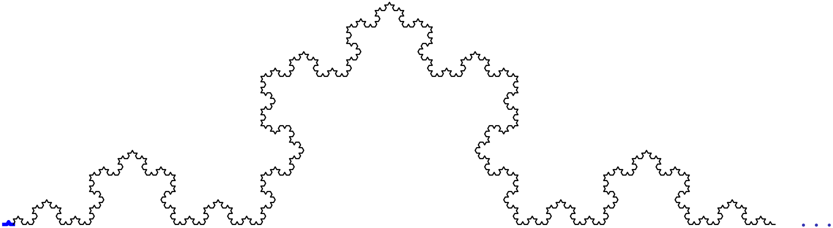

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Arithmetic subsequences of s

Arithmetic subsequences of $\mathbf{t}$ are automatic.

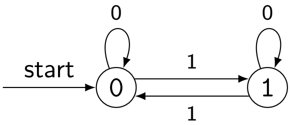


Figure: An automaton for $\mathbf{t}(n)$

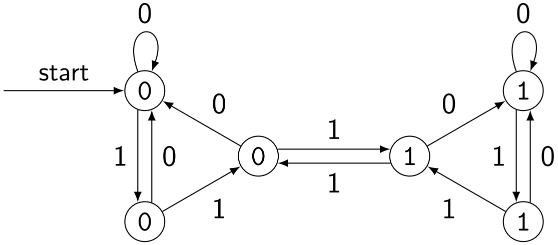
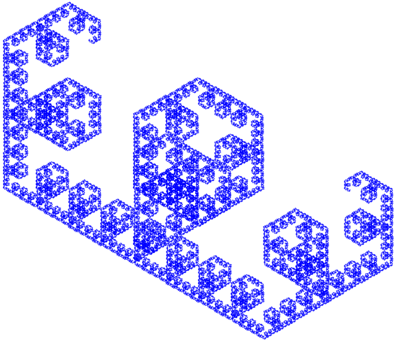


Figure: An automaton for $\mathbf{t}(3n)$

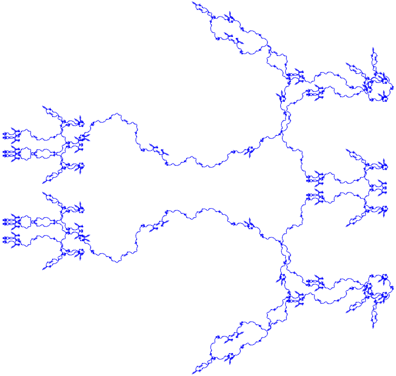
The sum of digits along arithmetic progressions

For all integers $d \geq 0$ and rationals x and y, the sequence

$n \mapsto \mathrm{e}\big(s(dn)x + ny\big)$ is 2-automatic. Partial sums yield interesting pictures.

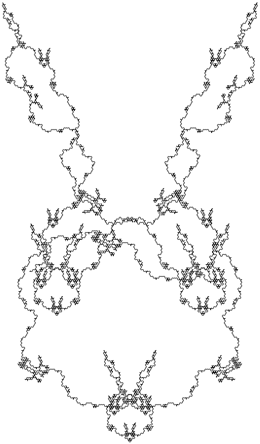


$\mathrm{e}(s(3n)/3)(-1)^n$
"Lévy"



$\mathrm{e}(s(7n)/3)(-1)^n$
"Rabbit"

The sum of digits along arithmetic progressions



$\mathrm{e}(s(7n)/3)(-1)^n$, closeup

Source: Wikipedia, "Rabbit"

The sum of digits along arithmetic progressions

The function s_q along arithmetic progressions is uniformly distributed in residue classes modulo m if $\gcd(q-1, m) = 1$. We state the following special case.

Theorem (Gelfond 1968)

Let $d \geq 1$ and a be integers. There is an absolute $\lambda < 1$ such that

$$\left|\{1 \leq n \leq x : \mathbf{t}(n) = 0, n \equiv a \bmod d\}\right| = \frac{x}{2d} + \mathcal{O}\left(x^{\lambda}\right).$$

Very sparse arithmetic subsequences of $\mathbf{t}$

The Thue–Morse sequence has mean value around $1/2$ along most very short arithmetic progressions — $\mathbf{t}$ (and $\mathbf{u}$) has "level of distribution 1".

Theorem (S. 2020)

For all $\varepsilon > 0$ we have

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} \mathbf{u}(n) \right| \le C x^{1-\eta}$$

for some C and $\eta > 0$ depending on ε, where $D = x^{1-\varepsilon}$.

In more relaxed language: let $R > 0$. As $N \to \infty$, the following holds.

Most $d \asymp N^R$ have the property that

$$\#\{0 \le n < N : \mathbf{t}(nd + a) = 0\}$$

is close to $N/2$ for all shifts a.

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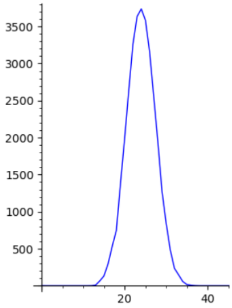
Problem ☕

Prove that for most $d \asymp N^R$,

$$m \mapsto \# \{0 \le n < N : s(nd + a) = m\}$$

closely follows a Gaussian for all shifts a.



2^{15} terms of $s\left(3^{21} n\right)$

Polynomials of higher degree: Gelfond's third problem

Let $S = s_q$ be the sum-of-digits function in base $q \geq 2$.

> *Finalement, signalons comme problème à résoudre l'estimation du nombre des valeurs du polynôme $P(t)$ ne prenant que des valeurs entières sur l'ensemble [...] des entiers rationels, pour lesquelles on a $S[P(n)] \equiv \ell \bmod m$.*

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That is, if P is a polynomial such that $P(\mathbb{N}) \subseteq \mathbb{N}$, we are interested in

$$A(q, P, m, \ell, x) := \#\{n < x : s_q(P(n)) \equiv \ell \bmod m\}.$$

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$$\mathbf{t} = \big(s_2(n) \bmod 2\big)_{n \geq 0}$$
$$= (\overset{\downarrow}{0}\overset{\downarrow}{1}10\overset{\downarrow}{1}0011\overset{\downarrow}{0}010110\overset{\downarrow}{1}0010110\overset{\downarrow}{0}\overset{\downarrow}{1}1010011001\overset{\downarrow}{0}110011010010\overset{\downarrow}{1}\cdots)$$

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- For "sufficiently large bases" q coprime to the leading coefficient of P, and $\gcd(q-1, m) = 1$, the equivalence $A(q, P, m, \ell, x) \sim x/m$ has been proved (Drmota–Mauduit–Rivat 2011).

The main result

Theorem (S. 2024+)

There exist real numbers $c > 0$ and C such that for all $x \geq 1$,

$$\left|\#\left\{n < x : \mathbf{t}\left(n^3\right) = 0\right\} - \frac{x}{2}\right| \leq Cx^{1-c}. \tag{2}$$

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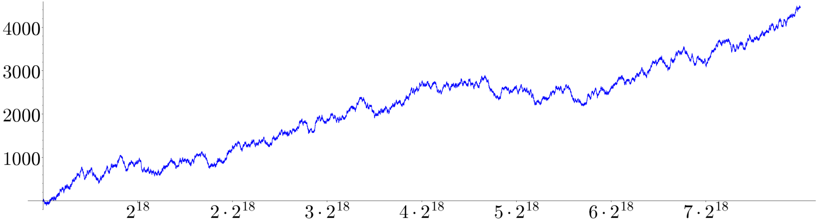

Partial sums of $\mathbf{u}(n^3)$

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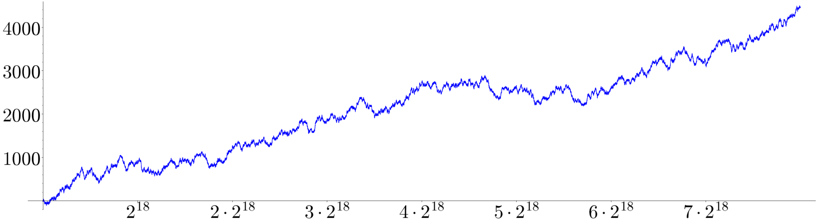

Partial sums of $\mathbf{u}\left(n^3\right)$

Caveat. Currently, c is not guaranteed to be greater than $2^{-500000}$.

Section 2

Sketch of the proof

Carry Lemma (Mauduit–Rivat 2009, 2010)

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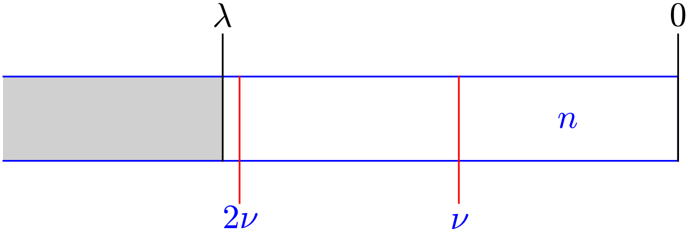
where r is *small* compared to 2^ν.
- The arguments $(n+r)^3$ and n^3 *usually* have the same digits with indices above

$$\lambda := \nu(2+\varepsilon).$$

- These digits can therefore be discarded.

Carry lemma: a picture



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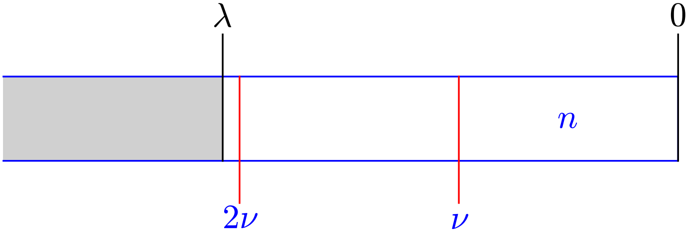

- In the correlation $\mathbf{u}\big((n+r)^3\big)\mathbf{u}\big(n^3\big)$, we may replace $\mathbf{u}$ by the *restricted Thue–Morse sequence* $\mathbf{u}^{[0,\lambda)}(n) := \mathbf{u}\big(n \bmod 2^\lambda\big)$.

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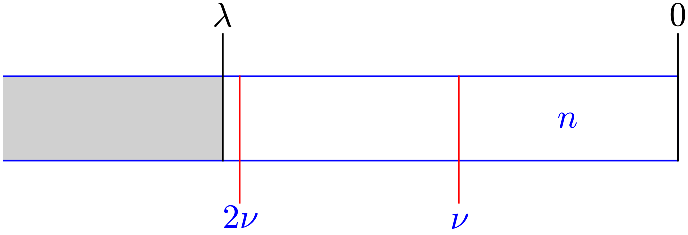

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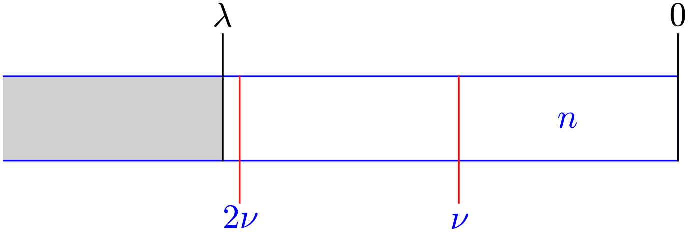

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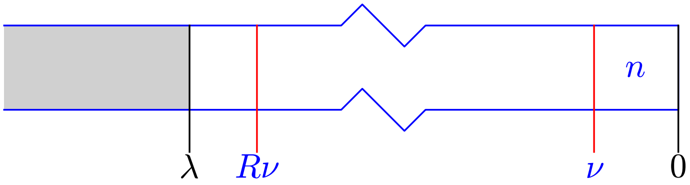
- The remaining window $[0,\lambda)$ of digits is about twice as long as the binary expansion of n.
- Therefore we cannot obtain uniform distribution of these digits, as n runs.

Too many significant digits

- A similar problem arises for sparse arithmetic progressions $nd + a$, where $n < 2^{\nu}$ and $d \gg 2^{R\nu}$: the binary digits of $(n + r)d + a$ and $nd + a$ usually differ up to index $\approx R\nu$.



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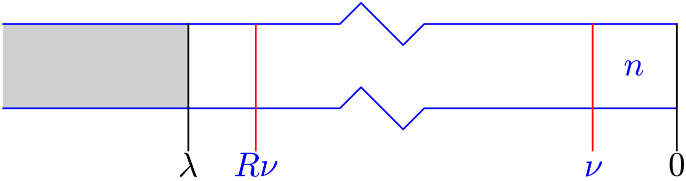

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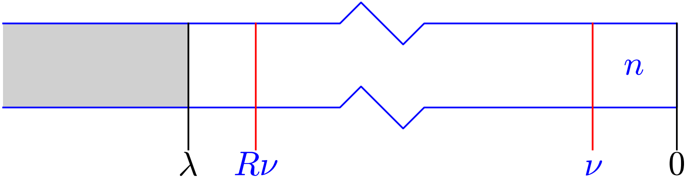

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Wikipedia: "Salami", by Aka (CC BY-SA 2.5)

Gowers norm

Iterated application $\bigl(\text{Cauchy–Schwarz, van der Corput}\bigr)^m$ leads to higher order correlations of the Thue–Morse sequence, more precisely, a *Gowers norm*. This was estimated by Konieczny (2019) (and for general automatic sequences, by Byszewski–Konieczny–Müllner 2023): for some $c = c(m) > 0$,

$$\frac{1}{2^{(m+1)\rho}} \sum_{n,r_1,\ldots,r_m<2^\rho} \prod_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} \mathbf{u}^{[0,\rho)}\Biggl(n + \sum_{1\le i\le m} \varepsilon_i r_i\Biggr) = O\bigl(\exp(-c\rho)\bigr).$$

Iterated van der Corput could so far not be used for removing sufficiently many digits of polynomial values $P(n)$, if $\deg P > 2$.

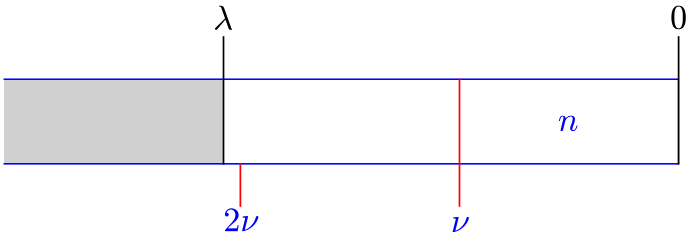
A trivial decomposition[1]

- Choose $\rho < \nu$ in such a way that $3\rho \geq \lambda$, and write

$$n = 2^\rho n_1 + n_0, \quad \text{where} \quad \begin{cases} 0 \leq n_1 < 2^{\nu-\rho}, \\ 0 \leq n_0 < 2^\rho. \end{cases}$$

- Expanding $n^3 \bmod 2^\lambda$, we see that the cubic term in n_1 disappears.

[1] Thanks to Michael Drmota, "maybe this can also be used for the cubes"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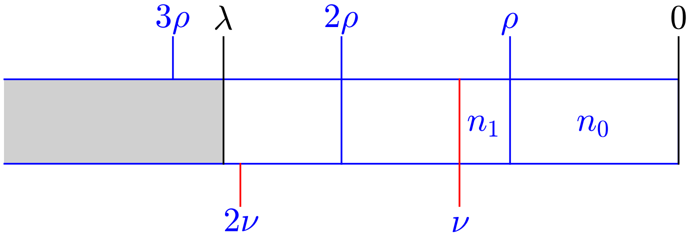

3ρ
λ
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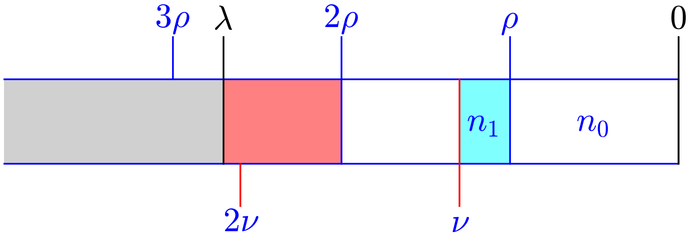
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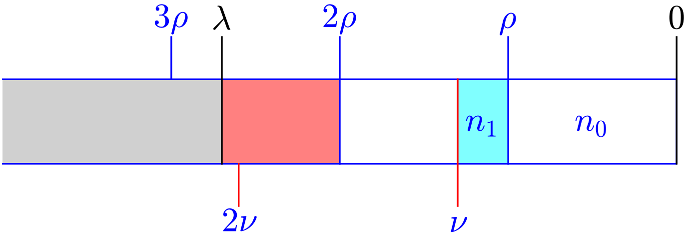
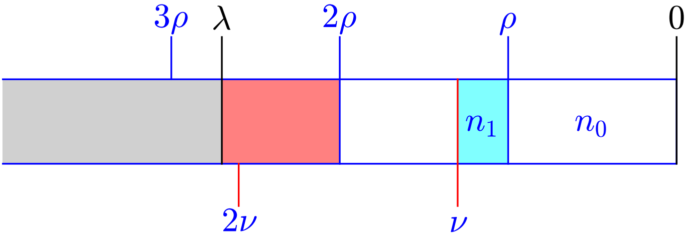


- On the critical interval $[2\rho, \lambda)$ of length $\kappa := \lambda - 2\rho$, the term n_1^2 is still relevant.



- On the critical interval $[2\rho, \lambda)$ of length $\kappa := \lambda - 2\rho$, the term n_1^2 is still relevant.
- We introduce an additional sum $\sum_{0 \le j < 2^\kappa}$ that parametrizes the digit combinations in this interval.

The critical interval of digits

We write

$$S_0 = \sum_{n<2^\nu} \mathbf{u}\big(n^3\big) = \sum_{n<2^\nu} \mathbf{u}^{[2\rho,\lambda)}\big(n^3\big)\,\mathbf{u}^{\mathbb{N}\setminus[2\rho,\lambda)}\big(n^3\big)$$

$$= \sum_{0\le j<2^\kappa} \mathbf{u}(j) \sum_{n<2^\nu} \mathbf{u}^{\mathbb{N}\setminus[2\rho,\lambda)}\big(n^3\big) \left[\!\left[\frac{n^3}{2^\lambda} \in \left[\frac{j}{2^\kappa}, \frac{j+1}{2^\kappa}\right) + \mathbb{Z} \right]\!\right].$$

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(1.) An additional sum of length 2^κ is introduced;

The critical interval of digits

We write

$$S_0 = \sum_{n<2^\nu} \mathbf{u}\bigl(n^3\bigr) = \sum_{n<2^\nu} \mathbf{u}^{[2\rho,\lambda)}\bigl(n^3\bigr)\,\mathbf{u}^{\mathbb{N}\setminus[2\rho,\lambda)}\bigl(n^3\bigr)$$

$$= \sum_{0\le j<2^\kappa} \mathbf{u}(j) \sum_{n<2^\nu} \mathbf{u}^{\mathbb{N}\setminus[2\rho,\lambda)}\bigl(n^3\bigr) \left[\!\left[\frac{n^3}{2^\lambda} \in \left[\frac{j}{2^\kappa}, \frac{j+1}{2^\kappa}\right) + \mathbb{Z}\right]\!\right].$$

(1.) An additional sum of length 2^κ is introduced;

(2.) The rightmost factor detects whether j corresponds to the digits of n^3 in the critical interval $\{\lambda-\kappa,\ldots,\lambda-1\}$.

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(2.) The rightmost factor detects whether j corresponds to the digits of n^3 in the critical interval $\{\lambda-\kappa, \ldots, \lambda-1\}$.

(3.) After cutting away also the digits with indices $\ge \lambda$ (carry lemma),

A LINEAR SUM-OF-DIGITS PROBLEM IN n_1 REMAINS.

Introducing trigonometric polynomials

- The “green” detecting term is approximated by a trigonometric polynomial T, evaluated at $(2^\rho n_1 + n_0)^3/2^\lambda$. The term n_1^3 does not appear in the argument of T.

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- Applying van der Corput’s inequality another time, the argument of T becomes linear in n_1, yielding a proper trigonometric polynomial in n_1 (cf. uniform distribution mod 1 of polynomial values).
- At this point all cubes and squares have been eliminated, at the cost of a much longer summation.

Decoupling the trigonometric part

- The trigonometric polynomial in n_1 is *decoupled* from the sum over n, using suitable arithmetic subsequences and summation by parts. (Note that "everything is linear"!)

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- The trigonometric component yields a geometric sum

$$\varphi_H(x) = \sum_{0 \le h < H} \mathrm{e}(hx) \ll \min\Big(H, \|x\|^{-1}\Big),$$

where $\|x\|$ is the distance of x to the nearest integer.

The average in x of $\varphi_H(x)$ is only $\log H$ in size!

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- Only arithmetic progressions play a role, which is is amenable to an iterated digit-elimination procedure [S2020]. This yields a gain N^{-c} for *some* $c > 0$, easily swallowing the logarithm.

Essence of the proof

The additional sum introduced for digit detection in the critical interval only contributes a logarithm. A linear digital problem remains, which can be handled by iterated digit block elimination.

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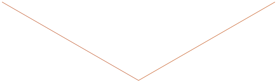
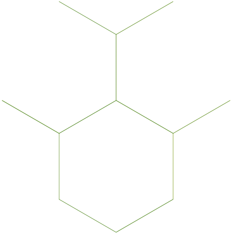
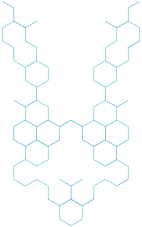
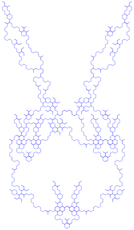
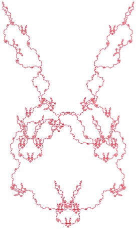




THANK YOU!

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van der Corput's inequality

Lemma

Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left|\sum_{n \in I} a_n\right|^2 \leq \frac{N + K(R-1)}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \sum_{\substack{n \in I \\ n+Kr \in I}} a_{n+Kr}\overline{a_n}.$$

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Instead of the original sum, we now have to estimate certain correlations (where KR will be small compared to N).

Higher degree polynomials

- Why not iterate the procedure of degree reduction?
- Note that

$$\int_0^1 \min\Bigl(H, \|x\|^{-1}\Bigr)\,\mathrm{d}x \asymp \log H,$$

while

$$\int_0^1 \Bigl|\min\Bigl(H, \|x\|^{-1}\Bigr)\Bigr|^2\,\mathrm{d}x \asymp H.$$