TOPOLOGY OF CRYSTALLOGRAPHIC TILES

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Abstract. We study self-affine tiles which tile the n-dimensional real vector space with respect to a crystallographic group. First we define classes of graphs that allow to determine the neighbors of a given tile algorithmically. In the case of plane tiles these graphs are used to derive a criterion for such tiles to be homeomorphic to a disk. As particular application, we will solve a problem of Gelbrich, who conjectured that certain examples of tiles which tile $\mathbb{R}^2$ with respect to the ornament group $p2$ are homeomorphic to a disk.

1. Introduction

The present paper is devoted to the study of self-affine tiles that tile $\mathbb{R}^n$ with respect to a crystallographic group. Before we describe our detailed aims we give some basic definitions.

Let $\mathcal{T} \subset \mathbb{R}^n$ be a compact set that is equal to the closure of its interior $\mathcal{T}^o$. If there is a family Γ of isometries such that

$$\mathbb{R}^n = \bigcup_{\gamma\in\Gamma} \gamma(\mathcal{T}) \quad \text{where} \quad \gamma(\mathcal{T}^o) \cap \delta(\mathcal{T}^o) = \emptyset \quad \text{for} \quad \gamma, \delta \in \Gamma \quad \text{with} \quad \gamma \neq \delta,$$

then we say that $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ is a *tiling* of $\mathbb{R}^n$ using a single *tile* $\mathcal{T}$. We also call $\gamma(\mathcal{T})$ a tile for each $\gamma \in \Gamma$. In the special case where Γ is isomorphic to $\mathbb{Z}^n$, the collection $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ is

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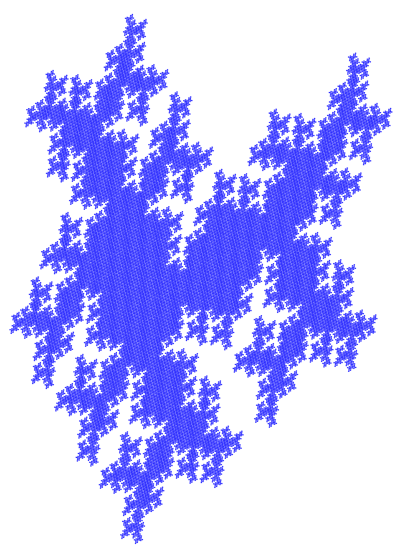


Figure 1. Example of a crystallographic reptile

a *lattice tiling* of $\mathbb{R}^n$. If, more generally, Γ is a crystallographic group, $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ is called a *crystallographic tiling* of $\mathbb{R}^n$. Recall that a *crystallographic group* in dimension n is a discrete cocompact subgroup Γ of the group $\mathrm{Isom}(\mathbb{R}^n)$ of all isometries on $\mathbb{R}^n$ with respect to a metric d.

The main object in this paper are crystallographic tilings whose tiles are self-affine sets.

Definition 1 (cf. [5])**.** A *crystallographic reptile* with respect to a crystallographic group Γ is a set $\mathcal{T} \subset \mathbb{R}^n$ with the following properties:

- The family $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ is a tiling of $\mathbb{R}^n$.
- There is an expanding affine map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$, and a finite collection $\mathcal{D} \subset \Gamma$ called *set of digits*, such that

$$g(\mathcal{T}) = \bigcup_{\delta \in \mathcal{D}} \delta(\mathcal{T}). \tag{1.1}$$

We will call a crystallographic reptile also simply a *crystile*.

Recall that the compact non-empty set $\mathcal{T}$ is uniquely defined by (1.1). This is easily seen by a fixed point argument (cf. Hutchinson [8]). Let $a \in \mathbb{R}^n$ be arbitrary. By iterating (1.1) we easily see that the crystile $\mathcal{T}$ can be written as

$$\mathcal{T} \;=\; \left\{ \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a),\ (\delta_j)_{j\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}} \right\}.$$

The choice of a is irrelevant since (1.1) implies that if $(\delta_j)_{j\in\mathbb{N}}$ is a sequence of digits, then for $a, a' \in \mathbb{R}^n$ we have

$$\lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a) \;=\; \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a'). \tag{1.2}$$

Remark 1. The following basic facts and some more details can also be found for instance in Gelbrich [5].

- By a Bieberbach theorem (see [4]), a group Γ is a crystallographic group in dimension n if and only if there is a normal subgroup $\Lambda \subset \Gamma$ with finite index, which is isomorphic to $\mathbb{Z}^n$, and which is a maximal abelian subgroup in Γ. The quotient group Γ/Λ is called the *point group*, and the group Λ is called the lattice. For a crystallographic group Γ, the lattice Λ consists of translations. In this paper, we always assume that $\Lambda = \mathbb{Z}^n$ and that the metric d of $\mathbb{R}^n$ is chosen in a way that each element $\gamma \in \Gamma$ is an isometry. So, if the point group is trivial then the group Γ is just the lattice $\mathbb{Z}^n$ and $\mathcal{T}$ is a lattice reptile.
- Let $\mathcal{T}, \mathcal{T}'$ be two crystiles with respect to crystallographic groups Γ, Γ', with expansions g, g' and digit sets $\{\gamma_1, \dots, \gamma_q\}, \{\gamma'_1, \dots, \gamma'_q\}$, respectively. An affine bijection $\phi : \mathcal{T} \to \mathcal{T}'$ is said to preserve pieces of level $m \in \mathbb{N}$ if for each sequence of indices $i_1, \dots, i_m$ there is a sequence $j_1, \dots, j_m$ such that ϕ is a bijection from $\left(g^{-1}\gamma_{i_1}\right) \circ \dots \circ \left(g^{-1}\gamma_{i_m}\right)\ (\mathcal{T})$ onto $\left(g'^{-1}\gamma'_{j_1}\right) \circ \dots \circ \left(g'^{-1}\gamma'_{j_m}\right)\ (\mathcal{T})$. We say that two crystiles $\mathcal{T}$ and $\mathcal{T}'$ are *isomorphic* iff there is an affine bijection $\phi : \mathcal{T} \to \mathcal{T}'$ preserving the pieces of all levels.
- For a crystile $\mathcal{T}$ w.r.t. Γ, each of the collections $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ and $\{g\gamma(\mathcal{T}) = g \circ \gamma \circ g^{-1}\left(g(\mathcal{T})\right) : \gamma \in \Gamma\}$ is a tiling of $\mathbb{R}^n$. We also say that the two sets $\mathcal{T}$ and $g(\mathcal{T})$ tile $\mathbb{R}^n$ under the action of the groups Γ and $g\Gamma g^{-1}$, respectively. Thus $\mathcal{D}$ is a complete set of right coset representatives of the subgroup $g\Gamma g^{-1}$.
- The reptile $\mathcal{T}$ is a self-affine set with respect to the IFS $\{g^{-1}\delta : \delta \in \mathcal{D}\}$. Hence, by replacing g by ${\delta_0}^{-1}g$ for some $\delta_0 \in \mathcal{D}$ and $\mathcal{D}$ by ${\delta_0}^{-1}\mathcal{D}$, we may assume that 1, the identity of Γ, is always contained in $\mathcal{D}$, because the iterated function system $\left\{\left(g^{-1}\delta_0\right)\left({\delta_0}^{-1}\delta\right) : \delta \in \mathcal{D}\right\}$ is equal to $\{g^{-1}\delta : \delta \in \mathcal{D}\}$.

Gelbrich [5] proved that for any $k \geq 2$ there are at most finitely many isomorphy classes of disk-like plane crystallographic reptiles with k digits. Here, a set is *disk-like* if it is homeomorphic to the closed unit disk. Sometimes, such a set is also called a topological disk.

In the same paper he gave all candidates for disk-like crystallographic reptiles with three digits for the case that the point group Γ/Λ has 2 elements. As $p2$ is the crystallographic group corresponding to such a point group (see [6]), we will shortly call these tiles $p2-$crystiles. Up to now no proof has been given for disk-likeness of those candidates.

The aim of this paper is to study topological properties of crystiles. We give an algorithmic way to construct all the tiles $\gamma(\mathcal{T})$ having non-empty intersection with the "central tile" $\mathcal{T}$ in a crystallograhic tiling and give some upper bound for the runtime of this algorithm. In fact, the set of the neighboring tiles forms the set of vertices of a graph which we call the neighborhood graph of the tiling. In the case of two dimensional crystiles this neighborhood graph (together with several Cayley-like graphs) is used in order to establish an algorithmic criterion which allows to check whether a given plane crystile is homeomorphic to a disk or not. This criterion will be applied to Gelbrich's example in order to show which of his candidates are really disk-like.

We mention that related questions have been already considered in literature. One of the first papers dealing with topological properties of iterated function systems is Hata [7]. Luo et al. [11] prove that a self-affine tile with nonempty interior is disk-like if its interior is connected. For the case of lattice tiles, Bandt and Wang [3] gave a criterion for disk-likeness in terms of the number of neighbors of the central tile. Disk-likeness for classes of lattice tiles are studied for instance by Akiyama and Thuswaldner [2] and by Luo and Zhou [12]. For properties of self-affine sets that are not homeomorphic to a disk we refer for instance to Ngai and Nguyen [13], Ngai and Tang [15, 14], and to the survey paper [1].

The present paper is organized as follows. In Section 2 we define several classes of graphs which will be needed in the sequel. Section 3 is devoted to the construction of the neighborhood graph. In Section 4 we state and prove our criterion for the disk-likeness of a plane crystile. Section 5 contains the treatment of Gelbrich's examples.

2. Definition of graphs

This section is devoted to the definition of sets and graphs that will play a role in many theorems and proofs in this paper.

In the sequel assume that $\mathcal{T}$ is a crystile with respect to the crystallographic group Γ. Denote by g its expanding map and by $\mathcal{D} \subset \Gamma$ its set of digits, which is supposed to contain 1, the identity of Γ. Then $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ is a crystallographic tiling.

The *set of neighbors* of $\mathcal{T}$ is defined by

$$\mathcal{S} := \{\gamma \in \Gamma \setminus \{1\} : \mathcal{T} \cap \gamma(\mathcal{T}) \neq \emptyset\}. \tag{2.1}$$

Because of the local finiteness of the tiling (recall that Γ is discrete and $\mathcal{T}$ is compact), $\mathcal{S}$ is a finite set. Moreover, for $\gamma \in \mathcal{S}$, let

$$B_\gamma := \mathcal{T} \cap \gamma(\mathcal{T}).$$

Among the possible neighbors of a tile, we mark out the following ones. An element $\gamma \in \mathcal{S}$ is called

- *vertex neighbor* if $B_\gamma = \{x\}$ for some $x \in \mathbb{R}^n$;
- *adjacent* or *edge neighbor* if B_γ contains a point of $\mathrm{int}(\mathcal{T} \cup \gamma(\mathcal{T}))$.

The set of adjacent neighbors is denoted by $\mathcal{A}$.

In order to construct the set $\mathcal{S}$ in the next section we require a set $\mathcal{R}$ which is related to the neighbors of certain approximations of $\mathcal{T}$. It is defined as follows. Let Q be an n-dimensional polyhedron that is the closure of a fundamental domain of Γ. The boundary of Q consists of a union of $(n-1)$-dimensional faces (hyperplanes of $\mathbb{R}^n$).
We can define $\mathcal{R}_0$ as the set of adjacent neighbors of the fundamental domain together with the

identity, *i.e.*,

$$\begin{aligned}\mathcal{R}_0 &:= \{1\} \cup \{\gamma \in \Gamma,\ \gamma(Q) \cap Q \text{ is an } (n-1)\text{-dimensional face}\} \\ &= \{1\} \cup \{\gamma \in \Gamma,\ \gamma(Q) \cap Q \text{ contains an inner point of } \gamma(Q) \cup Q\}\ .\end{aligned}$$

Starting from this set, we define a sequence $(\mathcal{R}_p)_{p\geq 1}$ recursively by

$$\mathcal{R}_p := \mathcal{R}_{p-1} \cup \{\gamma \in \Gamma,\ g\gamma g^{-1}\ \mathcal{D}\ \cap\ \mathcal{D}\,\mathcal{R}_{p-1}\ \neq \emptyset\} \tag{2.2}$$

and set: $\mathcal{R} = \bigcup_{p\geq 0} \mathcal{R}_p$.

Since g is an expanding mapping and Γ is a discrete group, the set $\mathcal{R}$ is finite: $\mathcal{R}_p$ eventually stabilizes, *i.e.*, there is a $p_0 \in \mathbb{N}$ such that $\mathcal{R}_p = \mathcal{R}_{p_0}$ for $p \geq p_0$. The obtention of some estimate for p_0 will be discussed in Remark 5.

Having defined these sets we are now in a position to give the definition of the graphs needed in the sequel.

Definition 2. For $M \subseteq \Gamma$ we define the graph $\mathbf{G}(M)$ as follows. The states $\mathbf{G}(M)$ are the elements of M. Moreover, there is an edge

$$\gamma \xrightarrow{\delta|\delta'} \gamma' \in \mathbf{G}(M) \text{ iff } \delta^{-1}\ g\gamma g^{-1}\ \delta'\ =\ \gamma' \text{ with } \gamma, \gamma' \in M \text{ and } \delta, \delta' \in \mathcal{D}.$$

The following special cases are of particular importance.

- The *neighborhood graph* $\mathbf{G}(\mathcal{S})$.
- The *contact graph* $\mathbf{G}(\mathcal{R})$.

Besides these graphs the following variants of Cayley graphs are needed.

Definition 3. The *adjacent graph* $\mathcal{G}_A(\Gamma)$ is defined to have Γ as set of states, and two distinct elements $\gamma_1, \gamma_2 \in \Gamma$ are incident if and only if $\gamma_1^{-1}\gamma_2 \in \mathcal{A}$. The subgraph $\mathcal{G}_A(M)$ for a subset M of Γ is then simply the restriction of $\mathcal{G}_A(\Gamma)$ to the set of states M.
The *double neighboring graph* $\mathcal{G}_2$ is the one having $\{\gamma\,(B_{\gamma'}) : \gamma \in \Gamma, \gamma' \in \mathcal{A}\}$ as set of states and in which two distinct elements $\gamma_1\left(B_{\gamma_1'}\right)$ and $\gamma_2\left(B_{\gamma_2'}\right)$ are incident whenever they intersect each other.

The following graph is a subgraph of $\mathcal{G}_2$. Its definition looks a bit awkward. We will explain it in Remark 2.

Definition 4. For each $\gamma \in \mathcal{A}$, denote by $\mathcal{V}_\gamma$ the set of states

$$\{\delta\left(B_{\delta^{-1}g\gamma g^{-1}\delta'}\right) :\ \text{if there is } \delta, \delta' \in \mathcal{D} \text{ with } \delta^{-1}g\gamma g^{-1}\delta' \in \mathcal{A}\}\ .$$

Then $\mathcal{G}_\gamma$ is the subgraph of the double neighboring graph $\mathcal{G}_2$ with set of vertices $\mathcal{V}_\gamma$.

3. Neighborhood and contact graphs

In the present section we will show how the neighborhood graph $\mathbf{G}(\mathcal{S})$ associated to a crystallographic tile can be used to characterize the boundary of the tile. Properties of the contact graph $\mathbf{G}(\mathcal{R})$ will also be given, and from this graph we will obtain the neighborhood graph algorithmically.
In this section, $\mathcal{T}$ denotes a crystallographic reptile (or *crystile*) of $\mathbb{R}^n$ with respect to a crystallographic group Γ; the expanding map is g and the *set of digits* $\mathcal{D} \subset \Gamma$ is assumed to contain 1, the identity map of Γ.

3.1. **Neighborhood graph.** The non overlapping property yields for the boundary of $\mathcal{T}$ that $\partial\mathcal{T} = \bigcup_{\gamma\in\mathcal{S}} B_\gamma$, with $\mathcal{S}$ and B_γ as in Section 2. The following holds for every $\gamma \in \mathcal{S}$.

$$\begin{aligned} g(B_\gamma) &= g(\mathcal{T} \cap \gamma(\mathcal{T})) \;=\; g(\mathcal{T}) \cap g\gamma(\mathcal{T}) \\ &= \textstyle\bigcup_{\delta\in\mathcal{D}} \delta(\mathcal{T}) \;\cap\; \bigcup_{\delta'\in\mathcal{D}} g\gamma g^{-1}\,\delta'(\mathcal{T}) \\ &= \bigcup_{\delta,\delta'\in\mathcal{D}} \delta(\underbrace{\mathcal{T} \;\cap\; \delta^{-1} g\gamma g^{-1}\,\delta'(\mathcal{T})}_{B_{\delta^{-1}\,g\gamma g^{-1}\,\delta'}}) \end{aligned}$$

Consequently, we obtain the following set equation for B_γ:

$$B_\gamma \;=\; \bigcup_{\delta,\delta'\in\mathcal{D}} g^{-1}\delta(\;B_{\delta^{-1}\,g\gamma g^{-1}\,\delta'}\;)\;.$$

Remark 2. This calculation gives an explanation to the graph $\mathcal{G}_\gamma$ defined in Definition 4. Indeed, the union of its set of vertices is equal to $g(B_\gamma)$. Thus $\mathcal{G}_\gamma$ contains some information about the connectivity of $g(B_\gamma)$ and, hence, of B_γ. Indeed, we will use this graph to show that B_γ is connected under certain circumstances.

Using the neighborhood graph defined in Definition 2 and the fact that $B_{\gamma'} \neq \emptyset$ iff $\gamma' \in \mathcal{S}$, the set equation above reduces to:

$$\partial\mathcal{T} \;=\; \bigcup_{\gamma\in\mathcal{S}} B_\gamma \text{ where } B_\gamma \;=\; \bigcup_{\substack{\delta\in\mathcal{D},\gamma'\in\mathcal{S},\\ \exists\,\delta'\in\mathcal{D},\;\gamma\xrightarrow{\delta|\delta'}\gamma'\,\in\mathbf{G}(\mathcal{S})}} g^{-1}\delta(B_{\gamma'})\;.$$

Remark 3. The graph $\mathbf{G}(\mathcal{S})$ has the following properties:

- To given $\gamma, \gamma' \in \mathcal{S}, \delta \in \mathcal{D}$, there is at most one $\delta' \in \mathcal{D}$ with $\gamma \xrightarrow{\delta|\delta'} \gamma'$ in $\mathbf{G}(\mathcal{S})$.
- The graph $\mathbf{G}(\mathcal{S})$ is left-resolving, *i.e.*,

$$\forall\;(\gamma',\delta') \in \mathcal{S}\times\mathcal{D},\;\exists!\;(\gamma,\delta)\in\mathcal{S}\times\mathcal{D}:\;\gamma \xrightarrow{\delta|\delta'} \gamma' \;\in \mathbf{G}(\mathcal{S}).$$

Indeed, if $(\gamma',\delta') \in \mathcal{S}\times\mathcal{D}$ then there exists exactly one $(\gamma,\delta) \in \Gamma\times\mathcal{D}$ such that the equality $\gamma'\delta'^{-1} = \delta^{-1}g\gamma g^{-1}$ holds. This is true because $\mathcal{D}$ is a complete set of right coset representatives of $g\Gamma g^{-1}$. Moreover,

$$\begin{aligned} &\mathcal{T}\cap\gamma'(\mathcal{T}) \neq \emptyset \\ \Rightarrow\;& \mathcal{T}\cap\delta^{-1}g\gamma g^{-1}\delta'(\mathcal{T}) \neq \emptyset \\ \Rightarrow\;& \underbrace{g^{-1}\delta(\mathcal{T})}_{\subset\mathcal{T}} \cap \underbrace{\gamma g^{-1}\delta'(\mathcal{T})}_{\subset\gamma(\mathcal{T})} \neq \emptyset \\ \Rightarrow\;& \mathcal{T}\cap\gamma(\mathcal{T}) \neq \emptyset\;. \end{aligned}$$

This implies that $\gamma \in \mathcal{S}$, and by the definition of $\mathbf{G}(\mathcal{S})$ we conclude that $\gamma \xrightarrow{\delta|\delta'} \gamma' \;\in \mathbf{G}(\mathcal{S})$.

One can wonder how many tiles of the tiling meet the tile $\mathcal{T}$ at a point of the boundary of $\mathcal{T}$.

Definition 5. For $\gamma_1,\dots,\gamma_L \;\in \mathcal{S}$ pairwise different, we call

$$V_L(\gamma_1,\dots,\gamma_L) = \{x \in \mathbb{R}^n,\; x \in \mathcal{T}\cap\gamma_1(\mathcal{T})\cap\ldots\cap\gamma_L(\mathcal{T})\}$$

the set of points (so-called L-vertices) that are common to $\gamma_1(\mathcal{T}),\dots,\gamma_L(\mathcal{T})$ and $\mathcal{T}$, and

$$V_L = \bigcup_{\{\gamma_1,\dots,\gamma_L\}\subseteq\mathcal{S}} V_L(\gamma_1,\dots,\gamma_L)$$

the set of L-vertices of $\mathcal{T}$.

Note that $V_1(\gamma) = B_\gamma$ holds.

There are characterizations of these sets using the neighborhood graph. Let us first characterize a point belonging to two tiles (say here to the central tile $\mathcal{T}$ and one of its neighbors). To this matter, we recall the following definition:

Definition 6. A walk in a graph G starting from a state γ of this graph is a sequence of edges

$$\gamma \xrightarrow{\delta|\delta'} \gamma_1 \xrightarrow{\delta_1|\delta_1'} \gamma_2 \xrightarrow{\delta_2|\delta_2'} \dots .$$

The number of edges in the walk is called the *length of the walk* (this can be infinite).

Characterization 3.1. *Let a be a point of $\mathbb{R}^n$, $(\delta_j)_{j\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ a sequence of digits and $\gamma \in \mathcal{S}$. Then the following assertions are equivalent.*

- $x = \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a) \in B_\gamma$.
- *There is an infinite walk in $\mathbf{G}(\mathcal{S})$ of the shape:*

$$\gamma \xrightarrow{\delta_1|\delta_1'} \gamma_1 \xrightarrow{\delta_2|\delta_2'} \gamma_2 \xrightarrow{\delta_3|\delta_3'} \dots \tag{3.1}$$

for some $\gamma_i \in \mathcal{S}$ and $\delta_i' \in \mathcal{D}$.

Proof. Suppose that $x = \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a) \in B_\gamma$, then x has also a representation of the shape:

$$x \;=\; \lim_{n\to\infty} \gamma g^{-1}\delta_1' \dots g^{-1}\delta_n'(a)$$

for some $\delta_i' \in \mathcal{D}$.
The elements $\gamma_1 = {\delta_1}^{-1} g\gamma g^{-1}\delta_1'$, $\gamma_2 = {\delta_2}^{-1} g\gamma_1 g^{-1}\delta_2', \dots$ can then successively be shown to belong to $\mathcal{S}$. Indeed, if $\gamma_1 = {\delta_1}^{-1} g\gamma g^{-1}\delta_1'$, then

$$\delta_1^{-1} g\,(x) \;=\; \underbrace{\lim_{n\to\infty} g^{-1}\delta_2 \dots g^{-1}\delta_n(a)}_{\in\mathcal{T}} \;=\; \underbrace{\lim_{n\to\infty} \gamma_1 g^{-1}\delta_2' \dots g^{-1}\delta_n'(a)}_{\in\gamma_1(\mathcal{T})},$$

and similarly for $\gamma_2, \gamma_3, \dots$ The elements $\gamma_1, \gamma_2, \dots$ now yield the required infinite walk in $\mathbf{G}(\mathcal{S})$, by the definition of the edges in this graph.

Conversely, if the infinite walk (3.1) in $\mathbf{G}(\mathcal{S})$ is given, by the definition of the edges of this graph the equality

$$g^{-1}\delta_1 \dots g^{-1}\delta_n\gamma_n(a) \;=\; \gamma g^{-1}\delta_1' \dots g^{-1}\delta_n'(a)$$

holds for every n and thus for $n \to \infty$. If now ρ is the greatest contraction factor of the contractions $g^{-1}\delta, \delta \in \mathcal{D}$, $\mathcal{S}$ being a finite set we can write

$$\forall n \in \mathbb{N},\; d\left(g^{-1}\delta_1 \dots g^{-1}\delta_n\gamma_n\,(a), g^{-1}\delta_1 \dots g^{-1}\delta_n\,(a)\right) \;\le\; \rho^n \max_{\gamma\in\mathcal{S}}\{d(a,\gamma(a))\}\;,$$

so

$$\underbrace{\lim_{n\to\infty} \gamma g^{-1}\delta_1' \dots g^{-1}\delta_n'\,(a)}_{\in\gamma(\mathcal{T})} \;=\; \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n\gamma_n\,(a) \;=\; \underbrace{\lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n\,(a)}_{\in\mathcal{T}}$$

and we are done. □

In a similar way we obtain the following generalization.

Characterization 3.2. *Let a be a point of $\mathbb{R}^n$ and $\gamma_{01}, \dots, \gamma_{0L} \in \mathcal{S}$ pairwise different. Furthermore let be $(\delta_j)_{j\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ a sequence of digits. Then the following assertions are equivalent.*

- $x = \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a) \in V_L(\gamma_{01}, \dots, \gamma_{0L})$.
- *There are L infinite walks in $\mathbf{G}(\mathcal{S})$ of the shape:*

$$\gamma_{0i} \xrightarrow{\delta_1|\delta_{1i}} \gamma_{1i} \xrightarrow{\delta_2|\delta_{2i}} \gamma_{2i} \xrightarrow{\delta_3|\delta_{3i}} \dots \quad (1 \le i \le L)$$

for some $\gamma_{1i}, \gamma_{2i} \dots \in \mathcal{S}$ and $\delta_{1i}, \delta_{2i} \dots \in \mathcal{D}$.

Characterization 3.3. *Let be $\gamma \in \Gamma$. Then:*

- *γ is a vertex neighbor iff every sequence $(\delta_j)_{j\in\mathbb{N}}$ of digits associated to an infinite walk in the graph $\mathbf{G}(\mathcal{S})$, $\gamma \xrightarrow{\delta_1|\delta_1'} \gamma_1 \xrightarrow{\delta_2|\delta_2'} \dots$ with $\gamma_1, \dots \in \mathcal{S}$ and $\delta_1', \delta_2', \dots \in \mathcal{D}$, leads to the same point $x = \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a)$.*
- *γ is an edge neighbor (i.e., $\gamma \in \mathcal{A}$) iff the set $B_\gamma \setminus V_2$ is nonempty iff there is a sequence $(\delta_j)_{j\in\mathbb{N}}$ of digits such that for each infinite walk in $\mathbf{G}(\mathcal{S})$ of the form*
$$\gamma_1 \xrightarrow{\delta_1|\delta_1'} \gamma_2 \xrightarrow{\delta_2|\delta_2'} \dots$$
where $\gamma_1, \gamma_2, \dots \in \mathcal{S}$ and $\delta_1', \delta_2', \dots \in \mathcal{D}$, we have $\gamma_1 = \gamma$.

3.2. **Contact graph.** Let Q be the closure of a fundamental domain of Γ. Q is a nonempty compact set of $\mathbb{R}^n$, so an iterative construction of $\mathcal{T}$ reads
$$\begin{cases} \mathcal{T}_0 & := & Q, \\ \mathcal{T}_p & := & \bigcup_{\delta\in\mathcal{D}} g^{-1}\delta(\mathcal{T}_{p-1}). \end{cases}$$
In the Hausdorff metric, we have $\mathcal{T}_p \to \mathcal{T}$ for $p \to \infty$ (cf. [8]).

By induction on p, one can readily prove the following result.

Proposition 3.4. *For every $p \geq 0$, $\{\gamma(\mathcal{T}_p), \gamma \in \Gamma\}$ is a tiling of $\mathbb{R}^n$.*

Definition 7. For each approximation $\mathcal{T}_p$, we denote by $B_{\gamma,p}$ the set $\mathcal{T}_p \cap \gamma(\mathcal{T}_p)$.

Using Proposition 3.4 we have
$$\partial\mathcal{T}_p = \bigcup_{\gamma\in\Gamma\setminus\{1\}} B_{\gamma,p} \ . \tag{3.2}$$
This is in fact a finite union because the tiling is locally finite.

Remark 4. For every $p \geq 0$, $g^p(\mathcal{T}_p)$ is a union of n-dimensional non-overlapping polyhedrons:
$$g^p(\mathcal{T}_p) = \bigcup_{\delta_0,\dots,\delta_{p-1}\in\mathcal{D}} g^{p-1}\delta_{p-1}g^{-(p-1)}g^{p-2}\delta_{p-2}g^{-(p-2)}\dots g\delta_1 g^{-1}\ \delta_0\ (Q)$$
where each $g^k\delta_k g^{-k}$ is an isometry of Γ. Thus for every $\gamma \in \Gamma$, $g^p\gamma(\mathcal{T}_p) = g^p\gamma g^{-p}\ g^p(\mathcal{T}_p)$ is also a union of n-dimensional non-overlapping polyhedrons and $g^p(B_{\gamma,p})$ is the intersection of two unions of polyhedrons that do not overlap. This also holds for $B_{\gamma,p}$, and, hence, this set is the union of $(n-1)$-dimensional faces.

The following result shows the correspondence between the boundary of $\mathcal{T}_p$ and the set $\mathcal{R}_p$ defined in (2.2).

Proposition 3.5. *If $B_{\gamma,p}$ contains an $(n-1)$-dimensional face then $\gamma \in \mathcal{R}_p$.*

Proof. We proceed by induction on p. First note that the result is obviously true for $p = 0$. We assume the result is true for a $p-1 \in \mathbb{N}$. The number of faces of $B_{\gamma,p}$ is also the number of faces of $g(B_{\gamma,p})$. We have
$$\begin{array}{rcl} g(B_{\gamma,p}) & = & \bigcup_{\delta,\delta'\in\mathcal{D}} \delta(\mathcal{T}_{p-1})\ \cap\ g\gamma g^{-1}\delta'(\mathcal{T}_{p-1}) \\ & = & \bigcup_{\delta,\delta'\in\mathcal{D}} \delta(B_{\delta^{-1}g\gamma g^{-1}\delta',p-1}) \end{array}$$
and by (2.2) this last union contains an $(n-1)$-dimensional face only if $\gamma \in \mathcal{R}_p$. □

Consequently, as $\partial\mathcal{T}_p$ is the union of its $(n-1)$-dimensional faces, from (3.2) follows that
$$\partial\mathcal{T}_p = \bigcup_{\gamma\in\mathcal{R}_p\setminus\{1\}} B_{\gamma,p}$$

with

$$B_{\gamma,p} = \bigcup_{\substack{\delta\in\mathcal{D},\gamma'\in\mathcal{R}_{p-1}\setminus\{1\}\\ \exists\delta'\in\mathcal{D},\ \gamma\xrightarrow{\delta|\delta'}\gamma'\ \in\mathbf{G}(\mathcal{R})}} g^{-1}\delta(B_{\gamma',p-1}).$$

Note that if $B_{\gamma,p}$ is not an $(n-1)$-dimensional face, then it must be already contained in other $(n-1)$-dimensional faces of other sets $B_{\gamma',p}$ with $\gamma'\neq\gamma$. This also holds for the sets $g^{-1}\delta(B_{\gamma',p-1})$ in the union above. Thus remembering that $\mathcal{R}_p\subset\mathcal{R}$ for every p and using Proposition 3.5, we get for the boundary of the approximations

$$\partial\mathcal{T}_p = \bigcup_{\gamma\in\mathcal{R}\setminus\{1\}} B_{\gamma,p}$$

with

$$B_{\gamma,p} = \bigcup_{\substack{\delta\in\mathcal{D},\gamma'\in\mathcal{R}\setminus\{1\}\\ \exists\delta'\in\mathcal{D},\ \gamma\xrightarrow{\delta|\delta'}\gamma'\ \in\mathbf{G}(\mathcal{R})}} g^{-1}\delta(B_{\gamma',p-1}). \tag{3.3}$$

There is a relation between the structure of the graph $\mathbf{G}(\mathcal{R})$ and the geometry of the sets $B_{\gamma,p}$:

Proposition 3.6. *Let $\gamma\in\mathcal{R}$. If all the walks in $\mathbf{G}(\mathcal{R})$ starting from γ are at most of length ℓ, then $B_{\gamma,p}$ has no $(n-1)$-dimensional face for $p>\ell$.*

Proof. Let us take $p>\ell$. Starting from the set equation (3.3) above and writing it for the sets $B_{\gamma',p-1}, B_{\gamma'',p-2},\dots$ that appear at each iteration, one comes after ℓ steps to

$$B_{\gamma,p} \;=\; \bigcup_{\gamma\xrightarrow{\delta}\gamma'\ \in\mathbf{G}(\mathcal{R})} \cdots \bigcup_{\gamma^{(\ell)}\xrightarrow{\delta^{(\ell)}}\gamma^{(\ell+1)}\ \in\mathbf{G}(\mathcal{R})} g^{-1}\delta\dots g^{-1}\delta^{(\ell)}\left(B_{\gamma^{(\ell+1)},p-(\ell+1)}\right),$$

but by assumption there is no edge $\gamma^{(\ell)}\to\gamma^{(\ell+1)}$ in $\mathbf{G}(\mathcal{R})$, so no $B_{\gamma^{(\ell+1)},p-(\ell+1)}$ and hence $B_{\gamma,p}$ can not have an $(n-1)$-dimensional face. □

3.3. **Finding neighbors.** Scheicher and Thuswaldner [16] gave an algorithm starting from $\mathbf{G}(\mathcal{R})$ to get the neighborhood graph $\mathbf{G}(\mathcal{S})$ for lattice tilings. In this section, we write this algorithm in a "crystallographic way". Examples of contact and neighborhood graphs obtained by this algorithm will be given in Section 5.

Definition 8. If G is a graph, we denote by $\mathrm{Red}(G)$ the graph emerging from G if all states of G that are not the starting point of a walk of infinite length are removed. Such a graph is called a *reduced graph*.

Definition 9. For two subgraphs G_1 and G_1' of $\mathbf{G}(\Gamma)$ we define the product graph $G_2 = G_1\otimes G_1'$ as follows.

- A state r_2 belongs to G_2 iff $r_2 = r_1r_1'$ or $r_2 = r_1'r_1$ for some $r_1\in G_1,\ r_1'\in G_1'$
- for $r_2,\ s_2$ states of G_2, and $\delta_1,\ \delta_2$ digits of $\mathcal{D}$, then there is an edge $r_2\xrightarrow{\delta_1|\delta_2} s_2\ \in G_2$

iff there are edges $r_1\xrightarrow{\delta_1|\delta_1'} s_1\ \in G_1$ and $r_1'\xrightarrow{\delta_1'|\delta_2} s_1'\ \in G_1'$
with $r_2 = r_1r_1',\ s_2 = s_1s_1'$ and $\delta_1'\in\mathcal{D}$

or there are edges $r_1'\xrightarrow{\delta_1|\delta_1'} s_1'\ \in G_1'$ and $r_1\xrightarrow{\delta_1'|\delta_2} s_1\ \in G_1$
with $r_2 = r_1'r_1,\ s_2 = s_1's_1$ and $\delta_1'\in\mathcal{D}$.

The conditions of the second item both lead to the same set of edges if $G_1 = G_1'$.
We write $\otimes_{i=1}^m G_1 = \underbrace{G_1\otimes\ldots\otimes G_1}_{m\text{ times}}$.

Definition 10. A subgraph $\mathbf{G}(M)$ of $\mathbf{G}(\Gamma)$ is said to have property (C) if for each pair $(\gamma', \delta) \in M \times \mathcal{D}$ there exists a unique pair $(\gamma, \delta') \in M \times \mathcal{D}$ such that $\gamma \xrightarrow{\delta|\delta'} \gamma' \ \in \mathbf{G}(M)$.

From the last three definitions and the fact that $\mathcal{D}$ is a complete set of right coset representatives of $g\Gamma g^{-1}$, we derive the following proposition:

Proposition 3.7. *If G_1 and G_1' are subgraphs of $\mathbf{G}(\Gamma)$ having property (C), then for r_2, s_2 states of $G_1 \otimes G_1'$ and $\delta_1, \delta_2 \in \mathcal{D}$,*

$$\text{there exists an edge } r_2 \xrightarrow{\delta_1|\delta_2} s_2 \ \in G_1 \otimes G_1' \quad \text{iff} \quad g r_2 g^{-1} \delta_2 = \delta_1 s_2.$$

Furthermore, $G_1 \otimes G_1'$ and hence $\mathrm{Red}(G_1 \otimes G_1')$ have property (C).

The proof of this result runs along the same lines as in the case of lattice tilings (cf. [16]) and we omit it.

This means that the product graph of two subgraphs satisfying property (C) is again a subgraph of $\mathbf{G}(\Gamma)$ and it has property (C), as well as its reduced graph.

Algorithm 3.8. We denote by $\mathbf{G}(S)$ the graph obtained from $\mathbf{G}(\mathcal{R})$ by the following algorithm.

$p := 1$
$A[1] := \mathrm{Red}(\mathbf{G}(\mathcal{R}))$
repeat
$p := p + 1,\ A[p] := \mathrm{Red}(A[p-1] \otimes A[1])$
until $A[p] = A[p-1]$
$\mathbf{G}(S) := A[p] \setminus \{1\}$

Proposition 3.9. *Algorithm 3.8 ends after finitely many steps and yields the neighborhood graph (i.e., $\mathbf{G}(S) = \mathbf{G}(\mathcal{S})$).*

In order to show this result we have to adapt the proof of the corresponding result in the case of lattice tilings of Scheicher and Thuswaldner [16]. First we need the following lemmata to get bounding sets for $\mathcal{S}$ (with respect to the inclusion) from below and above.

Lemma 3.10. *Each state of $\mathbf{G}(\mathcal{R}') := \mathrm{Red}(\mathbf{G}(\mathcal{R}))$ has infinitely many predecessors and infinitely many successors. Thus $\mathbf{G}(\mathcal{R}')$ is a union of cycles of $\mathbf{G}(\Gamma)$ and of walks connecting these cycles. Furthermore, $\mathbf{G}(\mathcal{R}')$ has property (C).*

This can be proved in the same way as in the lattice tiling case (cf. [16]).

Lemma 3.11. *The graph $\mathbf{G}(\mathcal{S} \cup \{1\})$ is the union of all cycles of $\mathbf{G}(\Gamma)$ and all walks connecting two of these cycles.*

Proof. Let γ be a state contained in a cycle of $\mathbf{G}(\Gamma)$, *i.e.*, there exists $\gamma \xrightarrow{\delta_1|\delta_1'} \gamma_1 \xrightarrow{\delta_2|\delta_2'} \dots \gamma_l \xrightarrow{\delta_l|\delta_l'} \gamma$ a cycle in $\mathbf{G}(\Gamma)$. Then for all $p \in \mathbb{N}$ we have

$$\gamma \ \left(g^{-1}\delta_1' \dots g^{-1}\delta_l'\right)^p \ = \ \left(g^{-1}\delta_1 \dots g^{-1}\delta_l\right)^p \ \gamma.$$

If we take $a \in \mathbb{R}^n$ and set $\delta_{j+pl} := \delta_j,\ \delta_{j+pl}' = \delta_j'$ for every $p \in \mathbb{N},\ j \in \{1, \dots l\}$, then we can write

$$\gamma\left(\lim_{n\to\infty} g^{-1}\delta_1' \dots g^{-1}\delta_n'(a)\right) \ = \ \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(\gamma(a)) \ = \ \lim_{n\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(a)$$

(the last equality follows from (1.2)). This means that $\gamma \in \mathcal{S} \cup \{1\}$.
All walks connecting two cycles of $\mathbf{G}(\Gamma)$ are also contained in $\mathbf{G}(\mathcal{S} \cup \{1\})$. This follows inductively, starting from the last state of the walk, that belongs to $\mathcal{S} \cup \{1\}$ as we just saw, and going the walk backwards: the existence of an edge $\gamma_1 \xrightarrow{\delta|\delta'} \gamma_2$ in $\mathbf{G}(\Gamma)$ with $\gamma_2 \in \mathcal{S} \cup \{1\}$ implies $\gamma_1 \in \mathcal{S} \cup \{1\}$ (see Remark 3).
No other state is contained in $\mathbf{G}(\mathcal{S} \cup \{1\})$, because $\mathcal{S}$ is finite and each state of $\mathbf{G}(\mathcal{S} \cup \{1\})$ must

have infinitely many predecessors and infinitely many successors. These states must be in a cycle or in a walk connecting two cycles. □

Corollary 3.12. $\mathrm{Red}(\mathbf{G}(\mathcal{R})) \subseteq \mathbf{G}(\mathcal{S} \cup \{1\})$.

This lower bound follows from Lemma 3.10 and 3.11. For the upper bound, we need one more lemma.

Lemma 3.13. *Let* $\mathbf{G}(\mathcal{R}') := \mathrm{Red}(\mathbf{G}(\mathcal{R}))$. *Then* $\mathcal{R}'$ *contains a generator set* $B := \{\gamma_1, \dots, \gamma_q\}$ *of* Γ. *By symmetry and because* $1 \to 1$ *is a cycle in* $\mathbf{G}(\mathcal{R})$, $\mathcal{R}'$ *even contains the set* $\{1\} \cup B \cup B^{-1}$.

Proof. $\mathcal{R}$ being a finite set, let $\ell := |\mathcal{R}|$. Then for $p > \ell$, the fact that $B_{\gamma,p}$ contains an $(n-1)$-dimensional face implies that γ belongs to $\mathcal{R}'$. Indeed, because of Proposition 3.6, there must be a walk of length $p > \ell$ in $\mathbf{G}(\mathcal{R})$ starting from γ. In this walk, a state $\gamma' \in \mathcal{R}$ has to appear at least twice (because $p > |\mathcal{R}|$). This provides a cycle in the walk which can be repeated to get an infinite walk starting from γ. Thus $\gamma \in \mathcal{R}'$. This allows the following description of the boundary of $\mathcal{T}_p$ for $p > \ell$:

$$\partial \mathcal{T}_p = \bigcup_{\gamma \in \mathcal{R}' \setminus \{1\}} B_{\gamma,p}$$

Now we show that $\mathcal{R}'$ generates Γ. Let $\alpha \in \Gamma$. Remember (Proposition 3.4) that $\{\gamma(\mathcal{T}_p), \gamma \in \Gamma\}$ is a tiling of $\mathbb{R}^n$; let $x \in \mathcal{T}_p$ and $y \in \alpha(\mathcal{T}_p)$ (but x, y not vertices of these tiles), one can draw a line from x to y avoiding the vertices of the tiles. $\mathcal{T}_p$ is compact, so this line passes through a finite number of tiles $\mathcal{T}_p, \alpha_1(\mathcal{T}_p), \dots, \alpha_q(\mathcal{T}_p)$ in this order. Two consecutive tiles have an $(n-1)$-dimensional face in common, so $\alpha_1, \alpha_1^{-1}\alpha_2, \dots, \alpha_q^{-1}\alpha$ are elements of $\mathcal{R}'$, thus α is a product of elements of $\mathcal{R}'$. □

Corollary 3.14. *The inclusion*

$$\mathrm{Red}(\mathbf{G}(\mathcal{R})^{p_0}) \supseteq \mathbf{G}(\mathcal{S} \cup \{1\})$$

holds for some positive integer p_0. *Furthermore,* $\mathrm{Red}(\mathbf{G}(\mathcal{R})^{p_0})$ *has property (C).*

Proof. If B is a generator set of Γ contained in $\mathcal{R}'$ (hence in $\mathcal{R}$ too), then the set of states of $\mathbf{G}(\mathcal{R})^p$ contains all elements of $(\{1\} \cup B \cup B^{-1})^p$ (see Lemma 3.13). As $\mathcal{S}$ is finite, there is a p_0 with $\mathbf{G}(\mathcal{R})^{p_0} \supset \mathbf{G}(\mathcal{S} \cup \{1\})$, and each state of $\mathbf{G}(\mathcal{S}) \cup \{1\}$ having infinitely many successors, the required inclusion holds. The second claim follows from Proposition 3.7. □

The following lemma will be useful in the conclusion of the proof of Proposition 3.9. It can be obtained in a similar way as in the case of lattice tilings, we refer the reader to [16, Section 5] for more details.

Lemma 3.15. *If* $\mathbf{G}(\mathcal{R}')$ *denotes* $\mathrm{Red}(\mathbf{G}(\mathcal{R}))$, *then the identity*

$$\mathrm{Red}(\mathbf{G}(\mathcal{R}')^p) = \underbrace{\mathrm{Red}(\dots \mathrm{Red}(\mathrm{Red}(\mathrm{Red}(\mathbf{G}(\mathcal{R}')) \otimes \mathbf{G}(\mathcal{R}')) \otimes \mathbf{G}(\mathcal{R}')) \dots \otimes \mathbf{G}(\mathcal{R}'))}_{p \text{ times}}$$

holds for every $p \in \mathbb{N}$.

Proof of Proposition 3.9. The proof of Proposition 3.9 is then similar as in [16]. Firstly, the algorithm terminates: choosing p_0 as in Corollary 3.14, we have by Lemma 3.15 that $A[p_0] = \mathrm{Red}(\mathbf{G}(\mathcal{R})^{p_0})$, thus $A[p_0] \supseteq \mathbf{G}(\mathcal{S} \cup \{1\})$. This implies in view of Lemma 3.11 that $A[p_0]$ contains each reduced finite subgraph of $\mathbf{G}(\Gamma)$ with the property that each of its states has a predecessor. Thus $A[p_0 + 1] \subseteq A[p_0]$, and the opposite inclusion being trivial we even have $A[p_0 + 1] = A[p_0]$. Hence the algorithm terminates for a $p_1 \leq p_0 + 1$, and we have $\mathbf{G}(S) = A[p_1] \setminus \{1\}$.
Secondly, $\mathbf{G}(S)$ is the neighborhood graph: note that by the definition of p_1 and of the algorithm, $A[p_1] = A[p_1 + 1] = \dots = A[p_0 + 1] = A[p_0]$ holds. Moreover, Lemma 3.11 indicates that $\mathbf{G}(\mathcal{S} \cup \{1\})$ contains all reduced finite subgraphs $\mathbf{G}(\Gamma)$ for which each state has a predecessor, and

Proposition 3.7 states that $A[p_0]$ has property (C), so that each state of $A[p_0]$ has a predecessor. Hence $A[p_0] \subseteq \mathbf{G}(\mathcal{S} \cup \{1\}) = A[p_1] = A[p_0]$, showing that $\mathbf{G}(S) = \mathbf{G}(\mathcal{S})$. □

We end up this section by giving an upper bound for the number of steps required by Algorithm 3.8 to compute the neighbor graph from the contact graph. We first give a bound for the number of neighbors, *i.e.*, for the cardinality of $\mathcal{S}$. Let $||\cdot||$ be the Euclidean norm on $\mathbb{R}^n$, with respect to which the elements of Γ are isometries. We denote also by $||\cdot||$ the induced matrix norm. We write $g(x) = Ax + t$, where A is the expanding matrix and t the translation vector associated to g. Similarly, we write for an isometry $\gamma \in \Gamma$: $\gamma(x) = A_\gamma x + t_\gamma$, where A_γ is an isometry matrix (in particular, $||A_\gamma|| = 1$), and t_γ is the translation vector of γ.

Proposition 3.16. *We have the following upper bound:*

$$|\mathcal{S}| \leq |\Gamma/\Lambda| \cdot \left(4 \cdot \max_{\delta \in \mathcal{D}} \{||t - t_\delta||\} \cdot \sum_{j=1}^{\infty} ||A^{-j}|| \right)^n .$$

Proof. We compute a bound M for the norms of the possible translational parts t_γ of the elements $\gamma \in \mathcal{S}$. Then, the volume of the hypercube of side size $2M$ will bound the number of these allowed translation parts. Multipliying this volume by the cardinality of the point group, we get the desired upper bound.

Let $\gamma \in \mathcal{S}$, then some point belongs to $\mathcal{T}$ and $\gamma(\mathcal{T})$. Thus there are sequences of digits (δ_j), (δ'_j), such that

$$\lim_{m\to\infty} g^{-1}\delta_1 \dots g^{-1}\delta_m(0) = \lim_{m\to\infty} \gamma g^{-1}\delta'_1 \dots g^{-1}\delta'_m(0).$$

This means for the translation part of γ that

$$t_\gamma = \lim_{m\to\infty}$$
$$A_\gamma \sum_{j=0}^{m-1} A_{\delta'_0} A^{-1} A_{\delta'_1} A^{-1} \dots A_{\delta'_j} A^{-1} (t_{\delta'_{j+1}} - t) \; - \; \sum_{j=0}^{m-1} A_{\delta_0} A^{-1} A_{\delta_1} A^{-1} \dots A_{\delta_j} A^{-1} (t_{\delta_{j+1}} - t),$$

where by convention $A_{\delta_0} = A_{\delta'_0} = 1$.

We recall now that, by Definition 1, $g\Gamma g^{-1} \subset \Gamma$, hence if an isometry $\gamma \in \Gamma$ is given, there is another isometry $\gamma' \in \Gamma$ such that $\gamma g^{-1} = g^{-1}\gamma'$. This property also holds for the linear parts: given A_γ, there is $A_{\gamma'}$ such that $A_\gamma A^{-1} = A^{-1} A_{\gamma'}$. Using this fact, one can rewrite the products of matrices in the first sum above in the following way: for each j, there are matrices $A_{\gamma_0^{(j)}}, A_{\gamma_1^{(j)}}, \dots, A_{\gamma_j^{(j)}}$ such that

$$A_{\delta'_0} A^{-1} A_{\delta'_1} A^{-1} \dots A_{\delta'_j} A^{-1} \; = \; A^{-(j+1)} A_{\gamma_0^{(j)}} A_{\gamma_1^{(j)}} \dots A_{\gamma_j^{(j)}},$$

and similarly for the other sum. Going to the norm and using the triangle inequality and the fact that isometries have norm 1, we obtain the following majoration:

$$||t_\gamma|| \leq 2 \;\cdot\; \sum_{j=1}^{\infty} ||A^{-j}|| \;\cdot\; \max_{\delta \in \mathcal{D}} \{||t_\delta - t||\} \quad =: M.$$

□

Remark 5. The bound in this lemma often applies to the number of elements in the contact set $\mathcal{R}$. Indeed, in practical applications, looking at a picture of the tiling, it is possible to guess an appropriate set $\mathcal{R}_0$ to start with, corresponding to a good first approximating polyedron of the central tile. In this case, we eventually obtain $\mathcal{R} \setminus \{1\} \subset \mathcal{S}$. Often, we even have that $\mathcal{R}$ is much smaller than $\mathcal{S}$. A better general upper bound can not be given, the cardinality of the contact set and the number of neighbors really depend on the geometry of the particular example (crystiles with arbitrary number of neighbors can be produced, see [2]).

Proposition 3.17. *Let $q := |\Gamma/\Lambda|$ be the cardinality of the point group. Then there is a basis $\mathcal{B}_L$ of the lattice Λ such that $\mathcal{B}_L \cup \mathcal{B}_L^{-1}$ are states of $\mathbf{G}(\mathcal{R}')^q$. Consider the translational parts of the elements of $\mathcal{S}$. If N is the maximal Euclidean norm of these translational parts, then $\mathcal{S}$*

is contained in the set of states of $\mathbf{G}(\mathcal{R}')^{Nq}$*. Hence, Algorithm 3.8 terminates after at most* Nq *steps.*

Proof. First note that by Lemma 3.15, the graph $\mathrm{Red}(\mathbf{G}(\mathcal{R}')^{Nq})$ will be exactly equal to the graph obtained when reduction happens after each step, like in Algorithm 3.8. This graph is moreover a subgraph of $\mathbf{G}(\mathcal{R}')^{Nq}$. Since q is the order of the point group and by Lemma 3.13 the set $\mathcal{R}'$ contains a generator for Γ, the product $\mathcal{R}'^{\,q} = \{r_1 \cdot \ldots \cdot r_q,\ r_i \in \mathcal{R}'\}$ contains a basis for the lattice Λ and is equal to the set of states of $\mathbf{G}(\mathcal{R}')^q$. It even contains all the elements of Γ whose translation vector has norm less than 1. Then, the set of states of $\mathbf{G}(\mathcal{R}')^{2q}$ contains all the elements of Γ whose translation vector as norm less than $1+1=2$. Consequently, the neighbor(s) with maximal translation vector will be reached after iterating at most N times this operation, as well as all the neighbors with smaller translation vector. □

Example. We eventually give the bound obtained above for the algorithm in a lattice example. For a lattice tile, we have $q=1$. We consider in the plane the expansion $g(x) = Ax$ with

$$A = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix}$$

and the digit set $\{(0,0)^T, \ldots, (4,0)^T\}$ (see [2]). We have $|\mathcal{S}| = 10$, and

$$\mathcal{R}' = \{\pm(4,1)^T, \pm(3,1)^T, \pm(1,0)^T\}$$

for a choice of $\mathcal{R}_0 = \{(0,0)^T, (\pm 1, 0), (0, \pm 1)^T\}$. The set of neighbors is

$$\mathcal{S} = \mathcal{R}' \cup \{\pm(6,2)^T, \pm(2,1)^T\}.$$

The greatest neighbor has Euclidean norm $2\sqrt{10}$. The bound for the number of steps in the above proposition is thus 6. The effective number of steps is 1.

4. Plane crystiles: criterion of disk-likeness

The aim of this section is to establish a criterion for the disk-likeness of plane crystiles.
We use the sets and graphs defined in Section 2.
We recall the following fundamental facts, easily seen from the definition of the set of neighbors $\mathcal{S}$ and the set of adjacent neighbors $\mathcal{A}$.

- In $\mathcal{A}$, there is a set of generators for Γ, and $\mathcal{A} \subseteq \mathcal{S}$.
- The boundary of $\mathcal{T}$ satisfies $\partial\mathcal{T} = \cup_{\gamma\in\mathcal{S}} B_\gamma = \cup_{\gamma\in\mathcal{A}} B_\gamma$.
- For each $\mathcal{A}' \subseteq \mathcal{S}$ with $\partial\mathcal{T} = \cup_{\gamma\in\mathcal{A}'} B_\gamma$, we have $\mathcal{A}' \supseteq \mathcal{A}$.

The first result is a proposition stated for general crystallographic tilings of the plane.

Proposition 4.1. *Suppose that* $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$ *is a crystallographic tiling of* $\mathbb{R}^2$*. Then,* $\mathcal{T}$ *is disk-like if and only if the following three conditions all hold.*

1. *The triple intersection* $V_2(\gamma_1, \gamma_2) = \mathcal{T} \cap \gamma_1(\mathcal{T}) \cap \gamma_2(\mathcal{T})$ *is either empty or a single point set for any distinct* $\gamma_1, \gamma_2 \in \mathcal{S}$*.*
2. *For each* $\gamma \in \mathcal{S}$*, the double edge* B_γ *is either a single point or a simple arc.*
3. *The subgraph of the double neighboring graph* $\mathcal{G}_2$ *with set of vertices* $\{B_\gamma : \gamma \in \mathcal{A}\}$ *consists of a simple loop.*

Since a crystallographic reptile $\mathcal{T}$ with respect to Γ induces a crystallographic tiling $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$, the disk-like question for crystiles is solved if we can verify the three items of Theorem 4.1, or disprove a single one. In general, Items (1) and (3) can be checked by concrete algorithms, while we still need to deal with Item (2). Recall that $\mathcal{T}$ is the attractor of an IFS $\{f_1, \ldots, f_k\}$ consisting of affine contractions and that $\mathcal{T}$ satisfies the open set condition. Thus, $\mathcal{T}$ is disk-like if and only if $\mathcal{T}^o$ is connected [11, Theorem 1.1 (*iii*)]. This will provide us an algorithm to solve the disk-like question of crystiles in $\mathbb{R}^2$, without verifying Item (2) of Proposition 4.1.

Indeed, for a crystallographic reptile $\mathcal{T}$ with respect to Γ and the corresponding crystallographic tiling $\{\gamma(\mathcal{T}) : \gamma \in \Gamma\}$, let g be the expanding map, and let the double neighboring graph $\mathcal{G}_2(\Gamma)$, the sets $\mathcal{V}_\gamma$ and the graph $\mathcal{G}_\gamma$ be as in Definition 4 of Section 2. Then the union of all the elements of $\mathcal{V}_\gamma$ is exactly the image $g(B_\gamma)$ of B_γ under the expansion map g. If the complementary set $\mathbb{R}^2 \setminus B_\gamma$ has some bounded component U then the region will be "enlarged" as $g(U)$, which is a bounded component of $\mathbb{R}^2 \setminus g(B_\gamma)$. Under simple assumptions on the graph $\mathcal{G}_\gamma$, we can exclude the existence of such a region U. This will eventually lead us to connectivity of $\mathcal{T}^o$ and hence disk-likeness of $\mathcal{T}$.

Now, we can state our disk-like criterion for crystiles as follows.

Theorem 4.2. *Let $\mathcal{T} \subset \mathbb{R}^2$ be a crystile with respect to Γ whose expanding map is g and whose digit set is $\mathcal{D}$. Then $\mathcal{T}$ is disk-like if and only if each of the following three conditions holds:*

(1) *The triple intersection $V_2(\gamma_1, \gamma_2) = \mathcal{T} \cap \gamma_1(\mathcal{T}) \cap \gamma_2(\mathcal{T})$ is either empty or a single point set for any disjoint pair $\gamma_1, \gamma_2 \in \mathcal{S}$.*
(2) *For each $\gamma \in \mathcal{A}$, the graph $\mathcal{G}_\gamma$ consists of a simple path.*
(3) *The subgraph of the double neighboring graph $\mathcal{G}_2$ with set of vertices $\{B_\gamma : \gamma \in \mathcal{A}\}$ consists of a simple loop.*

Before proving Proposition 4.1 and Theorem 4.2, let us recall some results from plane topology.

Given a set M in a topological space X, we say that M is a *cut* of X or *separates* X if its complement $X \setminus M = M^c$ is disconnected. If n points $x_1, \ldots, x_n, n \geq 2$, belong to pairwise distinct connected components of $X \setminus M$, we say that $x_1, \ldots, x_n$ are separated by M; if $n = 2$, we also say that M is a cut *between* x and y, or that M cuts between x and y in X.

A *continuum* is a connected compact Hausdorff space, and a locally connected continuum X is a *Janiszewski space* ([9, §61, I, p.505]) provided that the union $M \cup N$ of two continua $M, N \subset X$ is a cut of X whenever their intersection $M \cap N$ is not connected.

Lemma 4.3. [9, §61, I, Theorem 7] *Let B, C be two closed or open sets of a Janiszewski space X. If none of these sets is a cut between p and q and if $B \cap C$ is connected, then $B \cup C$ is not a cut between p and q either.*

Since the 2-dimensional sphere $\mathbb{S}^2$ is a Janiszewski space [9, §61, I, Theorem 2], we can infer two separation theorems for the plane $\mathbb{R}^2$ as follows.

Lemma 4.4. *If the common part of two continua M, N in the plane is disconnected, then there exist two points p_0, p_1 separated by $M \cup N$ but not by either M or N.*

Lemma 4.5. *Let M, N be compact sets in the plane and suppose that there exist $n(\geq 2)$ points separated by $M \cup N$ but not by either M or N. Then the common part $M \cap N$ has at least n components.*

Suppose that $M_1, M_2, \ldots, M_n, n \geq 3$, are compact sets in the plane. If $\#(M_i \cap M_{i+1}) = 1$ for $1 \leq i \leq n-1$ and $M_i \cap M_j = \emptyset$ for $|i-j| \geq 2$, we say that $M_1, M_2, \ldots, M_n$ form a *chain.* If $\#(M_i \cap M_{i+1}) = 1$ for $1 \leq i \leq n-1$, $\#(M_n \cap M_1) = 1$ and $M_i \cap M_j = \emptyset$ for $2 \leq |i-j| \leq n-2$, we say that $M_1, M_2, \ldots, M_n$ form a *circular chain.* By Lemmata 4.3 to 4.5, we have the following two corollaries which will be used in our proof for Theorem 4.2.

Corollary 4.6. *If $M_1, \ldots, M_n (n \geq 3)$ are compact sets in the plane which form a chain, then every two points separated by $\bigcup_j M_j$ are separated by a single M_j.*

Corollary 4.7. *If the continua $M_1, \ldots, M_n (n \geq 3)$ in the plane form a circular chain and each of them does not separate the plane, then $\mathbb{R}^2 \setminus \left(\bigcup_j M_j\right)$ is the union of two regions.*

We also recall a theorem of Torhorst.

Lemma 4.8. [9, §61, II, Theorem 4] *Let $M \subset \mathbb{S}^2$ be a locally connected continuum having no cutpoint and R a component of M^c. Then $\overline{R}$ is homeomorphic to a disk.*

Dealing with disk-like tiles in crystallographic tilings, we will need the following considerations. Recall that for a closed disk D, distinct points of the boundary are *accessible* from any given point of the interior of D by disjoint simple open arcs within D^o. More precisely, for distinct $a_1, \ldots, a_n \in \partial D$, $n \in \mathbb{N}$ and $p \in D^o$, there are pairwise disjoint simple open arcs $A_1, \ldots, A_n \subset D^o$ leading from p to $a_1, \ldots, a_n$, respectively. This fact is needed in order to prove the following lemmata:

Lemma 4.9. *Let D_1, D_2 be two closed disks with disjoint interiors such that there exists a bounded connected component Z of $(D_1 \cup D_2)^c$. Then $\overline{Z} \cap D_1 \cap D_2$ consists of two points a, b. Furthermore, $\overline{Z} \cap D_i$, $i = 1, 2$, are simple arcs meeting in their end points a and b, and there are $C_1 \subset D_1^o$, $C_2 \subset D_2^o$ simple open arcs from a to b such that $Z = \mathit{Interior}(C_1 \cup C_2 \cup \{a, b\}) \cap (D_1 \cup D_2)^c$. (Here the interior $\mathit{Interior}(C)$ of the simple closed curve C is defined as the bounded component of its complement.)*

Proof. Note that $D_1 \cup D_2$, union of disks with disjoint interiors, is a locally connected continuum with no cut points because its complement has a bounded component, so by Lemma 4.8 $\overline{Z}$ is disk-like, thus its boundary $\partial Z = \overline{Z} \cap (D_1 \cup D_2)$ is a simple closed curve and $Z = \mathrm{Interior}(\partial Z)$. The intersection $\overline{Z} \cap D_1 \cap D_2$ has at least two points because ∂Z is connected and has no cut point. Let us suppose that it contains three distinct points a, b, c. Then, choosing $p_i \in D_i^o$, $i = 1, 2$, and disjoint simple open arcs C_α^i from p_i to α within D_i^o for $i = 1, 2$ and $\alpha = a, b, c$, we obtain three disjoint simple open arcs from p_1 to p_2, namely $C_\alpha := \{\alpha\} \cup C_\alpha^1 \cup \left(-C_\alpha^2\right)$ for $\alpha = a, b, c$, with the property that $C_\alpha \subset D_1^o \cup D_2^o \cup \{\alpha\}$. Thus $\theta := \{p_1, p_2\} \cup \bigcup_\alpha C_\alpha$ is a theta-curve. Since $Z \subset \theta^c$ is connected, it must be entirely included in one open disk-like region B of θ^c. This implies that a, b, c are all on $\overline{B}$, contradicting the fact that these points lie on the distinct arcs of θ.
Thus $\overline{Z} \cap D_1 \cap D_2$ consists of exactly two points a, b, and the remaining assertions follow: $A_i := \overline{Z} \cap D_i$ for $i = 1, 2$ is a continuum and each point of A_i different from a, b is a cut point of A_i, so A_i is a simple arc on ∂Z (see [9, §49, IV, Theorem 4]). Every two simple open arcs C_1, C_2 from a to b within D_1^o, D_2^o are homotopic to A_1, A_2 within D_1, D_2 and have the required property. □

Lemma 4.10. *Let $\{\gamma(\mathcal{T}), \gamma \in \Gamma\}$ be a crystallographic tiling with disk-like tiles, and γ, γ' two elements of Γ. Then $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ has no bounded component. In other words, no pair of tiles can surround a third one.*

Proof.

(*i*) For each $\gamma, \gamma' \in \Gamma$, $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ has finitely many bounded connected components. Indeed, because of the tiling property, every component of $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ being open, it is intersected by the interior of at least one tile $\gamma''(\mathcal{T})$ with $\gamma'' \neq \gamma, \gamma'$, thus it contains the whole tile $\gamma''(\mathcal{T})$, since this tile is disk-like. For this reason also, disjoint components contain distinct tiles, and all of these tiles have the same Lebesgue-measure. But note that there is a bounded domain containing all the bounded components of $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$. Hence $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ can have only finitely many bounded components.

(*ii*) For each $\gamma \in \Gamma$, there are at most finitely many $\gamma' \in \Gamma$ such that $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ has a bounded component. Indeed, because the tiles are disk-like, if $\gamma' \notin \gamma\mathcal{S}$, then $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$ is connected and unbounded.

(*iii*) If Z is a bounded component of $(\gamma(\mathcal{T}) \cup \gamma'(\mathcal{T}))^c$, we denote by $N(\gamma, \gamma', Z)$ the number of tiles whose interior intersects Z. Note again that $\gamma''(\mathcal{T}^o) \subset Z$ as soon as $\gamma''(\mathcal{T}) \cap Z \neq \emptyset$, because of the disk-likeness of the tiles, thus this number is finite. Moreover, we have $N(\gamma', \gamma'\gamma, \gamma' Z) = N(1, \gamma, Z)$ for every $\gamma, \gamma' \in \Gamma$, and every component Z' of $(\gamma'(T) \cup \gamma'\gamma(\mathcal{T}))^c$ is obtained in this way (*i.e.*, $Z' = \gamma' Z$ with Z bounded component of $(\mathcal{T} \cup \gamma(\mathcal{T}))^c$).

(*iv*) Let

$$N := \begin{cases} \cdot \max\{N(1,\gamma,Z)\ :\ \gamma\in\Gamma, Z \text{ bounded component of } (\mathcal{T}\cup\gamma(\mathcal{T}))^c\} \\ \text{ if } (\mathcal{T}\cup\gamma(\mathcal{T}))^c \text{ has a bounded component for some } \gamma\in\Gamma, \\ \cdot 0 \text{ otherwise.} \end{cases}$$

Suppose that $N > 0$. Let γ, Z with $N = N(1,\gamma,Z)$. By Lemma 4.9, there exist disjoint points a, b in $\mathcal{T}\cap\gamma(\mathcal{T})$ and simple open arcs C_1, C_γ contained in $\mathcal{T}^o, \gamma(\mathcal{T}^o)$ respectively, such that $Z \subset \text{Interior}(C)$ with $C := C_1 \cup C_\gamma \cup \{a,b\}$ and such that for $\gamma' \notin \{1,\gamma\}$ with $\gamma'(\mathcal{T})\cap\text{Interior}(C) \neq \emptyset$, we have $\gamma'(\mathcal{T}^o) \subset Z$. Let $\gamma' \notin \{1,\gamma\}$ with $\gamma'(\mathcal{T})\cap\text{Interior}(C)\neq\emptyset$. Then $\gamma'(\mathcal{T})$ lies entirely in $\text{Interior}(C)\cup\{a,b\}$. Moreover, $\gamma'' := \gamma'\gamma$ and $Z' := \gamma' Z$ satisfy $N(\gamma',\gamma'',Z') = N$.

(*v*) We have $\gamma''(\mathcal{T})\cap Z = \emptyset$. Otherwise $\gamma''(\mathcal{T}) \subset \text{Interior}(C)\cup\{a,b\}$ and the bounded components of $(\gamma'(\mathcal{T})\cup\gamma''(\mathcal{T}))^c$ have to lie in Z, but they are all different from Z since they do not contain $\gamma'(\mathcal{T}^o)$ nor $\gamma''(\mathcal{T}^o)$ that are both in Z; in particular for Z' we have $N = N(\gamma',\gamma'',Z') < N(1,\gamma,Z) = N$, a contradiction.

(*vi*) We have $\gamma'' \notin \{1,\gamma\}$. For sure, $\gamma'' = \gamma'\gamma \neq \gamma$, and if $\gamma'' = 1$ we obtain the same contradiction as in item (*v*): the bounded components of $(\gamma'(\mathcal{T})\cup\mathcal{T})^c$ lie in Z but do not contain $\gamma'(\mathcal{T}^o)$, so $N = N(\gamma',1,Z') < N(1,\gamma,Z) = N$, the contradiction.

(*vii*) By Items (*v*) and (*vi*), $\gamma''(\mathcal{T}) \subset \text{Exterior}(C)\cup\{a,b\}$. Thus $\gamma'(\mathcal{T})\cap\gamma''(\mathcal{T}) = \{a,b\}$, because these tiles intersect in at least two points but are contained in $\text{Interior}(C)\cup\{a,b\}$ and $\text{Exterior}(C)\cup\{a,b\}$, respectively.

(*viii*) Consider a simple open arc $C_{\gamma''}$ from a to b within $\gamma''(\mathcal{T})$. We may assume that $C_1 \subset \text{Exterior}(C_{\gamma'}\cup\{a,b\}\cup C_{\gamma''})$.
We now have that $(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$ has a bounded component Z'' containing Z and $\gamma(\mathcal{T}^o)$. Indeed, each of these sets is in some bounded component of $(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$. Moreover, by Lemma 4.9, $\overline{Z}\cap\gamma(\mathcal{T})$ is a simple arc from a to b that does not intersect $\mathcal{T}$ except in a and b, and that lies in $\text{Interior}(C)\cup\{a,b\}$, so it does not intersect $\gamma''(\mathcal{T})$ except in a and b either. Thus one can find at least one point $c \in \overline{Z}\cap\gamma(\mathcal{T})\cap(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$, and this point connects the open disks Z and $\gamma(\mathcal{T}^o)$ within $(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$: indeed, $Z\cup\{c\}\cup\gamma(\mathcal{T}^o)$ is connected and lies in $(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$, thus the sets Z and $\gamma(\mathcal{T}^o)$ lie in the same bounded component of $(\mathcal{T}\cup\gamma''(\mathcal{T}))^c$.

(*ix*) From Item (*viii*), $N(1,\gamma'',Z'') > N(1,\gamma,Z) = N$, contradicting the maximality of N.

This means that the assumption $N > 0$ in Item (*iv*) is false. Hence no union of two tiles has a complement which contains a bounded component. □

We eventually recall that in a topological space X, the *quasi-component at the point* $p \in X$ is the intersection of all closed-open (or *clopen*) sets of X containing p.

Lemma 4.11. [9, §47, II, p.169, Theorem 2] *In compact spaces, the quasi-components are connected and coincide therefore with the components.*

Proof of Proposition 4.1. We first suppose that the three items hold and prove that $\mathcal{T}$ is disk-like.

Items (1) and (3) indicate that the compact sets $\{B_\gamma : \gamma\in\mathcal{A}\}$ form a circular chain, and Item (2) implies that for each $\gamma\in\mathcal{A}$, B_γ and B_γ^c are both connected. This means that $\{B_\gamma, \gamma\in\mathcal{A}\}$ form a circular chain of continua, each of which does not separate the plane. By Corollary 4.7, $\mathbb{R}^2\setminus\left(\bigcup_{\gamma\in\mathcal{A}} B_\gamma\right) = \mathbb{R}^2\setminus\partial\mathcal{T}$ is then the union of two connected sets; the bounded one is $\mathcal{T}^o$ and the unbounded one is $\mathbb{R}^2\setminus\mathcal{T}$, since $\mathcal{T} = \overline{\mathcal{T}^o}$ and $\mathcal{T}^o = \mathbb{R}^2\setminus\left(\bigcup_{\gamma\neq 1}\gamma(\mathcal{T})\right)$. Moreover, because $\partial\mathcal{T}$ is the union of arcs forming a circular chain, it has no cut point and Lemma 4.8 assures that $\mathcal{T}$ is disk-like.

Conversely, we assume that the tile $\mathcal{T}$ is disk-like and prove that the three items hold.

Proof of Item (1).

Let us assume that the triple intersection $\mathcal{T} \cap \gamma_1(\mathcal{T}) \cap \gamma_2(\mathcal{T})$ contains at least two distinct points, say a and b. Then, choosing a point p in $\mathcal{T}^o$, one can find two disjoint simple open arcs A and A' in $\mathcal{T}^o$ leading from p to a and b, respectively. $C := A \cup \{p\} \cup A'$ is then a simple open arc leading from a to b with $C \subset \mathcal{T}^o$. Similarly for the other tiles, one can find simple open arcs $C_1 \subset \gamma_1(\mathcal{T}^o)$, $C_2 \subset \gamma_2(\mathcal{T}^o)$, each of which joins a, b. Then $\theta := \{a, b\} \cup C \cup C_1 \cup C_2$ is a theta-curve whose complementary set consists of three regions. Assume with no loss of generality that C_1 does not intersect the unbounded component of $\mathbb{R}^2 \setminus \theta$. Then $\gamma_1(\mathcal{T}^o)$ entirely lies in the interior of the simple closed curve $C' = \{a, b\} \cup C \cup C_2$, indicating that $(\mathcal{T} \cup \gamma_2(\mathcal{T}))^c$ has a bounded component there, a contradiction to Lemma 4.10.

Proof of Item (2).

Suppose that $B_\gamma \neq \emptyset$. In view of Lemmata 4.4 and 4.10, B_γ must be connected. Thus B_γ is a connected subset of the simple closed curve $\partial\mathcal{T}$. If $B_\gamma = \partial\mathcal{T}$ this would imply that $\mathcal{T}$ is surrounded by $\gamma(\mathcal{T})$ which is impossible. Thus B_γ is homeomorphic to a (possibly degenerated) interval and the proof is done.

Item (3) now follows from the disk-likeness of $\mathcal{T}$ together with Items (1) and (2). $\square$

Proof of Theorem 4.2. The necessity part is a direct corollary of Theorem 4.1, so we just need to show the sufficiency part. More precisely, we will assume the three conditions and infer that the interior $\mathcal{T}^o$ of $\mathcal{T}$ is connected. By Corollary 4.7, we just need to show that B_γ does not separate the plane for each $\gamma \in \mathcal{A}$ and that $\{B_\gamma : \gamma \in \mathcal{A}\}$ is a collection of continua which form a circular chain.

Claim 1: B_γ^c is connected for every $\gamma \in \mathcal{A}$.
If it is not the case, consider the set

$$\mathcal{U} = \{U \subset \mathbb{R}^2 : \exists \gamma \in \mathcal{A} \text{ such that } U \text{ is a bounded connected component of } B_\gamma^c\},$$

and choose $U \in \mathcal{U}$ with maximal area, associated to B_γ for some $\gamma \in \mathcal{A}$. Recall that

$$g(B_\gamma) = \bigcup_{\delta(B_{\delta^{-1}g\gamma g^{-1}\delta'}) \in \mathcal{V}_\gamma} \delta(B_{\delta^{-1}g\gamma g^{-1}\delta'}),$$

where the set $\mathcal{V}_\gamma$ has been defined in Definition 4. Then $g(U)$, a bounded component of $g(B_\gamma)^c$, lies in the complement of every $\delta(B_{\delta^{-1}g\gamma g^{-1}\delta'}) \in \mathcal{V}_\gamma$, thus, by maximality of U, it must entirely lie in the unbounded component of $\delta(B_{\delta^{-1}g\gamma g^{-1}\delta'})^c$ for every $\delta(B_{\delta^{-1}g\gamma g^{-1}\delta'}) \in \mathcal{V}_\gamma$. Let be $p \in g(U)$ and q in the unbounded component of $g(B_\gamma)^c$. Items (1) and (2) imply that the elements of $\mathcal{V}_\gamma$ form a chain. Thus we can apply Corollary 4.6 to the sets of $\mathcal{V}_\gamma$ to obtain that the union of these sets, which is exactly $g(B_\gamma)$, does not cut between p and q, a contradiction to the choice of these points.

Claim 2: B_γ is connected for every $\gamma \in \mathcal{A}$.
Indeed, by Items (1) and (3), one can arrange the elements of $\mathcal{A}$ as $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that the compact sets $B_{\gamma_1}, \ldots, B_{\gamma_n}$ form a circular chain. Note that their union is the continuum ∂T. Without loss of generality, suppose that B_{γ_2} is disconnected. We denote by C_1 and C_3 the connected components of B_{γ_2} such that $\#C_1 \cap B_{\gamma_1} = 1 = \#C_3 \cap B_{\gamma_3}$. Then we have for each other component D of B_{γ_2} and for every $i \in \{1, 3, \ldots, n\}$ that $D \cap B_{\gamma_i} = \emptyset$.
If $C_1 = C_3 =: C$, then $C \neq B_{\gamma_2}$ and by Lemma 4.11 there exists a clopen subset P of B_{γ_2} with $C \subseteq P \subsetneq B_{\gamma_2}$. Thus the boundary can be written as

$$\partial\mathcal{T} = (B_{\gamma_2} \setminus P) \;\cup\; \left(P \;\cup \bigcup_{i \in \{1,3,\ldots,n\}} B_{\gamma_i} \right),$$

which is a separation of ∂T into two disjoint closed subsets, a contradiction to the connectedness of $\partial\mathcal{T}$.

If $C_1 \neq C_3$, one can write

$$\partial\mathcal{T} = C_1 \cup C_3 \cup E \;\; \cup \bigcup_{i\in\{1,3,\dots,n\}} B_{\gamma_i},$$

where E is the union of all connected components of B_{γ_2} different from C_1 and C_3. If $E = \emptyset$, note that the union $C_1 \cup C_3 \cup \bigcup_{i\in\{1,3,\dots,n\}} B_{\gamma_i}$ is not a cut of the space (use Claim 1 and apply Corollary 4.6 to the chain $C_3, B_{\gamma_3}, B_{\gamma_4}, \dots, B_{\gamma_n}, B_{\gamma_1}, C_1$). But this union is exactly $\partial\mathcal{T}$. This contradicts the fact that $\mathcal{T}$ is a tile. If $E \neq \emptyset$, let C be a component of B_{γ_2} distinct from C_1 and C_3. Then using Lemma 4.11 one can find clopen subsets P_1, P_3 of B_{γ_2} such that $C_1 \subseteq P_1, C_3 \subseteq P_3$ and $P_1 \cap C = P_3 \cap C = \emptyset$. Thus $P_1 \cup P_3 \subsetneq B_{\gamma_2}$ is a clopen subset of B_{γ_2}. This leads to the separation

$$\partial\mathcal{T} = (B_{\gamma_2} \setminus (P_1 \cup P_3)) \;\cup\; \left(P_1 \cup P_3 \;\cup \bigcup_{i\in\{1,3,\dots,n\}} B_{\gamma_i}\right)$$

of the boundary of $\mathcal{T}$ into two disjoint closed subsets, contradicting the connectivity of $\partial\mathcal{T}$.

From Claims 1 and 2 and Items (1) and (3) we obtain that the $\{B_\gamma, \gamma \in \mathcal{A}\}$ form a circular chain of continua, each of which does not separate the plane. The disk-likeness of $\mathcal{T}$ then follows as in the first part of the proof of Proposition 4.1. □

5. Application to p2-crystiles

5.1. **Introduction.** Now let Γ be the crystallographic group $p2$ in $\mathbb{R}^2$, *i.e.*,

$$\Gamma = \{a^p b^q c^r \;:\; p, q \in \mathbb{Z}\;,\; r \in \{0,1\}\}$$

where the isometries a, b, c are defined by:

$$a(x,y) = (x+1, y), \;\; b(x,y) = (x, y+1), \;\; c(x,y) = (-x, -y).$$

In this section, four examples of crystiles will be presented and we will answer the question of their disk-likeness. These examples are cases of 3-reptiles (*i.e.*, $|\mathcal{D}| = 3$) and correspond to disk-like candidates listed in Gelbrich [5].
For each example, we will proceed as follows:

(1) We compute the contact graph $\mathbf{G}(\mathcal{R})$, defined in Definition 2. Note that there are five types of fundamental domains of $p2$ given by Grünbaum and Shephard in [6, pp.288-290]. For each example, we have the possibility to choose one of these types for Q (hence for $\mathcal{R}_0$) to get the contact graph $\mathbf{G}(\mathcal{R})$.
(2) The neighborhood graph $\mathbf{G}(\mathcal{S})$ (Definition 2) is obtained by Algorithm 3.8.
(3) We use the neighborhood graph to give some informations concerning the crystile, about its sets of L-vertices (see Proposition 3.2), its vertex neighbors and its edge neighbors (set $\mathcal{A}$) (see Characterization 3.3).

The last part of the section is then devoted to the proof of the disk-likeness or non disk-likeness of the tiles presented in these examples by applying Theorem 4.2.

5.2. **Example 1.** This example corresponds to Gelbrich's picture [5, p.252, Fig.6 (i)]. This crystile will be shown to be disk-like. It is depicted in Figure 2 together with its subdivision tiles (images of the tile by the digits).

We take

$$g(x,y) = \left(y, -3x - \frac{1}{2}\right), \;\; \mathcal{D} = \{1, b, c\},$$

and the tile is defined by

$$g(\mathcal{T}) \;=\; \mathcal{T} \cup b(\mathcal{T}) \cup c(\mathcal{T}).$$

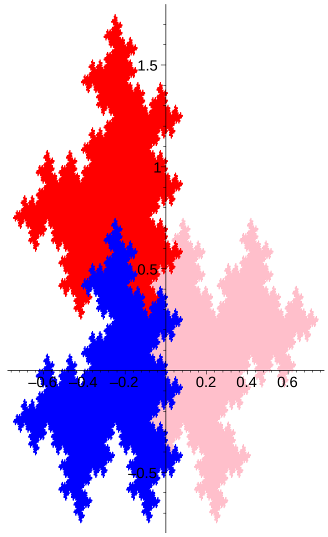


FIGURE 2. Crystile $\mathcal{T}$ of Example 1.

The requirement $g\Gamma g^{-1} \subset \Gamma$ (see Definition 1) is fulfilled because

$$gag^{-1} = b^{-3},\ gbg^{-1} = a,\ gcg^{-1} = b^{-1}c.$$

(1) We choose $\mathcal{R}_0 = \{1, b, b^{-1}, c, a^{-1}c\}$ (see the corresponding fundamental domain in Figure 5). This yields $\mathcal{R}_1 = \mathcal{R}_0 \cup \{a^{-1}b^{-1}c\}$, $\mathcal{R}_2 = \mathcal{R}_1 \cup \{b^{-1}c\} = \mathcal{R}_3$, so finally

$$\mathcal{R} = \{1, b, b^{-1}, c, a^{-1}c, a^{-1}b^{-1}c, b^{-1}c\}.$$

(2) The application of Algorithm 3.8 leads to $\mathcal{S} = \mathcal{R} \setminus \{1\}$, so the contact graph and the neighborhood graph are equal (up to the identity). They are depicted in Figure 6.

(3) *Sets of L-vertices.* Using Proposition 3.2, we read on the graph the sets of L-vertices and obtain the following results:
   – $\#V_2(b, c) = \#V_2(b, a^{-1}c) = \#V_2(b^{-1}, b^{-1}c) = \#V_2(b^{-1}, a^{-1}b^{-1}c)$
   $= \#V_2(c, b^{-1}c) = \#V_2(a^{-1}b^{-1}c, a^{-1}c) = 1$. (The other sets of 2-vertices are empty.)
   – $V_L = \emptyset$ for $L \geq 3$.

   Vertex and edge neighbors. One can use the graph of neighbors together with Characterization 3.3 to get that there is no vertex neighbor and that the set of edge neighbors is the whole set $\mathcal{S}$. Another way to show this will be given in the last part of this section (Proposition 5.4).

5.3. **Example 2.** This example corresponds to Gelbrich's picture [5, p.252, Fig.6 (b)]. This crystile will be shown to be disk-like. It is depicted here in Figure 3 together with its subdivision tiles. We take

$$g(x, y) = (-y, 3x + 1),\ \ \mathcal{D} = \{1, b, c\},$$

and the tile is defined by

$$g(\mathcal{T}) \ = \ \mathcal{T} \cup b(\mathcal{T}) \cup c(\mathcal{T}).$$

We have $g\Gamma g^{-1} \subset \Gamma$ because

$$gag^{-1} = b^3,\ gbg^{-1} = a^{-1},\ gcg^{-1} = b^2c.$$

(1) We choose $\mathcal{R}_0 = \{1, b, b^{-1}, c, bc, a^{-1}c\}$ (see Figure 5). This yields $\mathcal{R}_1 = \mathcal{R}_0$, so

$$\mathcal{R} = \{1, b, b^{-1}, c, bc, a^{-1}c\}.$$

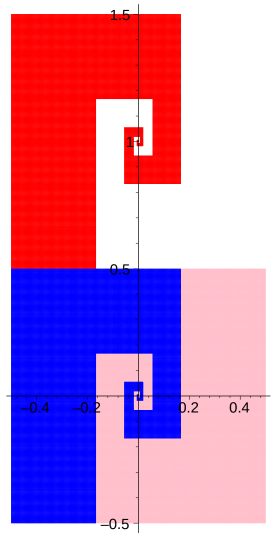


FIGURE 3. Crystile $\mathcal{T}$ of Example 2.

(2) The application of Algorithm 3.8 leads to $\mathcal{S} = \mathcal{R} \setminus \{1\} \cup \{a^{-1}bc, a^{-1}b^{-1}c\}$. The graphs are depicted in Figure 6.

(3) *Sets of L-vertices.* Using Proposition 3.2, we read off from the neighborhood graph the sets of L-vertices and obtain the following results:

- $\#V_2(b, a^{-1}c) = \#V_2(b, a^{-1}bc) = \#V_2(b^{-1}, a^{-1}b^{-1}c) = \#V_2(b^{-1}, a^{-1}c)$ $= \#V_2(a^{-1}c, a^{-1}bc) = \#V_2(a^{-1}c, a^{-1}b^{-1}c)$ $= \#V_2(b, bc) = \#V_2(b^{-1}, c) = \#V_2(c, bc) = 1$. (The sets of 2-vertices that are not listed are empty.)
- $\#V_3(b, a^{-1}c, a^{-1}bc) = \#V_3(b^{-1}, a^{-1}b^{-1}c, a^{-1}c) = 1$. (The sets of of 3-vertices that are not listed are empty.)
- $V_L = \emptyset$ for $L \geq 4$.

Remark. For each of the first six sets $V(s, s')$ in the first item, using the neighborhood graph in Figure 6 one gets only one possible sequence of labels for a walk starting from s and s', but for the other sets, one finds exactly two possible walks. Let us just consider $V_2(b, bc)$. Then the infinite walks starting from b and bc are $(1, 1, c, 1, c, 1, c, \ldots)$ and $(b, c, 1, c, 1, c, 1, \ldots)$. However, they represent the same point on the boundary of $\mathcal{T}$, because:

$$\lim_{n\to\infty} g^{-1}(g^{-1}g^{-1}c)^n(0,0) = \left(-\frac{1}{6}, \frac{1}{2}\right) = \lim_{n\to\infty} g^{-1}bg^{-1}c(g^{-}g^{-1}c)^n(0,0).$$

Vertex and edge neighbors. Looking at the graph in Figure 6, we see that there is exactly one infinite walk starting from the neighbors

$$\{a^{-1}bc, a^{-1}b^{-1}c\}.$$

This implies that these are vertex neighbors because of Characterization 3.3. One can also use Characterization 3.3 to obtain that the set of edge neighbors is $\mathcal{A} = \{1, b, b^{-1}, c, bc, a^{-1}c\}$, but another way will be given in Proposition 5.4.

5.4. **Example 3.** This example corresponds to Gelbrich's picture [5, p.253, Fig.8 (c)]. This crystile will be shown to be non disk-like. It is depicted in Figure 4 with its subdivision tiles. We take

$$g(x, y) = (-y, -3x - y), \quad \mathcal{D} = \{1, b, a^{-1}c\},$$

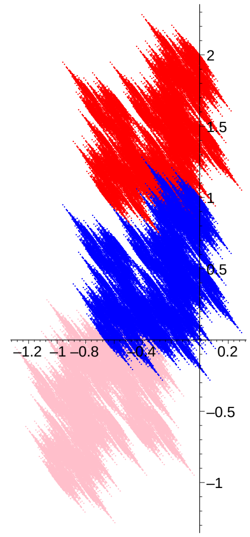


FIGURE 4. Crystile $\mathcal{T}$ of Example 3.

and the tile is defined by

$$g(\mathcal{T}) \;=\; \mathcal{T} \cup b(\mathcal{T}) \cup a^{-1}c(\mathcal{T}).$$

The property $g\Gamma g^{-1} \subset \Gamma$ holds because

$$ga^{-1}cg^{-1} = b^3c,\; gbg^{-1} = a^{-1}b^{-1},\; gcg^{-1} = c.$$

(1) We choose $\mathcal{R}_0 = \{1, b, b^{-1}, a^{-1}c, c, bc\}$ (see Figure 5). This yields $\mathcal{R}_1 = \mathcal{R}_0 \cup \{a^{-1}bc\}$, $\mathcal{R}_2 = \mathcal{R}_1$, so finally

$$\mathcal{R} = \{1, b, b^{-1}, a^{-1}c, c, bc, a^{-1}bc\}.$$

(2) The application of Algorithm 3.8 leads to

$$\mathcal{S} = \mathcal{R} \setminus \{1\} \;\cup \{a, a^{-1}, a^{-1}b, ab^{-1}, a^{-1}b^2c, b^2c, a^{-2}b^2c\}.$$

The graphs are depicted in Figure 7.

5.5. **Example 4.** This example corresponds to Gelbrich's picture [5, p.254, Fig.7 (a)], one of the two "not as convincing" pictures listed by Gelbrich. We represented it in Figure 1. We take

$$g(x, y) = (x - y, 3x + 1),\; \mathcal{D} = \{1, b, a^{-1}c\},$$

and the tile is defined by

$$g(\mathcal{T}) \;=\; \mathcal{T} \cup b(\mathcal{T}) \cup a^{-1}c(\mathcal{T}).$$

The property $g\Gamma g^{-1} \subset \Gamma$ holds because

$$ga^{-1}cg^{-1} = a^{-1}b^{-1}c,\; gbg^{-1} = a^{-1},\; gcg^{-1} = b^2c.$$

(1) We choose $\mathcal{R}_0 = \{1, b, b^{-1}, c, a^{-1}c, bc, a^{-1}bc\}$ (see Figure 5). This yields $\mathcal{R}_1 = \mathcal{R}_0$, so

$$\mathcal{R} = \{1, b, b^{-1}, a^{-1}c, c, bc, a^{-1}bc, a^{-1}bc\}.$$

(2) The application of Algorithm 3.8 leads to $\mathcal{S} = \mathcal{R}\backslash\{1\}$. The graphs are depicted in Figure 6.
(3) *Sets of L-vertices.* Using Proposition 3.2, we read on the graph the sets of L-vertices and obtain following results:
   - $\#V_2(b, bc) = \#V_2(b, a^{-1}bc) = \#V_2(b^{-1}, c) = \#V_2(b^{-1}, a^{-1}c)$ $= \#V_2(bc, c) = \#V_2(a^{-1}bc, a^{-1}c) \;=\; 1$. (The sets of 2-vertices that are not listed are empty.)
   - $V_L = \emptyset$ for $L \geq 3$.

Vertex and edge neighbors. One can use the graph of neighbors together with Characterization 3.3 to get that there is no vertex neighbor and that the set of edge neighbors is exactly the whole set $\mathcal{S}$. Indeed, the infinite walks

$$(a^{-1}c, b, b, \ldots), (1, b, b, \ldots), (b, b, b, \ldots), (a^{-1}c, a^{-1}c, 1, a^{-1}c, 1, \ldots),$$
$$(1, a^{-1}c, 1, a^{-1}c, \ldots), (a^{-1}c, 1, a^{-1}c, 1, \ldots)$$

are in $\mathbf{G}_1(\mathcal{S})$ iff their starting points are respectively $b, b^{-1}, c, a^{-1}c, bc, a^{-1}bc$. However, this result will also be obtained by Proposition 5.4.

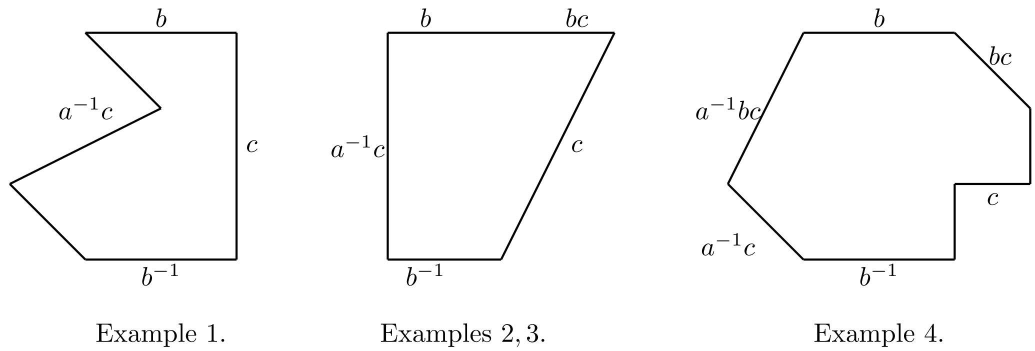


FIGURE 5. Fundamental domain Q.

5.6. **Application of the disk-likeness criterion.** We now answer the question of Gelbrich in our examples.

Proposition 5.1. *The following assertions hold.*

- *The tiles defined in Example* 1*, Example* 2 *and Example* 4 *are disk-like.*
- *The tile defined in Example* 3 *is non disk-like.*

To prove this, we will apply the criterion of Theorem 4.2 to the examples. To this matter, we want to identify the edge neighbors of the central tile $\mathcal{T}$ in another way than in Characterization 3.3. First we recall two results on the boundary connectedness of connected tiles.

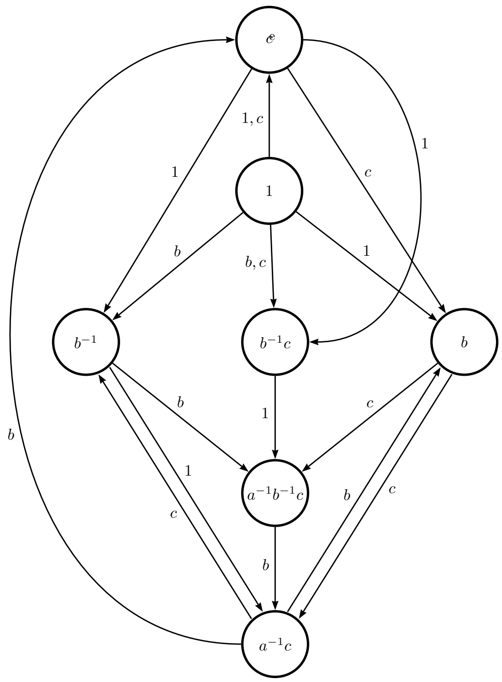
Lemma 5.2. [10, Theorem 1.1] *Let $f_1, \ldots, f_k$ be injective contractions on $\mathbb{R}^n$ $(n \geq 2)$ satisfying the open set condition, and let T be the attractor. Then the boundary ∂T of T is connected whenever T is.*

Remember that a compact set in $\mathbb{R}^n$ which coincides with the closure of its interior is said to *tile* $\mathbb{R}^n$ *by* $\mathbb{Z}^n$ if the translates of this set by $\mathbb{Z}^n$ cover $\mathbb{R}^n$ and the interiors of distinct tiles have no intersection.

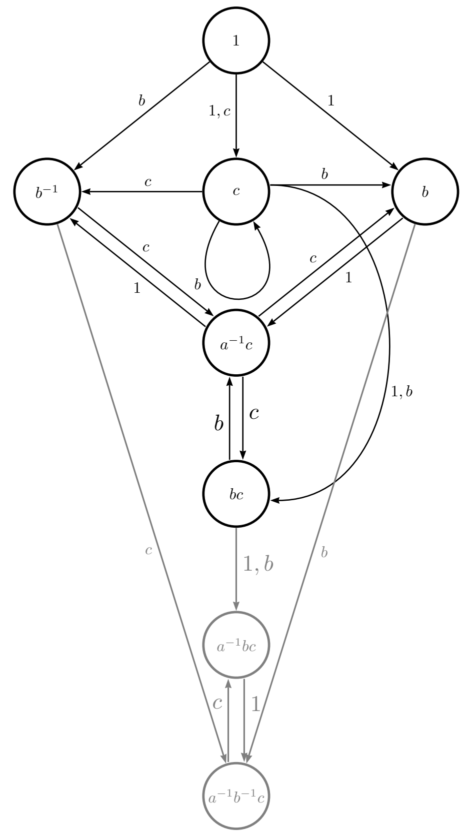
Lemma 5.3. [10, Theorem 3.1] *Let T be an arcwise connected compact set that tiles $\mathbb{R}^n$ by $\mathbb{Z}^n$, then its boundary ∂T is connected.*

Proposition 5.4. *Let $\mathcal{T}$ be a connected p2-tile. Let a, b be the translations of $\mathbb{R}^2$ defined by $a(x,y) = (x+1, y)$ and $b(x,y) = (x, y+1)$, c the $180°$-rotation of the plane $c(x,y) = (-x,-y)$.*

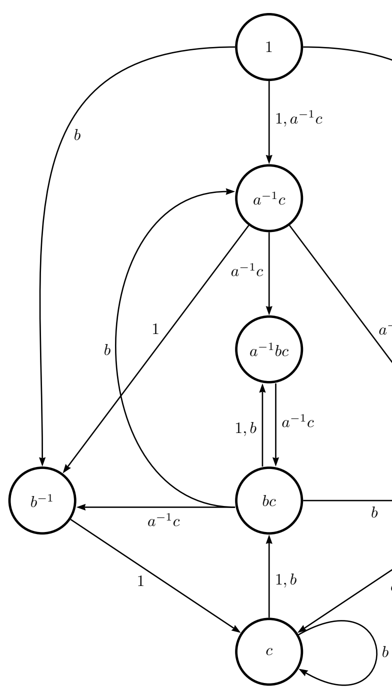
(i) *If $\mathcal{T}$ has six neighbors $\mathcal{S} = \{b, b^{-1}, c, a^{-1}c, a^{-1}b^{-1}c, b^{-1}c\}$, then $\mathcal{S}$ consists of edge neighbors.*
(ii) *If $\mathcal{T}$ has seven neighbors $\mathcal{S} = \{b, b^{-1}, c, bc, a^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}$, then $\{b, b^{-1}, c, bc, a^{-1}c\}$ are edge neighbors.*
(iii) *If $\mathcal{T}$ has eight neighbors $\mathcal{S} = \{b, b^{-1}, c, a^{-1}c, bc, b^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}$ (resp. $\mathcal{S} = \{c, bc, ac, a^{-1}bc, b, b^{-1}, ab^{-1}, a^{-1}b\}$), then $\{b, b^{-1}, c, a^{-1}c\}$ (resp. $\{c, bc, ac, a^{-1}bc\}$) are edge neighbors.*



Example 1.



Example 2.



Example 4.

FIGURE 6. Contact graphs $\mathbf{G}(\mathcal{R})$ (dark part) and neighborhood graphs $\mathbf{G}(\mathcal{S})$ (dark and dimmed parts).

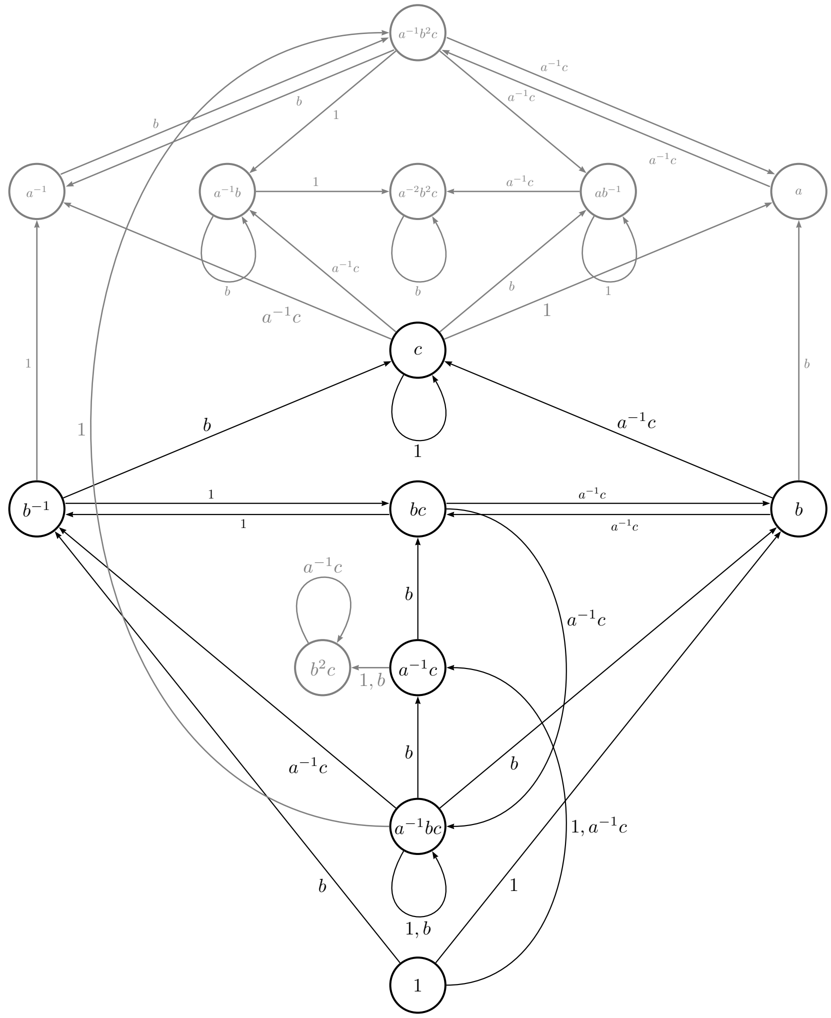


FIGURE 7. Example 3: contact graph $\mathbf{G}(\mathcal{R})$ (dark part) and neighborhood graph $\mathbf{G}(\mathcal{S})$ (dark and dimmed parts).

(*iv*) *If* $\mathcal{T}$ *has twelve neighbors* $\mathcal{S} = \{c, a^{-1}c, bc, abc, a^{-1}bc, a^{-1}b^{-1}c, a, a^{-1}, b, b^{-1}, ab, a^{-1}b^{-1}\}$, *then* $\{c, bc, a^{-1}c\}$ *are edge neighbors.*

Proof. Let us consider the case of six neighbors. We will show that the rotation c is an edge neighbor. Let

$$T' := \mathcal{T} \cup a^{-1}b^{-1}c(\mathcal{T}) .$$

Then T' is an arcwise connected compact set that tiles $\mathbb{R}^2$ by $\mathbb{Z}^2$, *i.e.*, that T' provides a lattice tiling of $\mathbb{R}^2$. We set

$$\begin{array}{rcl}
S_1 & = & \bigcup_{q\in\mathbb{Z}, r\in\{0,1\}} (ab)^q c^r(\mathcal{T}) \\
 & = & \bigcup_{q\in\mathbb{Z}} (ab)^q (T'), \\
\Omega_1 & = & \left(\bigcup_{q<q', r\in\{0,1\}} a^q b^{q'} c^r(\mathcal{T})\right) \\
 & = & \left(\bigcup_{q<q'} a^q b^{q'}(T')\right), \\
\Omega_2 & = & \left(\bigcup_{q>q', r\in\{0,1\}} a^q b^{q'} c^r(\mathcal{T})\right) \\
 & = & \left(\bigcup_{q>q'} a^q b^{q'}(T')\right).
\end{array}$$

The identity $\mathbb{R}^2 = S_1 \cup \Omega_1 \cup \Omega_2$ holds, and because of the assumption on the set $\mathcal{S}$, we have $\Omega_1 \cap \Omega_2 = \emptyset$.
The tile T' being an arcwise connected compact set, Lemma 5.3 assures that its boundary $\partial T'$ is connected. Let us suppose that $\partial T' \subseteq \Omega_1 \cup \Omega_2$, then we would obtain

$$\partial T' = (\partial T' \cap \Omega_1) \cup (\partial T' \cap \Omega_2),$$

which is a partition of $\partial T'$ into two relative closed sets that have empty intersection, a contradiction to the connectedness of $\partial T'$.
Consequently, $\partial T' \cap (S_1 \setminus (\Omega_1 \cup \Omega_2)) \neq \emptyset$, thus there is an $s \in \{ab, a^{-1}b^{-1}\}$ such that

$$(T' \cap s(T')) \setminus \bigcup_{(q,q') \neq (0,0), \pm(1,1)} a^q b^{q'}(T') \neq \emptyset, \tag{5.1}$$

since $ab(T')$ and $a^{-1}b^{-1}(T')$ are the only lattice tiles in S_1 that are in contact with T' (this follows from the assumption on $\mathcal{S}$).
By the assumption on $\mathcal{S}$, we have

$$T' \cap ab(T') = \mathcal{T} \cap c(\mathcal{T}) \text{ and } T' \cap a^{-1}b^{-1}(T') = a^{-1}b^{-1}c(\mathcal{T}) \cap a^{-1}b^{-1}(\mathcal{T}), \tag{5.2}$$

hence, also by this assumption, $(T' \cap ab(T')) \cap \big(T' \cap a^{-1}b^{-1}(T')\big) = \emptyset$.
Thus, if $s = ab$, we get from (5.1) and (5.2) that

$$(\mathcal{T} \cap c(\mathcal{T})) \setminus \bigcup_{\gamma \in \Gamma \setminus \{1,c\}} \gamma(\mathcal{T}) \neq \emptyset.$$

This indicates that c is an edge neighbor.
If $s = a^{-1}b^{-1}$, we obtain similarly that $a^{-1}b^{-1}(\mathcal{T})$ and $a^{-1}b^{-1}c(\mathcal{T})$ are edge neighbors, hence after translation by ab again that c is an edge neighbor.
The other neighbors of the six-neighbor case as well as the cases of seven, eight and twelve neighbors can be treated similarly. (Note that the case of translations, say b for instance, does not require the introduction of a substitution tile T': we can choose $S_1 = \bigcup_{q \in \mathbb{Z}} b^q(\mathcal{T})$ and use Lemma 5.2 to obtain that $\partial \mathcal{T}$ intersects S_1.) □

We are now able to examine the disk-likeness of the tiles presented in Examples 1 to 4 by checking the three items of Theorem 4.2.
For each example, we will have to compute the graph $\mathcal{G}_\gamma$ for every $\gamma \in \mathcal{A}$. The set of states of $\mathcal{G}_\gamma$ is

$$\{\delta\,(B_{\gamma'}) : \text{if there is an edge } \gamma \xrightarrow{\delta|\delta'} \gamma' \text{ in } \mathbf{G}(\mathcal{A})\}.$$

There is an edge in $\mathcal{G}_\gamma$ between two states $\delta\,(B_{\gamma'})$ and $\delta'\,(B_{\gamma''})$ iff

$$\mathcal{T} \cap \gamma''(\mathcal{T}) \cap \delta'^{-1}\delta(\mathcal{T}) \cap \delta'^{-1}\delta\gamma'(\mathcal{T}) \neq \emptyset.$$

This can easily be checked by looking at the sets V_2 and V_3.
We will also need the subgraph of $\mathcal{G}_2$ induced by the states $\{B_\gamma, \gamma \in \mathcal{A}\}$. Again, there is an edge between B_γ and $B_{\gamma'}$ in this graph iff

$$\mathcal{T} \cap \gamma(\mathcal{T}) \cap \gamma'(\mathcal{T}) \neq \emptyset,$$

which can be seen using the sets V_2.

Proof of Proposition 5.1.

- **Example 1**
  The first item of Theorem 4.2 is fulfilled as we check in Section 5.2.
  From Proposition 5.4, we get
  $$\mathcal{A} = \{b, b^{-1}, c, a^{-1}c, a^{-1}b^{-1}c, b^{-1}c\}.$$
  We obtain the graphs $\mathcal{G}_\gamma$ for $\gamma \in \mathcal{A}$ in Figure 8. Each of them consists of a simple path. So the second item is fulfilled.
  The subgraph of $\mathcal{G}_2$ induced by the states $\{B_\gamma : \gamma \in \mathcal{A}\}$ is represented on Figure 9: it is a

simple loop.
Thus the crystile of Example 1 is disk-like.

- **Example 2**
  The first item of Theorem 4.2 is fulfilled (see Section 5.3).
  From Proposition 5.4, we get:

$$\mathcal{A} = \{b, b^{-1}, c, a^{-1}c, a^{-1}b^{-1}c, b^{-1}c\}.$$

  We obtain the graphs $\mathcal{G}_\gamma$ in Figure 8. Each of them consists of a simple path. So the second item is fulfilled.
  The subgraph of $\mathcal{G}_2$ induced by the states $\{B_\gamma : \gamma \in \mathcal{A}\}$ is represented on Figure 9: it is a simple loop.
  Thus the crystile of Example 2 is disk-like.

- **Example 3**
  We show that the first item of the criterion of Theorem 4.2 is not fulfilled, in particular:

$$\#V_2(a^{-1}bc, b^{-1}) \geq 2\ .$$

  Indeed, the infinite walks

$$\begin{array}{ccccccccc} a^{-1}bc & \xrightarrow{1} & a^{-1}bc & \xrightarrow{b} & a^{-1}bc & \xrightarrow{b} & a^{-1}bc & \xrightarrow{b} & \dots \\ b^{-1} & \xrightarrow{1} & a^{-1} & \xrightarrow{b} & a^{-1}b^2c & \xrightarrow{b} & a^{-1} & \xrightarrow{b} & \dots \end{array}$$

  labelled by $(1\ b\ b\ b\ \dots)$ and

$$\begin{array}{ccccccccccccc} a^{-1}bc & \xrightarrow{b} & a^{-1}c & \xrightarrow{1} & b^2c & \xrightarrow{a^{-1}c} & b^2c & \xrightarrow{a^{-1}c} & b^2c & \xrightarrow{a^{-1}c} & \dots \\ b^{-1} & \xrightarrow{b} & c & \xrightarrow{1} & a & \xrightarrow{a^{-1}c} & a^{-1}b^2c & \xrightarrow{a^{-1}c} & a & \xrightarrow{a^{-1}c} & \dots \end{array}$$

  labelled by $(b\ 1\ a^{-1}c\ a^{-1}c\ a^{-1}c\ \dots)$ are in $\mathbf{G}(\mathcal{S})$ (look at Figure 7), this means by Proposition 3.2 that the points

$$x = \lim_{n\to\infty} g^{-1}(g^{-1}b)^n(0,0) = \left(-\frac{2}{3}, 1\right)$$

  and

$$y = \lim_{n\to\infty} g^{-1}bg^{-1}(g^{-1}a^{-1}c)^n(0,0) = \left(-\frac{4}{9}, \frac{1}{3}\right)$$

  are two distinct points of $V_2(a^{-1}bc, b^{-1})$.
  Thus the crystile of Example 3 is non disk-like.

- **Example 4**
  The first item of the criterion is fulfilled (see Section 5.5).
  From Proposition 5.4 (replace b by b^{-1}) we have

$$\mathcal{A} = \{b, b^{-1}, c, a^{-1}c, a^{-1}bc, bc\}.$$

  We obtain the graphs $\mathcal{G}_\gamma$ in Figure 8. Each of them consists of a simple path. So the second item is fulfilled.
  The subgraph of $\mathcal{G}_2$ induced by the states $\{B_\gamma : \gamma \in \mathcal{A}\}$ is represented in Figure 9: it is a simple loop.
  Thus the crystile of Example 4 is disk-like. □

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γ	$\mathcal{G}_\gamma$
b	$c(B_{a^{-1}b^{-1}c}) - c(B_{a^{-1}c})$
b^{-1}	$b(B_{a^{-1}b^{-1}c}) - B_{a^{-1}c}$
c	$B_{b^{-1}} - B_{b^{-1}c} - c(B_b)$
$a^{-1}c$	$c(B_{b^{-1}}) - b(B_c) - b(B_b)$
$a^{-1}b^{-1}c$	$b(B_{a^{-1}c})$
$b^{-1}c$	$B_{a^{-1}b^{-1}c}$

Example 1.

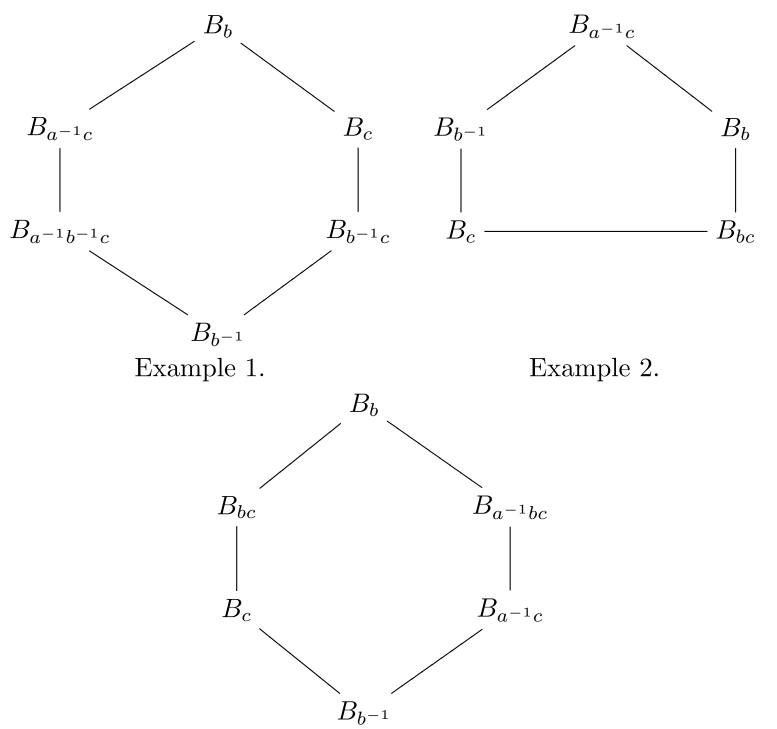
γ	$\mathcal{G}_\gamma$
b	$B_{a^{-1}c}$
b^{-1}	$c(B_{a^{-1}c})$
c	$b(B_b) - b(B_{bc}) - b(B_c) - B_{bc} - c(B_{b^{-1}})$
bc	$b(B_{a^{-1}c})$
$a^{-1}c$	$B_{b^{-1}} - c(B_{bc}) - c(B_b)$

Example 2.

γ	$\mathcal{G}_\gamma$
b	$a^{-1}c(B_c)$
b^{-1}	B_c
c	$b(B_{bc}) - b(B_c) - B_{bc}$
$a^{-1}c$	$B_{b^{-1}} - a^{-1}c(B_{a^{-1}bc}) - a^{-1}c(B_b)$
$a^{-1}bc$	$a^{-1}c(B_{bc})$
bc	$a^{-1}c(B_{b^{-1}}) - B_{a^{-1}bc} - b(B_{a^{-1}c}) - b(B_{a^{-1}bc}) - b(B_b)$

Example 4.

FIGURE 8. Subgraph $\mathcal{G}_\gamma$ of the double neighboring graph $\mathcal{G}_2$.



Example 4.

FIGURE 9. Restriction of $\mathcal{G}_2$ to the set of states $\{B_\gamma, \gamma \in \mathcal{A}\}$.

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