# CONTRACTIVITY OF THREE DIMENSIONAL SHIFT RADIX SYSTEMS WITH FINITENESS PROPERTY

HORST BRUNOTTE, PETER KIRSCHENHOFER, AND JÖRG M. THUSWALDNER

*Dedicated to Professor Attila Pethő on the occasion of his 60$^{th}$ birthday*

Abstract. Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. We define the *shift radix system* $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor) \qquad (\mathbf{a} = (a_1, \dots, a_d)).$$

The shift radix system $\tau_{\mathbf{r}}$ has the *finiteness property* if each $\mathbf{a} \in \mathbb{Z}^d$ is eventually mapped to $\mathbf{0}$ under iterations of $\tau_{\mathbf{r}}$.

The mapping $\tau_{\mathbf{r}}$ can be written as $\tau_{\mathbf{r}}(\mathbf{a}) = R(\mathbf{r})\mathbf{a} + \mathbf{v}(\mathbf{a})$ where $R(\mathbf{r})$ is a $d \times d$ matrix and $\mathbf{v}$ is a correction term. It has been conjectured that the fact that $\tau_{\mathbf{r}}$ has the finiteness property implies that all eigenvalues of $R(\mathbf{r})$ are strictly smaller than one in modulus.

The aim of the present paper is to prove this conjecture for the case $d = 3$.

## 1. Introduction

In 2005 Akiyama *et al.* [1] introduced so-called *shift radix systems*. Moreover, they showed that these simple dynamical systems are generalizations of several well-known notions of number systems such as beta-numeration and canonical number systems. To be more specific, let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ let[1]

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor),$$

where $\mathbf{r}\mathbf{a} = r_1 a_1 + \cdots + r_d a_d$, *i.e.*, the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. We call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short). If for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ such that the $k$-fold iterative application of $\tau_{\mathbf{r}}$ to $\mathbf{a}$ satisfies $\tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$ we say that the SRS $\tau_{\mathbf{r}}$ has the *finiteness property*[2].

It is easy to see that the mapping $\tau_{\mathbf{r}}$ can be viewed as a sum of a linear function and a small error term caused by the floor function $\lfloor \cdot \rfloor$ occurring in its definition. Indeed, for $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ we denote by

$$R(\mathbf{r}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix} \tag{1.1}$$

the companion matrix of the polynomial

$$\chi_{\mathbf{r}}(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1.$$

Then we can write

$$\tau_{\mathbf{r}}(\mathbf{a}) = R(\mathbf{r})\mathbf{a} + \mathbf{v}(\mathbf{a}), \tag{1.2}$$

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[1] For $y \in \mathbb{R}$ we denote by $\lfloor y \rfloor$ the largest $n \in \mathbb{Z}$ with $n \leq y$. Moreover, we set $\{y\} = y - \lfloor y \rfloor$.

[2] Note that our definition of SRS agrees with the one in [6] but slightly differs from the original one in [1]. Indeed, the SRS in [1] are the same as our SRS with finiteness property.

where $\mathbf{v}(\mathbf{a}) = (0, \dots, 0, \{\mathbf{ra}\})$. Note that $\chi_{\mathbf{r}}$ is the characteristic polynomial of $R(\mathbf{r})$. Moreover, given a vector $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$, calculating $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_{d+1})$ amounts to calculating the integer $a_{d+1}$. According to the definition of $\tau_{\mathbf{r}}$ this integer $a_{d+1}$ is given as the unique integral solution of the linear inequality

$$0 \le r_1 a_1 + \cdots + r_d a_d + a_{d+1} < 1. \tag{1.3}$$

This fact will be used throughout the present paper.

Known results on beta-transformations and canonical number systems suggest that the following conjecture is true.

**Conjecture 1.1** ([2, Conjecture 2, p. 23])**.** Let $\mathbf{r} \in \mathbb{R}^d$. If $\tau_{\mathbf{r}}$ has the finiteness property then $R(\mathbf{r})$ is contractive, *i.e.*, each of its eigenvalues has modulus strictly less than one.

It is clear that the finiteness property implies that each eigenvalue of $R(\mathbf{r})$ has modulus at most one because otherwise in view of (1.2) it is clear that $||\tau_{\mathbf{r}}^k(\mathbf{a})|| \to \infty$ for $k \to \infty$ if $||\mathbf{a}||$ is large. Thus it remains to check all parameters $\mathbf{r}$ giving rise to a matrix $R(\mathbf{r})$ whose eigenvalues have modulus at most one with equality in at least one case. We mention that Pethő [13] studied the case that some roots of $\chi_{\mathbf{r}}$ are roots of unity.

Moreover, the conjecture holds for all parameters corresponding to beta-numeration as well as to canonical number systems. This has been proved by employing algebraic methods (we refer to [9] for beta-numeration and to [12] for canonical number systems). However, these cases cover only countably many parameters $\mathbf{r}$. In general, these methods do not apply any more. The conjecture is trivial for $d = 1$ ([1, Proposition 4.1]) and has been proved for $d = 2$ ([3, Corollary 2.5]).

Our aim is to prove the conjecture for $d = 3$.

The following families of sets are needed for our studies. For $d \in \mathbb{N}$, $d \ge 1$ let

$$\begin{aligned} \mathcal{D}_d &:= \left\{\mathbf{r} \in \mathbb{R}^d \,:\, \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k\ge 0} \text{ is ultimately periodic}\right\} \quad \text{and} \\ \mathcal{D}_d^{(0)} &:= \left\{\mathbf{r} \in \mathbb{R}^d \,:\, \tau_{\mathbf{r}} \text{ has the finiteness property}\right\}. \end{aligned} \tag{1.4}$$

With this notation, our main result is contained in the following theorem.

**Theorem 1.2.** *If $\mathbf{r} \in \mathcal{D}_3^{(0)}$ then $R(\mathbf{r})$ is contractive,* i.e., *all roots of the polynomial $\chi_{\mathbf{r}}$ are strictly less than one in modulus.*

The set $\mathcal{D}_d$ is strongly related to contracting polynomials. In particular, let

$$\mathcal{E}_d := \left\{(r_1, \dots, r_d) \in \mathbb{R}^d \,:\, \rho(X^d + r_d X^{d-1} + \cdots + r_1) < 1\right\},$$

where we denote by $\rho(f)$ the maximum of the moduli of the roots of $f \in \mathbb{R}[X]$. In [1, Lemmas 4.1, 4.2 and 4.3] it is shown that

$$\operatorname{int}(\mathcal{D}_d) = \mathcal{E}_d. \tag{1.5}$$

Note that in view of the remarks after the statement of Conjecture 1.1 it is clear that $\mathcal{D}_d^{(0)} \subset \overline{\mathcal{E}_d}$. Thus the conjecture can be formulated as

$$\mathcal{D}_d^{(0)} \subset \mathcal{E}_d.$$

Therefore, it remains to prove Conjecture 1.1 for each $\mathbf{r} \in \partial\mathcal{E}_d$.

The proofs of Conjecture 1.1 for the cases $d = 1$ and $d = 2$ require explicit descriptions of the boundary of $\mathcal{E}_d$. In these cases the proof was established by constructing explicit orbits that do not end up at $\mathbf{0}$ for all $\mathbf{r}$ in the boundary of $\mathcal{E}_d$. In our case this seems no longer possible for all parameters $\mathbf{r} \in \partial\mathcal{E}_d$. We need to employ other methods to gain the proof of Theorem 1.2. Indeed, for some parameter regions we show that for large $n$ the set $\tau_{\mathbf{r}}^{-n}(\mathbf{0})$, where $\tau_{\mathbf{r}}^{-1}$ denotes the preimage of $\tau_{\mathbf{r}}$, has finite intersection with a subspace that is bounded by two hyperplanes. This allows us to conclude that some elements of this subspace belong to periodic orbits of $\tau_{\mathbf{r}}$ that do not end up at $\mathbf{0}$ without constructing these orbits explicitly.

For the sake of completeness we would also like to mention that Surer [14] studies a generalization of SRS by putting an additional translation vector in the floor function of the mapping $\tau_{\mathbf{r}}$, *i.e.*, he considers the mappings

$$\tau_{\mathbf{r},\varepsilon}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{r}\mathbf{a} + \varepsilon \rfloor).$$

These objects are called $\varepsilon$-SRS. In [7, Conjecture 13] Conjecture 1.1 has been formulated for $\varepsilon$-SRS. For $\varepsilon = \frac{1}{2}$ it is shown to be true for $d \le 3$ in [5, 10]. Indeed, $\frac{1}{2}$-SRS with finiteness property have been completely characterized for $d \le 3$. The methods used for proving these results are no longer applicable in our setting. Indeed, the situation in the case of SRS turns out to be much more complicated than it is for $\frac{1}{2}$-SRS.

## 2. Preliminary results

The present section contains several preliminary definitions and results needed in the proof of our main theorem. Moreover, already in this section we prove that certain regions of $\partial\mathcal{E}_d$ are not contained in $\mathcal{D}_d^{(0)}$.

For $\mathbf{r} \in \mathbb{R}^d$ we denote by

$$\mathrm{Per}(\mathbf{r}) = \{\mathbf{a} \in \mathbb{Z}^d \ : \ \text{there exists } k > 0 \text{ with } \tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{a}\}$$

the set of *(purely) periodic elements* of $\tau_{\mathbf{r}}$. For parameters $\mathbf{r} \in \mathcal{D}_d$ it is clear from the definition that $\tau_{\mathbf{r}}$ has the finiteness property if and only if $\mathrm{Per}(\mathbf{r}) = \{\mathbf{0}\}$.

Let $\mathbf{a}_1, \dots, \mathbf{a}_p \in \mathbb{Z}^d$ be given and suppose that $\tau_{\mathbf{r}}(\mathbf{a}_k) = \mathbf{a}_{k+1}$ $(1 \le k \le p-1)$ and $\tau_{\mathbf{r}}(\mathbf{a}_p) = \mathbf{a}_1$. Then we say that $\mathbf{a}_1, \dots, \mathbf{a}_p$ is a *cycle* or a *periodic orbit* of $\tau_{\mathbf{r}}$. Such a cycle is often denoted by

$$\mathbf{a}_1 \to \cdots \to \mathbf{a}_p \to \mathbf{a}_1.$$

The integer $p$ is called a *period* of the periodic element $\mathbf{a}_1$ (obviously, each element of a cycle is contained in $\mathrm{Per}(\mathbf{r})$ and therefore a periodic element).

A cycle $\mathbf{a}_1 \to \cdots \to \mathbf{a}_p \to \mathbf{a}_1$ of $\tau_{\mathbf{r}}$ is called *nontrivial* if $\mathbf{a}_1 \ne \mathbf{0}$. If $\mathbf{r} \in \mathcal{D}_d$ it is immediate from the definitions that $\tau_{\mathbf{r}}$ admits a nontrivial cycle if and only if $\tau_{\mathbf{r}}$ does not have the finiteness property. Of course, given an arbitrary $\mathbf{r} \in \mathbb{R}^d$ it is sufficient to exhibit a nontrivial period in order to conclude that $\tau_{\mathbf{r}}$ does not have the finiteness property. This criterium will be used throughout the paper.

We are now preparing the proof of our main theorem. As we already know that $\mathcal{D}_3^{(0)} \subset \overline{\mathcal{E}_3}$ it remains to check that $\mathcal{D}_3^{(0)} \cap \partial\mathcal{E}_3 = \emptyset$. The boundary of $\mathcal{E}_d$ has been investigated in [11] (see also [8]). In particular, it is shown in [11] that

$$\partial\mathcal{E}_d = E_d^{(1)} \cup E_d^{(-1)} \cup E_d^{(\mathbb{C})},$$

where $E_d^{(1)}$ and $E_d^{(-1)}$ are subsets of hyperplanes and $E_d^{(\mathbb{C})}$ is a hypersurface. Here, $E_d^{(1)}$ and $E_d^{(-1)}$ correspond to polynomials having 1 and $-1$ as a root, respectively, while $E_d^{(\mathbb{C})}$ is the closure of the set of all parameters corresponding to polynomials having a nonreal root of modulus 1. The set $\mathcal{E}_3$ is depicted in Figure 1.

*Remark* 2.1. For the sake of completeness we mention that this correspondence is induced by the bijective mapping

$$\mathbf{r} \in \mathbb{R}^d \mapsto \chi_{\mathbf{r}} \in \mathbb{R}[X].$$

If $\beta$ is a nonrational algebraic number with at least one conjugate on the unit circle then the minimal polynomial of $\beta$ belongs to $E_d^{(\mathbb{C})}$. In particular, if $d$ is odd and $\beta$ is a Salem number of degree $d+1$ with minimal polynomial $M$ then we can write $M = (X-\beta)f$ with $f \in E_d^{(\mathbb{C})}$.

For the case $d = 3$ in [11, Section 6.2] the parameterizations

$$E_3^{(1)} = \{(s, s+t+st, st+t+1) \ : \ -1 \le s,t \le 1\} \quad \text{and} \tag{2.1}$$

$$E_3^{(-1)} = \{(-s, s-t-st, st+t-1) \ : \ -1 \le s,t \le 1\} \tag{2.2}$$

are derived. In order to get a parametrization of $E_d^{(\mathbb{C})}$ we note that a real polynomial of degree three whose roots are at most one in modulus and which has at least one nonreal root is of the form

$$\chi_{\mathbf{r}}(X) = (X+t)(X^2+sX+1) = X^3 + (s+t)X^2 + (st+1)X + t$$

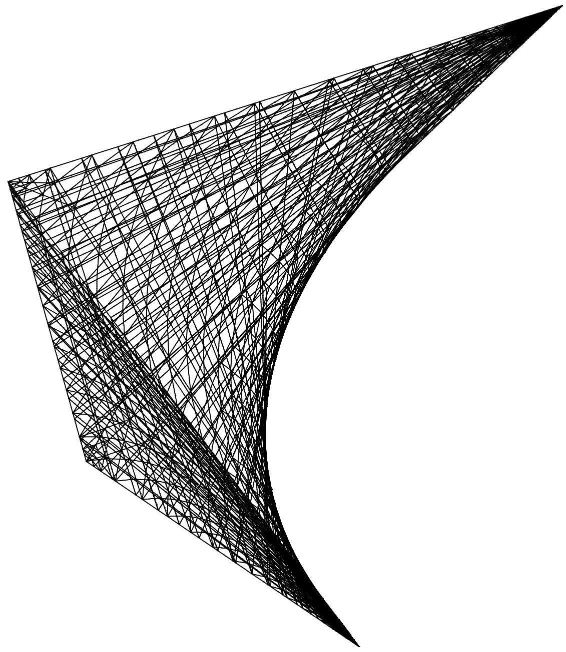

FIGURE 1. The set $\mathcal{E}_3$. The boundary of $\mathcal{E}_3$ consists of three pieces. The two hyperplanes are the sets $E_d^{(1)}$ and $E_d^{(-1)}$, while the quadratic surface is the set $E_d^{(\mathbb{C})}$.

with $-2 < s < 2$ and $-1 \le t \le 1$. Thus

$$E_3^{(\mathbb{C})} = \{(t, st+1, s+t) \mid -2 \le s \le 2,\ -1 \le t \le 1\}.$$

What we have to show now is that

$$\mathcal{D}_3^{(0)} \cap (E_3^{(1)} \cup E_3^{(-1)} \cup E_3^{(\mathbb{C})}) = \emptyset.$$

The most difficult part consists in showing that $\mathcal{D}_3^{(0)} \cap E_3^{(\mathbb{C})} = \emptyset$. The other two parts of $\partial\mathcal{E}_3$ can be treated in an easy way. We start with $E_3^{(1)}$.

**Lemma 2.2.** *We have* $\mathcal{D}_3^{(0)} \cap E_3^{(1)} = \emptyset$.

*Proof.* Let $\mathbf{r} \in E_3^{(1)}$. Then by (2.1) there exist $s, t \in [-1, 1]$ such that $\mathbf{r} = (s, s+t+st, st+t+1)$. We will show that for this parameter $\mathbf{r}$ the mapping $\tau_{\mathbf{r}}$ has $(1, -1, 1)$ as a periodic point. Indeed, by (1.3) we have $\tau_{\mathbf{r}}((1, -1, 1)) = (-1, 1, i)$ with $i \in \mathbb{Z}$ given as the solution of

$$0 \le s - (s+t+st) + (st+t+1) + i < 1 \quad \Longleftrightarrow \quad 0 \le i+1 < 1.$$

As $i = -1$ is the only choice we get $\tau_{\mathbf{r}}((1, -1, 1)) = (-1, 1, -1)$. Going one step further we get $\tau_{\mathbf{r}}((-1, 1, -1)) = (1, -1, i)$ with

$$0 \le -s + (s+t+st) - (st+t+1) + i < 1 \quad \Longleftrightarrow \quad 0 \le i-1 < 1$$

and therefore $\tau_{\mathbf{r}}((-1, 1, -1)) = (1, -1, 1)$. Thus $(1, -1, 1) \to (-1, 1, -1) \to (1, -1, 1)$ is a cycle of $\tau_{\mathbf{r}}$. Therefore $\mathbf{r} \notin \mathcal{D}_3^{(0)}$. Since $\mathbf{r}$ was an arbitrary element of $E_3^{(1)}$ the lemma is proved. □

The region $E_3^{(-1)}$ is settled in the following lemma.

**Lemma 2.3.** *We have* $\mathcal{D}_3^{(0)} \cap E_3^{(-1)} = \emptyset$.

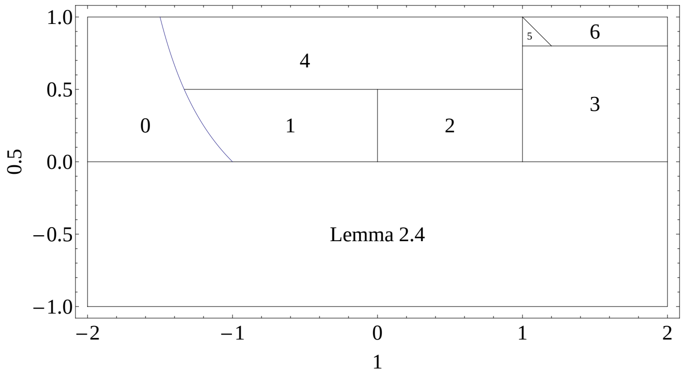


FIGURE 2. This figure shows how the rectangle $\{(s,t)\ :\ s \in (-2,2),\, t \in (-1,1)\}$ is subdivided by the regions $\mathcal{C}_i$ and the region treated in Lemma 2.4. The region $\mathcal{C}_i$ is labeled by $i$ in this figure. ($0 \le i \le 6$; note that $\mathcal{C}_0$ also contains the lines $s = 0$ and $s = 1$.)

*Proof.* Let $\mathbf{r} \in E_3^{(-1)}$. Then by (2.2) there exist $s, t \in [-1,1]$ such that $\mathbf{r} = (-s, s-t-st, st+t-1)$. We will show that for this parameter $\mathbf{r}$ the mapping $\tau_{\mathbf{r}}$ has $(1,1,1)$ as a periodic point. Indeed, we have $\tau_{\mathbf{r}}((1,1,1)) = (1,1,i)$ with $i \in \mathbb{Z}$ given as the solution of

$$0 \le -s + (s-t-st) + (st+t-1) + i < 1 \quad \Longleftrightarrow \quad 0 \le i - 1 < 1.$$

As $i = 1$ is the only choice we get $\tau_{\mathbf{r}}((1,1,1)) = (1,1,1)$. Thus $(1,1,1) \to (1,1,1)$ is a cycle of $\tau_{\mathbf{r}}$. Therefore $\mathbf{r} \notin \mathcal{D}_3^{(0)}$. Since $\mathbf{r}$ was an arbitrary element of $E_3^{(-1)}$ the lemma is proved. $\square$

Thus in all what follows we may restrict ourselves to parameters $\mathbf{r} \in E_3^{(\mathbb{C})}$, *i.e.*, $\mathbf{r} = (t, st+1, s+t)$ with $t \in [-1,1]$ and $s \in [-2,2]$. The following lemma contains the case $t \le 0$, which is a special case of a known result.

**Lemma 2.4.** *If $t \le 0$ then $(t, st+1, s+t) \notin \mathcal{D}_3^{(0)}$.*

*Proof.* This is a special case of [4, Theorem 2.1]. $\square$

*Remark* 2.5. We note that the parameters treated in Lemma 2.4 contain all parameters corresponding to Salem numbers of degree four.

Moreover, observe that the parameters corresponding to $t = 1$, $s = 2$ or $s = -2$ are related to polynomials having 1 or $-1$ as a root. Thus all these parameters are contained in $E_3^{(1)} \cup E_3^{(-1)}$ and are therefore treated in Lemmas 2.2 and 2.3. We can therefore restrict ourselves to $\mathbf{r} = (t, st+1, s+t)$ with $t \in (0,1)$ and $s \in (-2,2)$.

Thus only the parameters $\mathbf{r}$ specified in Lemma 2.6 remain to be examined. We split the remaining parameters into seven parts.

**Lemma 2.6.** *Set*

$$\begin{aligned}
\mathcal{C}_0 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; 0 < t \le 1, \; -2 < s < -\frac{1+2t}{1+t} \; \text{or} \; s \in \{0,1\}\right\}, \\
\mathcal{C}_1 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; 0 < t \le \frac{1}{2}, \; -\frac{1+2t}{1+t} \le s < 0\right\}, \\
\mathcal{C}_2 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; 0 < t \le \frac{1}{2}, \; 0 < s < 1\right\}, \\
\mathcal{C}_3 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; 0 < t \le \frac{4}{5}, \; 1 < s < 2\right\}, \\
\mathcal{C}_4 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; \frac{1}{2} < t < 1, \; -\frac{1+2t}{1+t} \le s < 1, s \neq 0\right\}, \\
\mathcal{C}_5 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; \frac{4}{5} < t < 1, \; 1 < s < 2-t\right\}, \\
\mathcal{C}_6 &:= \left\{(t, st+1, s+t) \in E_3^{(\mathbb{C})} \; : \; \frac{4}{5} < t < 1, \; 2-t \le s < 2\right\}.
\end{aligned}$$

*Then* $\mathcal{D}_3^{(0)} \cap (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_6) = \emptyset$ *implies Theorem 1.2.*

*Proof.* This is an immediate consequence of Lemmas 2.2, 2.3 and 3.1. □

The subdivision of the rectangle $\{(s,t) \; : \; s \in (-2,2), \, t \in (-1,1)\}$ is illustrated in Figure 2.

Thus it remains to deal with the parameters $\mathbf{r} \in \mathcal{C}_0 \cup \ldots \cup \mathcal{C}_6$ to finish the proof of our main result. For the parameters contained in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_5$ we show that the preimages of the origin under the iterates of $\tau_{\mathbf{r}}$ cannot cover the whole lattice $\mathbb{Z}^3$. To treat the elements of $\mathcal{C}_0 \cup \mathcal{C}_4 \cup \mathcal{C}_6$ we exhibit several periodic elements of $\mathbb{Z}^3$. We proceed that way because for $\mathbf{r} \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_5$ the structure of the orbits is hard to survey and we were not able to treat these parameters by constructing explicit nontrivial orbits.

## 3. Considering concrete orbits of $\tau_{\mathbf{r}}$

In this section we deal with the elements of $\mathcal{C}_0$, $\mathcal{C}_4$ and $\mathcal{C}_6$. We can show that these elements are not contained in $\mathcal{D}_3^{(0)}$ by exhibiting concrete nontrivial orbits. This is the easier part of the proof.

First we establish the following lemma.

**Lemma 3.1.** *Let* $\mathcal{C}_0$ *be given as in Lemma 2.6. Then*

$$\mathcal{C}_0 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* We shall check the three following items.

(i) If

$$0 < t < 1 \text{ and } -2 \le s < -\frac{1+2t}{1+t} \tag{3.1}$$

then $(1,1,1)$ is a fixed point of $\tau_{(t,st+1,s+t)}$.

Indeed, in view of (1.3) this fixed point exists if and only if the inequalities $0 \le t+(st+1)+s+t+1 < 1$ are satisfied. Observing that $t+(st+1)+s+t+1 = (s+2)t+s+2$ this follows immediately from (3.1).

(ii) For $s = 0$ we see that $(1,0,-1)$ is a periodic element (of period 4), indeed we get

$$(1,0,-1) \to (0,-1,0) \to (-1,0,1) \to (0,1,0) \to (1,0,-1).$$

The existence of this cycle is equivalent to the set of inequalities (note that $s = 0$)

$$\begin{gathered}
0 \le t - t < 1, \\
0 \le -1 + 1 < 1, \\
0 \le -t + t < 1, \\
0 \le 1 - 1 < 1
\end{gathered}$$

which are obviously satisfied.

(iii) For $s=1$ we check that $(0,1,-1)$ is a periodic element (of period 3), indeed we get

$$(0,1,-1)\to(1,-1,0)\to(-1,0,1)\to(0,1,-1).$$

The existence of this cycle is equivalent to the set of inequalities (note that $s=1$)

$$\begin{gathered}0\le(t+1)-(t+1)<1,\\ 0\le t-(t+1)+1<1,\\ 0\le -t+(t+1)-1<1\end{gathered}$$

which are obviously satisfied.

Putting (i), (ii), and (iii) together yields the result. □

The following lemma settles the parameters contained in $\mathcal{C}_4$.

**Lemma 3.2.** *Let $\mathcal{C}_4$ be given as in Lemma 2.6. Then*

$$\mathcal{C}_4\cap\mathcal{D}_3^{(0)}=\emptyset.$$

*Proof.* Let $\mathbf{r}=(t,st+1,s+t)$ be an element of $\mathcal{C}_4$. We have to split the parameter region $\frac12<t<1$, $-\frac{1+2t}{1+t}\le s<1$ into several parts. For each of these parts we exhibit a nontrivial orbit. The candidates for these orbits were found by extensive computer calculations. Moreover, verifying that a given cycle occurs for a particular range of parameters can easily be done using a short Mathematica program. However, as we shall exemplify for the first case, the calculations are not too hard and can also be done without the aid of a computer.

**Case 1, $-\frac{1+2t}{1+t}\le s<-\frac{2+2t}{1+2t}$:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$\begin{gathered}(-1,0,1)\to(0,1,2)\to(1,2,2)\to(2,2,1)\to(2,1,0)\to\\ (1,0,-1)\to(0,-1,-1)\to(-1,-1,0)\to(-1,0,1).\end{gathered}$$

To verify this, in view of (1.3) we need to check that the inequalities

$$\begin{gathered}0\le s+2<1,\\ 0\le st+2s+2t+3<1,\\ 0\le 2st+2s+3t+3<1,\\ 0\le 2st+s+3t+2<1,\\ 0\le 2t+st<1,\\ 0\le -s-1<1,\\ 0\le -st-s-t-1<1,\\ 0\le -st-t<1\end{gathered}$$

are satisfied for all $s,t$ satisfying $-\frac32<s<-\frac{2+2t}{1+2t}$ and $\frac12<t<1$. First note that for $\frac12<t<1$ we have $-\frac{2+2t}{1+2t}\in\left(-\frac32,-\frac43\right)$. Thus we have $s\in(-\frac32,-\frac43)$. This implies that the inequalities in the first line are satisfied. The lower bound in the second line follows from

$$st+2s+2t+3=(s+2)t+2s+3>\left(-\frac32+2\right)\cdot\frac12+2\left(-\frac32\right)+3=\frac14\ge 0;$$

the upper bound of the second line is true because of

$$st+2s+2t+3=(s+2)t+2s+3<\left(-\frac43+2\right)\cdot 1+2\left(-\frac43\right)+3=1.$$

The other inequalities can be checked in a similar way.

**Case 2, $-\frac{2+2t}{1+2t}\le s<-\frac{t}{1+t}$:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$\begin{gathered}(-1,0,1)\to(0,1,2)\to(1,2,1)\to(2,1,0)\to\\ (1,0,-1)\to(0,-1,-1)\to(-1,-1,0)\to(-1,0,1).\end{gathered}$$

**Case 3,** $-\frac{t}{1+t} \le s < -\frac{t}{1+2t}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(1,-1,-1) \to (-1,-1,1) \to (-1,1,2) \to (1,2,0) \to (2,0,-2) \to \\ (0,-2,0) \to (-2,0,2) \to (0,2,1) \to (2,1,-1) \to (1,-1,1).$$

**Case 4,** $-\frac{t}{1+2t} \le s < -\frac{t}{2+2t}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(1,2,0) \to (2,0,-2) \to (0,-2,0) \to (-2,0,2) \to (0,2,1) \to (2,1,-2) \to (1,-2,-1) \to \\ (-2,-1,2) \to (-1,2,2) \to (2,2,-1) \to (2,-1,-2) \to (-1,-2,1) \to (-2,1,2) \to (1,2,0).$$

**Case 5,** $-\frac{t}{2+2t} \le s < 0$**:** This case has to be split up further.

**Case 5.1,** $t < \frac{\sqrt{5}-1}{2}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1-1,1) \to (-1,1,2) \to (1,2,-1) \to (2,-1,-1) \to (-1,-1,1).$$

**Case 5.2,** $t \ge \frac{\sqrt{5}-1}{2}$**:** We need even more subcases here.

**Case 5.2.1,** $s > \frac{-1+t}{1+t}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1-1,1) \to (-1,1,2) \to (1,2,-1) \to (2,-1,-1) \to (-1,-1,1).$$

**Case 5.2.2,** $s \le \frac{-1+t}{1+t}$**:** Note that this case only occurs for $t \ge \frac{2}{3}$ because otherwise $\frac{-1+t}{1+t} < -\frac{t}{2+2t}$ and the range for $s$ is empty.

**Case 5.2.2.1,** $\frac{2}{3} \le t \le \frac{1}{4}(\sqrt{17}-1)$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1,0,1) \to (0,1,1) \to (1,1,-1) \to (1,-1,-1) \to (-1,-1,1) \to (-1,1,2) \to (1,2,-1) \to \\ (2,-1,-1) \to (-1,-1,0) \to (-1,0,2) \to (0,2,0) \to (2,0,-1) \to (0,-1,0) \to (-1,0,1).$$

**Case 5.2.2.2,** $\frac{1}{4}(\sqrt{17}-1) < t < 1$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(1,-1,-1) \to (-1,-1,1) \to (-1,1,2) \to (1,2,-1) \to (2,-1,-1) \to (-1,-1,0) \to (-1,0,2) \to \\ (0,2,0) \to (2,0,-1) \to (0,-1,-1) \to (-1,-1,2) \to (-1,2,1) \to (2,1,-1) \to (1,-1,-1).$$

**Case 6,** $0 < s < \frac{1}{1+t}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1,0,1) \to (0,1,0) \to (1,0,-1) \to (0,-1,1) \to \\ (-1,1,1) \to (1,1,-1) \to (1,-1,0) \to (-1,0,1).$$

**Case 7,** $\frac{1}{1+t} < s < 1$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1,0,1) \to (0,1,0) \to (1,0,-1) \to (0,-1,1) \to (-1,1,1) \to (1,1,-2) \to \\ (1,-2,1) \to (-2,1,1) \to (1,1,-1) \to (1,-1,0) \to (-1,0,1).$$

□

The following lemma treats $\mathcal{C}_6$.

**Lemma 3.3.** *Let* $\mathcal{C}_6$ *be given as in Lemma 2.6. Then*

$$\mathcal{C}_6 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* Let $\mathbf{r} = (t, st+1, s+t)$ be an element of $\mathcal{C}_6$. Again we have to split the parameter region $\frac{4}{5} < t < 2-t$, $s < 2$ into several parts. For each of these parts we exhibit a nontrivial orbit.

**Case 1,** $2-t \le s < -\frac{3}{2}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(2,-2,1) \to (-2,1,1) \to (1,1,-2) \to (1,-2,2) \to (-2,2,0) \to (2,0,-2) \to \\ (0,-2,3) \to (-2,3,-2) \to (3,-2,0) \to (-2,0,2) \to (0,2,-2) \to (2,-2,1).$$

**Case 2,** $s = -\frac{3}{2}$**:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1,0,1) \to (0,1,-1) \to (1,-1,1) \to (-1,1,0) \to (1,0,-1) \to (0,-1,2) \to (-1,2,-2) \to \\ (2,-2,-1) \to (-2,1,1) \to (1,1,-2) \to (1,-2,2) \to (-2,2,-1) \to (2,-1,0) \to (-1,0,1).$$

**Case 3, $-\frac{3}{2} < s < 2$:** If $(s,t)$ is in this range, $\tau_{(t,st+1,s+t)}$ admits the periodic orbit

$$(-1,0,1) \to (0,1,-1) \to (1,-1,1) \to (-1,1,0) \to (1,0,-1) \to$$
$$(0,-1,2) \to (-1,2,-2) \to (2,-2,2) \to (-2,2,-1) \to (2,-1,0) \to (-1,0,1).$$

$\square$

## 4. Calculating preimages of $\tau_{\mathbf{r}}$

The present section is devoted to the treatment of the sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_5$. Indeed, we prove that the parameters contained in these sets cannot belong to $\mathcal{D}_3^{(0)}$.

We first deal with the preimage of $\tau_{\mathbf{r}}$ for certain $\mathbf{r} \in \mathcal{D}_3$. We describe these preimages in a form that is adapted to later use.

In order to simplify the notation we will sometimes write $\tau_{\mathbf{r}}^{-1} S$ instead of $\tau_{\mathbf{r}}^{-1}(S)$.

**Lemma 4.1.** *Let $s,t \in \mathbb{R}$ and $t > 0$. Then the preimage of $(x_1,x_2,x_3) \in \mathbb{Z}^3$ under $\tau_{(t,st+1,s+t)}$ is given by*

$$\left\{ (i,x_1,x_2) \in \mathbb{Z}^3 \;:\; \frac{-x_1 - sx_2 - x_3}{t} \le i + sx_1 + x_2 < \frac{-x_1 - sx_2 - x_3 + 1}{t} \right\}.$$

*Proof.* Using (1.3) we find

$$\left\{ (i,x_1,x_2) \in \mathbb{Z}^3 \;:\; \frac{-(st+1)x_1 - (s+t)x_2 - x_3}{t} \le i < \frac{-(st+1)x_1 - (s+t)x_2 - x_3 + 1}{t} \right\}$$

which immediately implies the result. $\square$

In particular we are interested in the preimages mentioned in the following corollary.

**Corollary 4.2.** (i) $\tau_{(t,st+1,s+t)}^{-1}(\mathbf{0}) = \left\{ (i,0,0) \in \mathbb{Z}^3 \;:\; 0 \le i < \frac{1}{t} \right\}$.

(ii) *For $i \in \mathbb{Z}$ we have*

$$\tau_{(t,st+1,s+t)}^{-1}(i,0,0) = \left\{ (j,i,0) \in \mathbb{Z}^3 \;:\; -\frac{i}{t} \le j + si < \frac{-i+1}{t} \right\}. \tag{4.1}$$

(iii) *We have the particular preimages*

$$\tau_{(t,st+1,s+t)}^{-1}(-2,1,0) = \left\{ (\ell,-2,1) \in \mathbb{Z}^3 \;:\; \frac{2-s}{t} \le \ell - 2s + 1 < \frac{3-s}{t} \right\}, \tag{4.2}$$

$$\tau_{(t,st+1,s+t)}^{-1}(-1,1,-1) = \left\{ (\ell,-1,1) \in \mathbb{Z}^3 \;:\; \frac{2-s}{t} \le \ell - s + 1 < \frac{3-s}{t} \right\}, \tag{4.3}$$

$$\tau_{(t,st+1,s+t)}^{-1}(-1,1,0) = \left\{ (\ell,-1,1) \in \mathbb{Z}^3 \;:\; \frac{1-s}{t} \le \ell - s + 1 < \frac{2-s}{t} \right\}, \tag{4.4}$$

$$\tau_{(t,st+1,s+t)}^{-1}(-1,1,1) = \left\{ (\ell,-1,1) \in \mathbb{Z}^3 \;:\; \frac{-s}{t} \le \ell - s + 1 < \frac{1-s}{t} \right\}, \tag{4.5}$$

$$\tau_{(t,st+1,s+t)}^{-1}(0,0,1) = \left\{ (\ell,0,0) \in \mathbb{Z}^3 \;:\; -\frac{1}{t} \le \ell < 0 \right\}, \tag{4.6}$$

$$\tau_{(t,st+1,s+t)}^{-1}(0,1,0) = \left\{ (\ell,0,1) \in \mathbb{Z}^3 \;:\; \frac{-s}{t} \le \ell + 1 < \frac{1-s}{t} \right\}, \tag{4.7}$$

$$\tau_{(t,st+1,s+t)}^{-1}(0,1,1) = \left\{ (\ell,0,1) \in \mathbb{Z}^3 \;:\; \frac{-1-s}{t} \le \ell + 1 < \frac{-s}{t} \right\}, \tag{4.8}$$

$$\tau_{(t,st+1,s+t)}^{-1}(0,2,1) = \left\{ (\ell,0,2) \in \mathbb{Z}^3 \;:\; \frac{-1-2s}{t} \le \ell + 2 < \frac{-2s}{t} \right\}, \tag{4.9}$$

$$\tau_{(t,st+1,s+t)}^{-1}(1,-1,1) = \left\{ (\ell,1,-1) \in \mathbb{Z}^3 \;:\; \frac{s-2}{t} \le \ell + s - 1 < \frac{s-1}{t} \right\}, \tag{4.10}$$

$$\tau_{(t,st+1,s+t)}^{-1}(1,1,0) = \left\{ (\ell,1,1) \in \mathbb{Z}^3 \;:\; \frac{-1-s}{t} \le \ell + s + 1 < \frac{-s}{t} \right\}, \tag{4.11}$$

$$\tau^{-1}_{(t,st+1,s+t)}(1,1,1) = \left\{ (\ell,1,1) \in \mathbb{Z}^3 \;:\; \frac{-2-s}{t} \le \ell + s + 1 < \frac{-1-s}{t} \right\}, \tag{4.12}$$

*and*

$$\tau^{-1}_{(t,st+1,s+t)}(2,1,1) = \left\{ (\ell,2,1) \in \mathbb{Z}^3 \;:\; \frac{-3-s}{t} \le \ell + 2s + 1 < \frac{-2-s}{t} \right\}. \tag{4.13}$$

*Proof.* Each assertion follows immediately from Lemma 4.1 by inserting the appropriate values. □

Now we describe the action of $\tau_{\mathbf{r}}$ for $\mathbf{r} \in \mathcal{D}_3^{(0)} \cap \partial\mathcal{D}_3$ in a form which makes it accessible to further calculation. Of course, we pursue the goal to show that such a vector $\mathbf{r}$ cannot exist.

**Lemma 4.3.** *Let $\mathbf{r} = (t, st+1, s+t) \in E_3^{(\mathbb{C})}$ with $s \in (-2,2)$ be given. Then for all $\mathbf{x} \in \mathbb{Z}^3$ there is some $\varepsilon \in [0,1)$ such that*

$$\tau_{\mathbf{r}}(\mathbf{x}) = V\Lambda V^{-1}\mathbf{x} + (0,0,\varepsilon) \tag{4.14}$$

*where*

$$\Lambda = \begin{pmatrix} -t & 0 & 0 \\ 0 & \Re(\lambda) & \Im(\lambda) \\ 0 & -\Im(\lambda) & \Re(\lambda) \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 1 & 0 \\ -t & \Re(\lambda) & \Im(\lambda) \\ t^2 & \Re(\lambda^2) & \Im(\lambda^2) \end{pmatrix} \tag{4.15}$$

*and $\lambda$ is a root of the polynomial $X^2 + sX + 1$ with $\Im(\lambda) \neq 0$. The first row of the matrix $V^{-1}$ is given by*

$$\frac{1}{1+t^2+2t\Re(\lambda)} \, (1, -2\Re(\lambda), 1).$$

*Proof.* Clearly, $-t, \lambda, \overline{\lambda}$ are the eigenvalues of the $R(\mathbf{r})$, and we have

$$R(\mathbf{r}) = V\Lambda V^{-1}.$$

Now (1.2) yields (4.14). The assertion on the inverse of $V$ can easily be checked by direct calculation. □

A crucial role is played by the set $L = L_{\mathbf{r}}$ which we introduce in the following lemma.

**Lemma 4.4.** *Let $\mathbf{r} = (t, st+1, s+t) \in E_3^{(\mathbb{C})}$ with*

$$0 < t < 1, \qquad -2 < s < 2, \qquad s \neq 0,$$

*$V \in \mathbb{R}^{3\times 3}$ the matrix defined by (4.15) and*

$$L = \left\{ \mathbf{x} \in \mathbb{Z}^3 \;:\; \left|(V^{-1}\mathbf{x})_1\right| > \frac{1}{(1+t)^2(1-t)} \right\}.$$

(i) *Set*

$$f(s,t) := \frac{1-st+t^2}{(1+t)^2(1-t)}. \tag{4.16}$$

*Then for $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{Z}^3$ we have*

$$\mathbf{x} \in L \iff |x_1 + sx_2 + x_3| > f(s,t).$$

(ii) *For all $n \in \mathbb{N}$ we have*

$$\tau_{\mathbf{r}}^{-n}(L) \subseteq L.$$

*Proof.* (i) By Lemma 4.3 we have

$$(V^{-1}\mathbf{x})_1 = \frac{1}{1-st+t^2} \begin{pmatrix} 1 \\ s \\ 1 \end{pmatrix} \cdot \mathbf{x},$$

hence $L$ is bounded by the two planes

$$\pm\,\mathbf{u} + \sigma_1 \begin{pmatrix} 1 \\ \Re(\lambda) \\ \Re(\lambda^2) \end{pmatrix} + \sigma_2 \begin{pmatrix} 0 \\ \Im(\lambda) \\ \Im(\lambda^2) \end{pmatrix} \quad (\sigma_1,\sigma_2 \in \mathbb{R}) \quad \text{with} \quad \mathbf{u} = \frac{1}{(1+t)^2(1-t)} \begin{pmatrix} 1 \\ -t \\ t^2 \end{pmatrix}.$$

In view of (here "$\times$" denotes the vector product)

$$\begin{pmatrix} 1 \\ \Re(\lambda) \\ \Re(\lambda^2) \end{pmatrix} \times \begin{pmatrix} 0 \\ \Im(\lambda) \\ \Im(\lambda^2) \end{pmatrix} = \begin{pmatrix} \Im(\lambda) \\ -2\Re(\lambda)\Im(\lambda) \\ \Im(\lambda) \end{pmatrix}$$

a normal vector on these planes is given by $\mathbf{n} = (1, s, 1)$, hence, $\mathbf{x}$ belongs to $L$ if and only if

$$|\mathbf{x} \cdot \mathbf{u}| > |\mathbf{n} \cdot \mathbf{u}|.$$

Finally we observe

$$|\mathbf{n} \cdot \mathbf{u}| = \frac{\left|1 - st + t^2\right|}{(1+t)^2(1-t)} = \frac{1 - st + t^2}{(1+t)^2(1-t)}.$$

(ii) Using induction it suffices to prove the statement for $n = 1$. Let $\mathbf{y} \in \tau_{\mathbf{r}}^{-1}(L)$, hence, $\mathbf{y} \in \mathbb{Z}^3$ and there is some $\mathbf{x} \in L$ with $\tau_{\mathbf{r}}(\mathbf{y}) = \mathbf{x}$. Using (i) and (4.14) we find

$$\frac{1 - st + t^2}{(1+t)^2(1-t)} < \left|(V^{-1}\mathbf{x})_1\right| = \left|(\Lambda V^{-1}\mathbf{y})_1\right| = \left|-t(V^{-1}\mathbf{y})_1\right| = t\left|(V^{-1}\mathbf{y})_1\right| < \left|(V^{-1}\mathbf{y})_1\right|$$

which implies our assertion. □

Using Lemma 4.4 we will show now that $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_5$ do not contain elements of $\mathcal{D}_3^{(0)}$. In particular, let us assume that there is some $\mathbf{r} \in \mathcal{D}_3^{(0)} \cap E_3^{(\mathbb{C})}$. We intend to show that in this case the set

$$B = B_{\mathbf{r}} = \Big(\bigcup_{n \geq 1} \tau_{\mathbf{r}}^{-n}(\mathbf{0})\Big) \setminus L$$

is bounded which is a contradiction because for $\mathbf{r} \in \mathcal{D}_3^{(0)}$ we would find

$$\bigcup_{n \geq 1} \tau_{\mathbf{r}}^{-n}(\mathbf{0}) = \mathbb{Z}^3,$$

hence, $B = \mathbb{Z}^3 \setminus L$ which is unbounded. As $\tau_{\mathbf{r}}^{-n}(\mathbf{0})$ is bounded for each fixed $n$, in view of Lemma 4.4 (ii) it is sufficient to find some positive integer $n$ and a finite set $T = T_{\mathbf{r}} \subset \mathbb{Z}^3 \setminus L$ satisfying

$$\tau_{\mathbf{r}}^{-n}(\mathbf{0}) \setminus L \subseteq T \qquad \text{and} \qquad \tau_{\mathbf{r}}^{-1}(T) \setminus L \subseteq T. \tag{4.17}$$

The set $\mathcal{C}_1$ is settled in the following lemma.

**Lemma 4.5.** *Let $\mathcal{C}_1$ be given as in Lemma 2.6. Then*

$$\mathcal{C}_1 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* Recall that $\mathcal{C}_1$ is the set of all parameters $\mathbf{r} = (t, st+1, s+t)$ satisfying $-\frac{1+2t}{1+t} \leq s < 0$ and $0 < t \leq \frac{1}{2}$. We distinguish two cases.

(i) The case $-1 \leq s < 0$: For this interval we have $f(s,t) < \frac{8}{5}$ where $f$ is defined as in (4.16). By Corollary 4.2 (i) we get

$$\tau_{(t,st+1,s+t)}^{-1}(\mathbf{0}) = \left\{(i,0,0) \in \mathbb{Z}^3 \ : \ 0 \leq i < \frac{1}{t}\right\}.$$

By Lemma 4.4 (i) we get that $(i,0,0) \notin L$ yields $i \in \{0,1\}$. Thus

$$\tau_{(t,st+1,s+t)}^{-1}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0)\}. \tag{4.18}$$

By Corollary 4.2 (ii) we have

$$\tau_{(t,st+1,s+t)}^{-1}(1,0,0) = \left\{(i,1,0) \in \mathbb{Z}^3 \ : \ -\frac{1}{t} \leq i+s < 0\right\}.$$

By Lemma 4.4 (i) the assumption $(i,1,0) \notin L$ implies $|i+s| \leq \frac{8}{5}$. Thus easy calculations yield

$$\tau_{(t,st+1,s+t)}^{-1}(1,0,0) \setminus L \subseteq \{(0,1,0),(-1,1,0)\}. \tag{4.19}$$

Combining (4.18) and (4.19) we get

$$\tau^{-2}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(-1,1,0)\}. \tag{4.20}$$

Thus we have to consider the preimages of the elements $(0,1,0)$ and $(-1,1,0)$. Combining (4.7) and Lemma 4.4 (i) and taking into account that $-1 \le s < 0$ we easily find

$$\tau^{-1}_{(t,st+1,s+t)}(0,1,0) \setminus L \subseteq \left\{(i,0,1) \;:\; -s \le i+1 < -s + \frac{1}{t},\;\; |i+1| < \frac{8}{5}\right\} = \{(0,0,1)\}. \tag{4.21}$$

Moreover, in the same way we obtain

$$\tau^{-1}_{(t,st+1,s+t)}(-1,1,0) \setminus L \subseteq \left\{(i,-1,1) \;:\; \frac{1-s}{t} \le \ell - s + 1 < \frac{2-s}{t},\;\; |\ell - s + 1| < \frac{8}{5}\right\} = \emptyset. \tag{4.22}$$

Combining (4.18), (4.19), (4.20), (4.21), and (4.22) yields

$$\tau^{-3}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(-1,1,0),(0,0,1)\}.$$

Now we have to treat $(0,0,1)$. Arguing as above and using (4.6) yields

$$\tau^{-1}_{(t,st+1,s+t)}(0,0,1) \setminus L \subseteq \{(-1,0,0)\}$$

which gives

$$\tau^{-4}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(-1,1,0),(0,0,1),(-1,0,0)\}. \tag{4.23}$$

It remains to treat the element $(-1,0,0)$. Since by (4.1) we have

$$\tau^{-1}_{(t,st+1,s+t)}(-1,0,0) \setminus L = \emptyset$$

we get

$$\tau^{-5}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(-1,1,0),(0,0,1),(-1,0,0)\}. \tag{4.24}$$

Thus choosing

$$T := \{(0,0,0),(1,0,0),(0,1,0),(-1,1,0),(0,0,1),(-1,0,0)\}$$

equations (4.23) and (4.24) yield that $\tau_{\mathbf{r}}^{-1}(T) \setminus L \subseteq T$. Therefore this case is finished in view of (4.17).

(ii) The case $-\frac{1+2t}{1+t} < s < -1$:
Since for $0 < t \le \frac{1}{2}$ we have $-\frac{4}{3} \le -\frac{1+2t}{1+t}$ we may assume $-\frac{4}{3} \le s < -1$. Easy calculations show that we have $f(s,t) < 2$ in this parameter region. By Corollary 4.2 (i) and Lemma 4.4 (i) we get

$$\tau^{-1}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L = \{(0,0,0),(1,0,0)\}.$$

Now by Corollary 4.2 (ii) we have

$$\tau^{-1}_{(t,st+1,s+t)}(1,0,0) = \left\{(i,1,0) \in \mathbb{Z}^3 \;:\; -\frac{1}{t} \le i + s < 0\right\}.$$

By Lemma 4.4 (i) the assumption $(i,1,0) \notin L$ implies $|i+s| < 2$. Thus, since $-\frac{4}{3} \le s < -1$, easy calculations yield

$$\tau^{-1}_{(t,st+1,s+t)}(1,0,0) \setminus L \subseteq \{(0,1,0),(1,1,0)\}.$$

Thus

$$\tau^{-2}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}. \tag{4.25}$$

The new elements are $(0,1,0)$ and $(1,1,0)$. Firstly, observe that by (4.7)

$$\tau^{-1}_{(t,st+1,s+t)}(0,1,0) \setminus L \subseteq \left\{(i,0,1) \;:\; -\frac{s}{t} - 1 \le i < \frac{1-s}{t} - 1,\;\; |i+1| < 2\right\} = \emptyset. \tag{4.26}$$

Secondly, by (4.11) we have

$$\tau^{-1}_{(t,st+1,s+t)}(1,1,0)\setminus L \subseteq \left\{(i,1,1)\;:\; \frac{-1-s}{t} \le i+s+1 < \frac{-s}{t},\;\; |i+s+1|<2\right\}. \tag{4.27}$$

Observing that the conditions on $i$ in the set defined in (4.27) imply

$$0 < -\frac{1+s}{t} \le i+s+1 < 2$$

which is satisfied for $i \in \{1,2\}$ we arrive at

$$\tau^{-1}_{(t,st+1,s+t)}(1,1,0)\setminus L \subseteq \{(1,1,1),(2,1,1)\}. \tag{4.28}$$

From (4.25), (4.26) and (4.28) we deduce

$$\tau^{-3}_{(t,st+1,s+t)}(\mathbf{0})\setminus L \subseteq \{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(1,1,1),(2,1,1)\}. \tag{4.29}$$

We have to scrutinize $(1,1,1)$ and $(2,1,1)$. Firstly, by (4.12)

$$\begin{aligned}\tau^{-1}_{(t,st+1,s+t)}(1,1,1)\setminus L &\subseteq \left\{(i,1,1)\;:\; \frac{-2-s}{t} \le i+s+1 < \frac{-1-s}{t},\;\; |i+s+1|<2\right\}\\ &\subseteq \{(-1,1,1),(0,1,1),(1,1,1),(2,1,1)\}.\end{aligned} \tag{4.30}$$

Moreover, by (4.13)

$$\begin{aligned}\tau^{-1}_{(t,st+1,s+t)}(2,1,1)\setminus L &\subseteq \left\{(i,2,1)\;:\; \frac{-3-s}{t} \le i+2s+1 < \frac{-2-s}{t},\;\; |i+2s+1|<2\right\}\\ &\subseteq \{(0,2,1)\}.\end{aligned} \tag{4.31}$$

From (4.29), (4.30), and (4.31) we get

$$\begin{aligned}\tau^{-4}_{(t,st+1,s+t)}(\mathbf{0})\setminus L \subseteq\; &\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(1,1,1),(2,1,1),\\ &(-1,1,1),(0,1,1),(0,2,1)\}.\end{aligned} \tag{4.32}$$

In the next step we have to deal with the elements $(-1,1,1)$, $(0,1,1)$, and $(0,2,1)$. Firstly, (4.5) implies

$$\tau^{-1}_{(t,st+1,s+t)}(-1,1,1)\setminus L = \emptyset, \tag{4.33}$$

secondly, by (4.8)

$$\tau^{-1}_{(t,st+1,s+t)}(0,1,1)\setminus L \subseteq \left\{(i,0,1)\;:\; -\frac{s+1}{t} \le i+1 < -\frac{s}{t},\;\; |i+1|<2\right\} \subseteq \{(0,0,1)\} \tag{4.34}$$

and thirdly, (4.9) implies

$$\tau^{-1}_{(t,st+1,s+t)}(0,2,1)\setminus L = \emptyset. \tag{4.35}$$

Combining (4.32), (4.33), (4.34), and (4.35) now yields

$$\begin{aligned}\tau^{-5}_{(t,st+1,s+t)}(\mathbf{0})\setminus L \subseteq\; &\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(1,1,1),(2,1,1),\\ &(-1,1,1),(0,1,1),(0,2,1),(0,0,1)\}.\end{aligned}$$

It remains to deal with $(0,0,1)$. But since by (4.6) we get

$$\tau^{-1}_{(t,st+1,s+t)}(0,0,1)\setminus L = \emptyset,$$

thus this case is also finished because (4.17) is true for the choice

$$\begin{aligned}T = \;&\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(1,1,1),(2,1,1),\\ &(-1,1,1),(0,1,1),(0,2,1),(0,0,1)\}.\end{aligned}$$

□

The proofs of the next two lemmas are completely analogous.

**Lemma 4.6.** *Let $\mathcal{C}_2$ be given as in Lemma 2.6. Then*

$$\mathcal{C}_2 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* Recall that $\mathcal{C}_2$ consists of the elements $\mathbf{r} = (t, st+1, s+t)$ with $0 < s < 1$, $0 < t \le \frac{1}{2}$. One easily checks that for this parameter range we get ($f$ is defined as in (4.16))

$$f(s,t) < 10/9,$$

hence, analogously as in the proof of Lemma 4.5 we get

$$\tau^{-1}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{(0,0,0),(1,0,0)\}.$$

We distinguish two subcases.

(i) The case $s \le 8/9$:

In this case we have

$$\tau^{-1}_{(t,st+1,s+t)}(1,0,0) \setminus L \subseteq \{(-1,1,0)\}$$

because by Corollary 4.2 (ii) we have to study

$$i+s<0, \qquad |i+s| < 10/9,$$

which easily implies $i = -1$. Using (4.4) we further have

$$\tau^{-1}_{(t,st+1,s+t)}(-1,1,0) \setminus L = \emptyset.$$

Therefore this case is settled by (4.17) with $n = 2$ and $T = \{\mathbf{0}, (1,0,0), (-1,1,0)\}$.

(ii) The case $s > 8/9$:

By Corollary 4.2 (ii) we have

$$\tau^{-1}_{(t,st+1,s+t)}(1,0,0) \setminus L \subseteq \{(-1,1,0),(-2,1,0)\}$$

In the next step we consider the elements in $\tau^{-1}_{(t,st+1,s+t)}(-1,1,0) \setminus L$, thus by (4.4) and Lemma 4.4 (i) the integer solutions of

$$|i-s+1| < 10/9, \qquad \frac{1-s}{t} \le i-s+1 .$$

This yields $i > s-1 > -1$, hence $i \ge 0$, and further $i < 2$ because otherwise

$$10/9 > 2-s+1 > 2.$$

Therefore $i \in \{0,1\}$. But $i = 0$ is impossible because this would imply

$$\frac{1-s}{t} \le 1-s.$$

Thus

$$\tau^{-1}_{(t,st+1,s+t)}(-1,1,0) \setminus L \subseteq \{(1,-1,1)\}$$

Similarly, using (4.2), we find

$$\tau^{-1}_{(t,st+1,s+t)}(-2,1,0) \setminus L = \emptyset.$$

Moreover, by (4.10) we have

$$\tau^{-1}_{(t,st+1,s+t)}(1,-1,1) \setminus L \subseteq \{(-1,1,-1)\},$$

and (4.3) yields

$$\tau^{-1}_{(t,st+1,s+t)}(-1,1,-1) \setminus L = \emptyset.$$

Again we are done by (4.17).

□

**Lemma 4.7.** *Let $\mathcal{C}_3$ be given as in Lemma 2.6. Then*

$$\mathcal{C}_3 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* Recall that $\mathcal{C}_3$ consists of the elements $\mathbf{r} = (t, st+1, s+t)$ with $1 < s < 2$, $0 < t \le \frac{4}{5}$. We distinguish two cases.

(i) The case $t \le \sqrt{3}-1$: For this interval we have $f(s,t)<1$ (where $f$ is defined in (4.16)), hence,

$$\tau^{-1}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L = \{\mathbf{0}\}\,. \tag{4.36}$$

(ii) The case $t > \sqrt{3}-1$: Here we have to consider two subcases. For $s \ge 4/3$ we again find $f(s,t)<1$ and (4.36) holds again. Now let $s<4/3$. One easily checks that for this parameter range we get

$$f(s,t) < 13/10\,.$$

By Corollary 4.2 (i) and Lemma 4.4 (i) we get

$$\tau^{-1}_{(t,st+1,s+t)}(\mathbf{0}) \setminus L \subseteq \{\mathbf{0},(1,0,0)\}\ .$$

Then by (4.1) with $i=1$ we find

$$\tau^{-1}_{(t,st+1,s+t)}(1,0,0) \setminus L \subseteq \{(-2,1,0)\}$$

because $-2$ is the only integer solution of

$$-\frac{1}{t} \le j+s < 0\,.$$

Finally,

$$\tau^{-1}_{(t,st+1,s+t)}(-2,1,0) \setminus L = \emptyset$$

because by (4.2) and Lemma 4.4 (i) we have to study the inequalities

$$\frac{2-s}{t} \le 1-2s+1 \qquad \text{and} \qquad |i-2s+1| < \frac{13}{10}\,.$$

These yield

$$i \ge \frac{5(2-s)}{4} + 2s - 1,$$

hence $i \ge 3$, and further

$$\frac{13}{10} > 4-2s$$

implying the absurd inequality

$$\frac{8}{3} > 2s > 4 - \frac{13}{10}.$$

In all cases, choosing $T$ appropriately, we are done by (4.17). □

## 5. Treatment of the region near $(s,t)=(1,1)$

In this section we deal with the region $\mathcal{C}_5$. This requires to study preimages $\tau_{\mathbf{r}}^{-n}(\mathbf{0})$ for large values of $n$.

We define the following elements. For $n \ge 0$ set

$$\begin{array}{lcllcl} d_n^{(1)} &=& (n,-(n-1),n-1), & d_n^{(4)} &=& (-(n+2),n+2,-(n+1)),\\ d_n^{(2)} &=& (-(n+1),n,-(n-1)), & d_n^{(5)} &=& (n+2,-(n+2),n+2),\\ d_n^{(3)} &=& (n+2,-(n+1),n), & d_n^{(6)} &=& (-(n+2),n+2,-(n+2)). \end{array} \tag{5.1}$$

The basis for the treatment of $\mathcal{C}_5$ is contained in the following lemma.

**Lemma 5.1.** *Let $\mathbf{r}=(t,st+1,s+t)\in\mathcal{C}_5$ be given. For $n>0$ and $k\in\{1,\ldots,6\}$ let $d_n^{(k)} = (d_{n1}^{(k)},d_{n2}^{(k)},d_{n3}^{(k)})$ be as in (5.1). Then*

$$\begin{array}{lcll} d_n^{(k)} \notin L & \Longrightarrow & \tau_{\mathbf{r}}^{-1}\{d_n^{(k)}\} = \{d_n^{(k+1)}\} & (k\in\{1,\ldots,5\}),\\ d_n^{(6)} \notin L & \Longrightarrow & \tau_{\mathbf{r}}^{-1}\{d_n^{(6)}\} = \{d_{n+3}^{(1)}\}. & \end{array}$$

*Proof.* Note that Lemma 4.4 implies that $d_n^{(k)} \notin L$ holds if and only if $|d_{n1}^{(k)} + s\, d_{n2}^{(k)} + d_{n3}^{(k)}| \le f(s,t)$.

By Lemma 4.1 the preimage $\tau_{\mathbf{r}}^{-1}\{d_n^{(k)}\}$ is given by the set

$$\left\{(i, d_{n1}^{(k)}, d_{n2}^{(k)}) \;:\; \frac{-(st+1)d_{n1}^{(k)} - (s+t)d_{n2}^{(k)} - d_{n3}^{(k)}}{t} \le i < \frac{-(st+1)d_{n1}^{(k)} - (s+t)d_{n2}^{(k)} - d_{n3}^{(k)} + 1}{t}\right\}.$$

Thus, if $k \in \{1,\dots,5\}$ then $\tau_{\mathbf{r}}^{-1}\{d_n^{(k)}\} = \{d_n^{(k+1)}\}$ holds if and only if the inequalities

$$\begin{aligned} \frac{-(st+1)d_{n1}^{(k)} - (s+t)d_{n2}^{(k)} - d_{n3}^{(k)}}{t} &> d_{n,1}^{(k+1)} - 1, \\ \frac{-(st+1)d_{n1}^{(k)} - (s+t)d_{n2}^{(k)} - d_{n3}^{(k)} + 1}{t} &\le d_{n1}^{(k+1)} + 1 \end{aligned} \tag{5.2}$$

are satisfied. Moreover, $\tau_{\mathbf{r}}^{-1}\{d_n^{(6)}\} = \{d_{n+3}^{(1)}\}$ holds if and only if the inequalities

$$\begin{aligned} \frac{-(st+1)d_{n1}^{(6)} - (s+t)d_{n2}^{(6)} - d_{n3}^{(6)}}{t} &> d_{n+3,1}^{(1)} - 1, \\ \frac{-(st+1)d_{n1}^{(6)} - (s+t)d_{n2}^{(6)} - d_{n3}^{(6)} + 1}{t} &\le d_{n+3,1}^{(1)} + 1 \end{aligned} \tag{5.3}$$

are satisfied. Therefore, the lemma is proved if we can show that for positive $n$ the inequality $|d_{n1}^{(k)} + s\, d_{n2}^{(k)} + d_{n3}^{(k)}| \le f(s,t)$ implies (5.2) for $k \in \{1,\dots,5\}$ and (5.3) for $k = 6$.

The proof is split up into the six possible values for $k$. We start with $k = 1$. Suppose that $n \ge 0$ and

$$|n - s(n-1) + (n-1)| \le f(s,t). \tag{5.4}$$

Then we have to show that

$$\begin{aligned} \frac{-(st+1)n + (s+t)(n-1) - n + 1}{t} &> -n - 2, \\ \frac{-(st+1)n + (s+t)(n-1) - n + 2}{t} &\le -n. \end{aligned} \tag{5.5}$$

As $(s,t)$ are parameter values of the set $\mathcal{C}_5$ we can write $s = 1 + \alpha$ and $t = 1 - \beta$ with $0 < \alpha < \beta < \frac{1}{5}$. Now (5.4) reads

$$n(1-\alpha) + \alpha \le \frac{1 - (\alpha+\beta)(1-\beta)}{4\beta - 4\beta^2 + \beta^3}$$

which, combined with the fact that $n > 0$, is equivalent to

$$0 < n \le \frac{1 - (\alpha+\beta)(1-\beta)}{(1-\alpha)(4\beta - 4\beta^2 + \beta^3)} - \frac{\alpha}{1-\alpha}. \tag{5.6}$$

On the other hand, (5.5) is equivalent to

$$\frac{\beta - \alpha}{\beta(1-\alpha)} \le n < \frac{1 - \alpha - \beta}{(1-\alpha)\beta}. \tag{5.7}$$

As, for $\alpha, \beta$ in the indicated range, we have

$$\frac{\beta - \alpha}{\beta(1-\alpha)} < 1$$

and (as easy calculations show)

$$\frac{1 - (\alpha+\beta)(1-\beta)}{(1-\alpha)(4\beta - 4\beta^2 + \beta^3)} - \frac{\alpha}{1-\alpha} < \frac{1-\alpha-\beta}{(1-\alpha)\beta}$$

the lemma is proved for the case $k = 1$. The cases $k = 2,\dots,6$ can be shown in the same way and we omit the details. $\square$

The region $\mathcal{C}_5$ is settled in the following lemma.

**Lemma 5.2.** *Let $\mathcal{C}_5$ be given as in Lemma 2.6. Then*

$$\mathcal{C}_5 \cap \mathcal{D}_3^{(0)} = \emptyset.$$

*Proof.* Arguing along the lines of the lemmas in the previous section it is easy to see that for each $\mathbf{r} \in \mathcal{C}_5$ we have $\tau_{\mathbf{r}}^{-2}\{\mathbf{0}\} = \tau_{\mathbf{r}}^{-1}\{\mathbf{0}\} \cup \{(-2,1,0)\}$.

Using the notation of Lemma 5.1 we see that $(-2,1,0) = d_1^{(2)}$. Thus, Lemma 5.1 completely describes the sets $\tau_{\mathbf{r}}^{-n}\{(-2,1,0)\}$ as long as they are not contained in $L$. As $|n-s(n-1)+n-1|$ is not uniformly bounded in $n$, the definition of $L$ implies that $d_n^{(1)} \in L$ for $n$ large enough. Therefore, there is some $n_0$ such that $\tau_{\mathbf{r}}^{-n_0+1}\{(-2,1,0)\} \cap L = \emptyset$ and $\tau_{\mathbf{r}}^{-n_0}\{(-2,1,0)\} \subset L$ (Indeed, each of these sets contains exactly one element in view of Lemma 5.1.) Thus, Lemma 4.4 (ii) implies that $\tau_{\mathbf{r}}^{-n}\{(-2,1,0)\} \subset L$ for $n \geq n_0$. From this inclusion we get that

$$B = \bigcup_{n\geq 1} \tau_{\mathbf{r}}^{-n}(\mathbf{0}) \setminus L = \bigcup_{n=1}^{n_0+1} \tau_{\mathbf{r}}^{-n}(\mathbf{0}) \setminus L,$$

which is a bounded set. This proves the lemma by the remarks after the proof of Lemma 4.4. □

## 6. Conclusion

The proof of the main theorem now follows by collecting the results that we achieved in the previous sections.

*Proof of the main theorem.* This follows immediately from Lemmas 2.6, 4.5, 4.6, 4.7, 5.2, 3.2 and 3.3. □

*Remark* 6.1. We expect that a similar result can also be proved for three dimensional $\varepsilon$-SRS in the sense of Surer [14]. With similar methods (but considerably more computational effort) it should be possible to settle Conjecture 1.1 for the case $d = 4$. However, for proving the conjecture in full generality, new ideas are needed.

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Haus-Endt-Strasse 88, D-40593 Düsseldorf, GERMANY
*E-mail address*: `brunoth@web.de`

Chair of Mathematics and Statistics, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
*E-mail address*: `Peter.Kirschenhofer@unileoben.ac.at`

Chair of Mathematics and Statistics, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
*E-mail address*: `Joerg.Thuswaldner@unileoben.ac.at`