ON LINEAR COMBINATIONS OF UNITS WITH BOUNDED COEFFICIENTS

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Abstract. Starting with a paper of Jacobson form the 1960s, many authors became interested in characterizing all algebraic number fields in which each integer is the sum of pairwise distinct units. Although there exist many partial results for number fields of low degree, a full characterization of these number fields is still not available. Narkiewicz and Jarden posed an analogous question for sums of units that are not necessarily distinct.

In this paper we propose a generalization of these problems. In particular, for a given rational integer n we consider the following problem. Characterize all number fields for which every integer is a linear combination

$$a_1\varepsilon_1 + \cdots + a_\ell\varepsilon_\ell$$

of finitely many units ε_i in a way that the coefficients $a_i \in \mathbb{N}$ are bounded by n.

The paper gives several partial results on this problem. In our proofs we exploit the fact that these representations are related to symmetric beta expansions with respect to Pisot bases.

1. Introduction

The representation of algebraic integers as sums of distinct units goes back to Jacobson [11] who observed that the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ have the property that every algebraic integer can be written as a sum of distinct units. He conjectured that these are the only quadratic fields with this property, which was proved some years later by Śliwa [17]. These results were extended to cubic and quartic fields by Belcher [5, 6]. The problem of characterizing all number fields in which every integer is a sum of distinct units is still unsolved. It is listed as Problem 18 in Narkiewicz' [15, 539ff] problem list.

More recently Jarden and Narkiewicz [12] showed that not every algebraic integer in a given number field can be written as the sum of a bounded number of units that are not necessarily distinct. Jarden and Narkiewicz also ask for a characterization of all number fields having the property that every integer can be written as the sum of (not necessarily distinct) units. This problem is known as the *unit sum number problem* for number fields and was considered for quadratic [5, 3], complex cubic [20], and complex quartic fields [9, 21].

In the present paper we want to consider a generalization of both of these problems. To this matter we introduce the following definition.

Definition 1.1. *Let $\mathfrak{o}$ be some order of a number field and $\alpha \in \mathfrak{o}$ some algebraic integer. Assume that α can be written as a linear combination*

$$\alpha = a_1\varepsilon_1 + \cdots + a_\ell\varepsilon_\ell, \tag{1.1}$$

where $\varepsilon_1, \dots, \varepsilon_\ell \in \mathfrak{o}^*$ are distinct units and $a_1 \geq \cdots \geq a_\ell > 0$ are integers. If (in case there exist more than one representations of the form (1.1)) a_1 in (1.1) is chosen as small as possible, we call $\omega(\alpha) = a_1$ the unit sum height *of α. In addition we define $\omega(0) = 0$ and $\omega(\alpha) = \infty$ if α admits no representation as a finite sum of units. Moreover, we define*

$$\omega(\mathfrak{o}) = \max\{\omega(\alpha) \,:\, \alpha \in \mathfrak{o}\}$$

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if the maximum exists. If the maximum does not exist we write

$$\omega(\mathfrak{o}) = \begin{cases} \omega & \text{if } \omega(\alpha) < \infty \text{ for each } \alpha \in \mathfrak{o}, \\ \infty & \text{if there exists } \alpha \in \mathfrak{o} \text{ such that } \omega(\alpha) = \infty. \end{cases}$$

If $\mathfrak{o}$ is the maximal order of a number field k we also write $\omega(k) := \omega(\mathfrak{o})$.

The purpose of the present paper is to investigate the unit sum height. In particular we are interested in upper bounds for $\omega(k)$ for any given field k. By the above-mentioned result of Jacobson we know that $\omega(\mathbb{Q}(\sqrt{2})) = \omega(\mathbb{Q}(\sqrt{5})) = 1$. On the other hand it is easy to see that $\omega(\mathbb{Z}) = \omega$. Moreover, it is clear that every order $\mathfrak{o}$ that is not generated by its units has unit sum height $\omega(\mathfrak{o}) = \infty$. Therefore we are only interested in orders that are generated by their units. We call these orders *unit generated orders*, UG orders for short.

Belcher [6] found a simple sufficient condition for the maximal order $\mathfrak{o}$ of a number field having unit sum height 1 (this condition is *a fortiori* valid for any order). In particular he proved that an UG order $\mathfrak{o}$ has unit sum height 1, provided that 2 can be written as the sum of two distinct units. Note that the converse of this assertion is not true. With the methods introduced in this paper we are able to find fields that do not satisfy "Belcher's test" but have unit sum height 1. Such examples can be found in the family of Shank's simplest cubic fields, *i.e.*, fields $\mathbb{Q}(\alpha)$ where α is a root of the polynomial

$$X^3 - (a-1)X^2 - (a+2)X - 1,$$

with $a \in \mathbb{Z}$. In particular, take $a = 2$. Then, because $a^2 + a + 7 = 13$ is square-free in this case we have $\mathfrak{o} = \mathbb{Z}[\alpha]$ by the considerations in [14, p.53]. Thus Theorem 7.4 and the remark in its proof imply that $\omega(k) = \omega(\mathbb{Z}[\alpha]) = 1$ despite 2 cannot be written as the sum of two distinct units.

In the next section we give some notations and state our results. Section 3 contains results on symmetric beta expansions with respect to Pisot numbers which will be needed later on. In Section 4 we prove a theorem which is strongly related to the theory of beta expansions (Theorem 2.1) that works very well in the case of unit rank 1. In Section 5 we derive some consequences of Theorem 2.1 and apply symmetric beta expansions to our problems. In particular we completely characterize $\omega(\mathfrak{o})$ for all real quadratic and purely cubic fields. In most cases with unit rank greater than one Theorem 2.1 provides only crude bounds for ω. Better bounds are obtained by using Theorem 2.10 which goes beyond the framework of beta expansions. This result is proved in Section 6. In the last section we deal with Shank's simplest cubic fields in order to demonstrate the potential of our method.

2. Notations and Statement of Results

In the present section we give the basic notions and definitions that we need throughout the paper. Moreover, we state our main results.

Let k be a field of degree n and signature (r, s) and let us fix the real embeddings $\sigma_1, \dots, \sigma_r$ and the complex embeddings $\sigma_{r+1} = \bar{\sigma}_{r+s+1}, \dots, \sigma_{r+s} = \bar{\sigma}_{r+2s}$ of k. For $\alpha \in k$ we denote the Galois conjugates of α by $\alpha^{(i)} = \sigma_i \alpha$ and by convention we write $\alpha = \alpha^{(1)}$ and $\alpha' = \alpha^{(2)}$.

We recall that an algebraic integer α is a *Pisot number* if $\alpha > 1$ and all conjugates of α have absolute value less than 1.

Let $\mathfrak{o}$ be some fixed order of k. For our purposes the following notation for a "cylinder" will be used.

$$T(x_1, \dots, x_{r+s}) = \{y \in \mathfrak{o} \ : \ |y^{(i)}| \le x_i\}.$$

With this notation we are in a position to state our first result.

Theorem 2.1. *Let $k \neq \mathbb{Q}$ be a real number field and $\mathfrak{o}$ some order of k. Let $\varepsilon \in \mathfrak{o}^*$ be a Pisot unit of k and put $w_1 = \left\lceil \frac{\varepsilon - 1}{2} \right\rceil$. Then for each $\alpha \in \mathfrak{o}$ there exists $N \in \mathbb{N}$ such that*

$$\alpha\varepsilon^N = \beta + \sum_{i=0}^{n} a_i \varepsilon^i, \tag{2.1}$$

with $|a_i| \le w_1$ *and* β *is contained in the finite set*

$$\mathcal{B}_\varepsilon := T\left(\frac{1}{2}, \frac{w_1}{1-|\varepsilon^{(2)}|}, \cdots, \frac{w_1}{1-|\varepsilon^{(r+s)}|}\right).$$

The elements of $\mathcal{B}_\varepsilon \setminus \{0\}$ *will be called* critical points*.*

Remark 2.2. Note that each real number field k contains a Pisot unit. This is an immediate consequence of the proof of Dirichlet's Unit Theorem, see [13, p. 104–109] (we also refer the reader to the slight variant in [16, I, §7, Exercise 6]).

Remark 2.3. This theorem is strongly related to beta numeration with respect to Pisot numbers (see *e.g.* Frougny and Solomyak [10]). Indeed, it is a slight variation of the fact that beta transformations related to symmetric beta expansions (see Akiyama and Scheicher [1] for a definition and basic properties of these objects) admit ultimately periodic orbits. Indeed, Theorem 2.1 even gives an upper bound for the elements contained in the periods. For a survey on beta numeration see *e.g.* Barat *et al.* [4].

This theorem has a number of interesting consequences. An immediate consequence is the following corollary.

Corollary 2.4. *Let the assumptions and notations of Theorem 2.1 be in force and set*

$$w_2 = \max\{\omega(\beta) \,:\, \beta \in \mathcal{B}_\varepsilon\},$$

where we assume the ordering $0 < 1 < 2 < \cdots < \omega < \infty$*. Then* $\omega(\mathfrak{o}) \le w_1 + w_2$*, with the convention that* $l + \infty = \infty$*. (Note that neither* w_1 *nor* w_2 *can be equal to* ω*.)*

The following result will be derived from Theorem 2.1 at the beginning of Section 5.

Corollary 2.5. *Let the assumptions and notations of Theorem 2.1 be in force. Let* $\mathfrak{o}$ *be some UG order of* k*. Then* $\omega(\mathfrak{o})$ *is finite.*

In the case of real quadratic and real cubic fields having signature $(1,1)$ we are able to give more precise statements. In particular, for quadratic fields we even give an exact formula. Moreover, in the following corollary we recover theorems due to Śliwa [17] and Belcher [6] for the quadratic and respectively for the real cubic case with signature $(1,1)$.

Corollary 2.6. *The following two assertions hold:*

- *There are only finitely many real quadratic fields* k *satisfying* $\omega(k) = 1$*.*
- *There are only finitely many cubic fields* k *with signature* $(1,1)$ *satisfying* $\omega(k) = 1$*.*

Indeed we are able to show more. Corollary 2.6 will be a special case of the following two theorems.

For the case of quadratic fields we get the following complete characterization of $\omega(k)$.

Theorem 2.7. *Let* $k = \mathbb{Q}(\sqrt{d})$ *be a quadratic field with* $d > 1$ *square-free.*

- *If* $d = a^2 \pm 1$ *for some* $a \in \mathbb{N}$ *and* $d \not\equiv 1 \mod 4$*, then we have* $\omega(k) = a$*.*
- *If* $d = a^2 - 4$ *for some* $a \in \mathbb{N}$ *and* $d \equiv 1 \mod 4$*, then we have* $\omega(k) = \frac{a-1}{2}$*.*
- *If* $d = a^2 + 4$ *for some* $a \in \mathbb{N}$ *and* $d \equiv 1 \mod 4$*, then we have* $\omega(k) = \frac{a+1}{2}$*.*

All the other real quadratic fields are not generated by their units.

For the case of cubic number fields with signature $(1,1)$ we do not know a nice characterization (in terms of the discriminant or the regulator) of the fields having a maximal order that is UG. For this reason, we cannot give a complete characterization like in the quadratic case. However, we shall prove the following partial result.

Theorem 2.8. *Let* k *be a cubic field with signature* $(1,1)$ *and let* $\varepsilon > 1$ *be a fundamental unit of* k*. If the maximal order of* k *is UG, then* $\omega(k) = \left\lceil \frac{\varepsilon-1}{2} \right\rceil$*.*

In particular, for each constant $C > 0$ *there exist only finitely many cubic fields with signature* $(1,1)$ *satisfying* $\omega(k) \le C$*.*

On the other hand we know by the work of Tichy and Ziegler [20] which maximal orders of pure cubic fields are UG. Using this characterization we obtain the following result.

Theorem 2.9. *Let $d>1$ be a cube-free integer. Then the maximal order $\mathfrak{o}$ of $k=\mathbb{Q}(\sqrt[3]{d})$ is UG if and only if d is square-free, $d \not\equiv \pm 1 \mod 9$ and $d=a^3\pm 1$ for some integer a or $d=28$. Moreover we have*

- $\omega(k)=\frac{3a^2}{2}$ *if a is even,*
- $\omega(k)=\frac{3a^2+1}{2}$ *if a is odd and $d=a^3+1$ and $d\neq 28$,*
- $\omega(k)=\frac{3a^2-1}{2}$ *if a is odd and $d=a^3-1$,*
- $\omega(k)=3$ *if $d=28$.*

We want to point out that Corollary 2.4 gives sharp bounds only in case of unit rank 1. In case of higher unit rank the bound is in most cases far from the truth. However, if k is real and provides at least one non-trivial $\mathbb{Q}$-automorphism we can significantly improve our result.

Theorem 2.10. *Let k be a real number field with signature (r,s) with at least one non-trivial $\mathbb{Q}$-automorphism σ and let $\mathfrak{o}$ be some order of k. Let $\varepsilon=\varepsilon^{(1)}\in\mathfrak{o}^*$ be a Pisot unit with conjugates $\varepsilon^{(1)},\dots,\varepsilon^{(n)}$ and let us assume that $\varepsilon^{(2)}=\varepsilon'=\sigma\varepsilon$ and $\varepsilon^{(r)}=\sigma^{-1}\varepsilon$. Define the set*

$$\mathcal{B}_{\varepsilon,\sigma}=T\left(\frac{1}{2},\frac{w}{1-|\varepsilon^{(2)}|},\dots,\frac{w}{1-|\varepsilon^{(r-1)}|},w+\frac{(w-1+|\varepsilon|^B)|)|\varepsilon^{(r)}|}{1-|\varepsilon^{(r)}|},\right.$$
$$\left.\frac{w}{1-|\varepsilon^{(r+1)}|},\dots,\frac{w}{1-|\varepsilon^{(r+s)}|}\right),$$

where

$$B=\left\lfloor-\frac{\log\varepsilon-\log 2-\log w}{\log|\varepsilon'|}\right\rfloor+1 \quad \textit{and} \quad w=\left\lceil\varepsilon\cdot\frac{1-|\varepsilon'|}{2|\varepsilon'|}\right\rceil.$$

Then for each $\alpha\in\mathfrak{o}$ there exists an integer N such that

$$\alpha\varepsilon^N=\beta+\sum_{i=0}^{m}\sum_{j=0}^{n_i}a_{ij}\varepsilon^i(\varepsilon')^j$$

for some positive integers $m,n_1,\dots,n_m$, integer coefficients $|a_{ij}|\leq w$ and $\beta\in\mathcal{B}_{\varepsilon,\sigma}$.

Note that the set $\mathcal{B}_{\varepsilon,\sigma}$ in Theorem 2.10 is finite and effectively computable.

3. Preliminaries on symmetric beta expansions

In the present section we review results on symmetric beta numeration that will be used in subsequent sections. All these results are taken from Akiyama and Scheicher [1].

Let $\beta>1$ be a Pisot number with minimal polynomial $A(x)=x^d-a_1x^{d-1}-\ldots-a_d\in\mathbb{Z}[x]$ and define the real numbers $r_1,\dots,r_{d-1}$ by

$$r_{d-j+1}=\frac{a_j}{\beta}+\frac{a_{j+1}}{\beta^2}+\cdots+\frac{a_d}{\beta^{d-j+1}} \qquad (2\leq j\leq d).$$

In other words, $r_1,\dots,r_{d-1}$ are the coefficients of the polynomial defined by

$$A(x)=(x-\beta)(x^{d-1}+r_{d-1}x^{d-2}+\cdots+r_1)$$

Let D be the closed triangle with vertices $(-\frac{1}{2},0),(\frac{1}{2},1),(\frac{1}{2},-1)$, L_1 the straight line connecting $(-\frac{1}{2},0)$ and $(\frac{1}{2},-1)$, and L_2 the straight line connecting $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},1)$.

Then the following result holds.

Proposition 3.1 (see [1]). *For Pisot numbers of degree two and three we get the following results.*

- *Let β be a quadratic Pisot number. If $r_1\in(-\frac{1}{2}\frac{1}{2}]$ then each $\alpha\in\mathbb{Z}[\beta]$ admits an expansion of the shape*

$$\alpha=\sum_{j=n}^{m}a_j\beta^j$$

with $|a_j|\leq\left\lceil\frac{\beta-1}{2}\right\rceil$ and $n,m\in\mathbb{Z}$.

- *Let β be a cubic Pisot number.*
  If $(r_1, r_2) \in D \setminus (L_1 \cup L_2)$ then each $\alpha \in \mathbb{Z}[\beta]$ admits an expansion of the shape
$$\alpha = \sum_{j=n}^{m} a_j \beta^j$$
  with $|a_j| \leq \left\lceil \frac{\beta-1}{2} \right\rceil$ and $n, m \in \mathbb{Z}$.

Note that in the quadratic case we have $r_1 = -\beta'$ while in the cubic case we get $r_1 = \beta^{(2)}\beta^{(3)}$ and $r_2 = -\beta^{(2)} - \beta^{(3)}$. In the sequel we will use these representation of r_1 and r_2.

4. Proof of Theorem 2.1

In this section we provide the proof of Theorem 2.1. The proof is in the spirit of many proofs occurring in the theory of numeration when proving that certain number systems admit eventually periodic digit expansion. Here, the Pisot unit ε plays the role of the "base" and the coefficients a_i of the linear combination in (2.1) are the "digits". We start with the following lemma.

Lemma 4.1. *Let $x, \varepsilon \in \mathbb{R}$ with $\varepsilon > 1$ and set $w_1 = \left\lceil \frac{\varepsilon-1}{2} \right\rceil$. Then there exist rational integers $n, a_0, \ldots, a_n$ with $|a_i| \leq w_1$ such that*
$$\left| x - \sum_{i=0}^{n} a_i \varepsilon^i \right| \leq \frac{1}{2}.$$

Proof. The proof is easy when we have a greedy digit expansion in mind. For completeness we give a proof by induction. If $|x| \leq \frac{1}{2}$ the proof is obvious. Assume that for each $|x| \leq \frac{\varepsilon^n}{2}$
$$\left| x - \sum_{i=0}^{n-1} a_i \varepsilon^i \right| \leq \frac{1}{2} \tag{4.1}$$
holds with a_i as specified in the lemma. We shall prove the analogous assertion for n replaced by $n+1$. Let $|x| \leq \frac{\varepsilon^{n+1}}{2}$. Then, by the definition of w_1 there exists $a_n \in \mathbb{Z}$ with $|a_n| \leq w_1$ such that
$$|x - a_n \varepsilon^n| \leq \frac{\varepsilon^n}{2}.$$
The result now follows by induction, if we replace x by $x - a_n\varepsilon^n$ in (4.1). □

To prove Theorem 2.1 let $\alpha \in \mathfrak{o}$ be arbitrary. Since ε is a Pisot number, for each $\delta > 0$ there exists $N \in \mathbb{N}$ such that $\left|\alpha^{(i)}(\varepsilon^{(i)})^N\right| < \delta$ holds for $i = 2, \ldots, n$. In other words, apart form $\alpha\varepsilon^N$ itself all conjugates of $\alpha\varepsilon^N$ are small. We now want to approximate $\alpha\varepsilon^N$ by a sum of units. To this matter let us apply Lemma 4.1 to $x = \alpha\varepsilon^N$. This yields n, a_0, …, a_n such that
$$\left| \alpha\varepsilon^N - \sum_{i=0}^{n} a_i \varepsilon^i \right| \leq \frac{1}{2}. \tag{4.2}$$
Write $\beta = \alpha\varepsilon^N - \sum_{i=0}^{n} a_i \varepsilon^i$. Then, taking conjugates, we have
$$|\beta^{(i)}| \leq w_1 \sum_{k=0}^{n} |(\varepsilon^{(i)})^k| + \delta < \frac{w_1}{1 - |\varepsilon^{(i)}|} + \delta \tag{4.3}$$
for $2 \leq i \leq r+s$. Therefore we have proved that $\beta = \alpha\varepsilon^N - \sum_{i=0}^{n} a_i\varepsilon^i$ lies in the set
$$T\left(\frac{1}{2}, \frac{w_1}{1-|\varepsilon^{(2)}|} + \delta, \cdots, \frac{w_1}{1-|\varepsilon^{(r+s)}|} + \delta\right).$$
Let (r, s) be the signature of k and recall that the natural embedding $\mathfrak{o} \to \mathbb{R}^r \times \mathbb{C}^{2s}$, $y \mapsto (y^{(1)}, \ldots, y^{(n)})$ of $\mathfrak{o}$ is a discrete set in $\mathbb{R}^r \times \mathbb{C}^{2s}$. Therefore there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ we have
$$T\left(\frac{1}{2}, \frac{w_1}{1-|\varepsilon^{(2)}|} + \delta, \cdots, \frac{w_1}{1-|\varepsilon^{(r+s)}|} + \delta\right) = T\left(\frac{1}{2}, \frac{w_1}{1-|\varepsilon^{(2)}|}, \cdots, \frac{w_1}{1-|\varepsilon^{(r+s)}|}\right).$$

Choosing δ small enough proves Theorem 2.1.

5. Quadratic and cubic fields

This section is devoted to the consequences of Theorem 2.1. A first consequence is that every UG order of a real number field has finite unit sum height (see Corollary 2.5). Indeed, note that $\mathcal{B}_\varepsilon$ is a finite set and that the unit weights of associated elements are equal. Therefore let us write $w_2 = \max\{\omega(\beta) : \beta \in \mathcal{B}_\varepsilon\}$. But by Theorem 2.1 we know that $\omega(\alpha\varepsilon^N - \beta) \leq w_1$ and therefore $\omega(\alpha) \leq w_1 + w_2$ for all $\alpha \in \mathfrak{o}$, hence Corollary 2.5 is proved.

Now let us draw our attention to quadratic fields and cubic fields with signature $(1, 1)$.

Lemma 5.1. *Let k be a real quadratic field or a real cubic field with signature $(1, 1)$ and let the notations and assumptions of Theorem 2.1 be in force. Then we have $w_1 \leq \omega(k) \leq w_1 + w_2$.*

Proof. The upper bound has been already proved. For the proof of the lower bound we may assume that $w_1 \geq 2$ and, hence, $\varepsilon > 3$ is the fundamental unit. Let us consider sums of the form

$$S = \sum_{j=n+1}^{m} a_j \varepsilon^j$$

with $a_j \in \mathbb{Z}$, $|a_j| \leq w_1 - 1$ for $n + 1 \leq j \leq m$ and $m \geq n + 1$. Let us assume for a moment that $m > n + 1$ and that $a_m \geq 1$. Then we have

$$S \geq \varepsilon^m - \varepsilon^{n+1}(w_1 - 1)\frac{\varepsilon^{m-n-1} - 1}{\varepsilon - 1} > \varepsilon^m - \varepsilon^{n+1}\frac{\varepsilon - 1}{2} \cdot \frac{\varepsilon^{m-n-1} - 1}{\varepsilon - 1} + \varepsilon^{n+1}\frac{\varepsilon^{m-n-1} - 1}{\varepsilon - 1} > \frac{\varepsilon^m}{2} + \frac{\varepsilon^{n+1}}{2} > \varepsilon^{n+1}.$$

This yields ε^{n+1} is the unique minimal element (in absolute values) in the set of all sums of the form S considered above.

Let us assume that every algebraic integer has unit sum height $\leq w_1 - 1$. In particular, also the rational integer

$$N := \left\lfloor (w_1 - 1) \sum_{i=-\infty}^{n} \varepsilon^i \right\rfloor + 1 = \left\lfloor \frac{\varepsilon^{n+1}(w_1 - 1)}{\varepsilon - 1} \right\rfloor + 1$$

has unit sum height $\leq w_1 - 1$. By the construction of N some unit $\geq \varepsilon^{n+1}$ contributes to the representation of N as a sum of units with coefficients $\leq w_1 - 1$ in absolute values. Therefore we conclude by the minimality of ε^{n+1} (in the sense discussed above) that $N - \varepsilon^{n+1} > -N$, *i.e.* $2N > \varepsilon^{n+1}$, since otherwise all possible “reductions“ with units $\geq \varepsilon^{n+1}$ (in absolute values) would lead to an algebraic integer larger than N (in absolute values).

But on the other hand we have

$$2N < \varepsilon^{n+1}\left(\frac{2(w_1 - 1)}{\varepsilon - 1} + \frac{2}{\varepsilon^{n+1}}\right) < \varepsilon^{n+1}$$

for sufficiently large n, by the definition of w_1. □

Remark 5.2. Note that for real quadratic fields or cubic fields with signature $(1, 1)$ the set $\mathcal{B}_\varepsilon$ is rather small and also the elements contained in $\mathcal{B}_\varepsilon$ are “small”. This often guarantees that w_2 stays small and hence w_1 is a good guess for the unit sum height (cf. Theorem 2.7 or 2.8).

5.1. **Quadratic fields.**

In this subsection we intend to prove Theorem 2.7. In particular we have to distinguish the four cases $d \in \{a^2 \pm 1, a^2 \pm 4\}$. Note that these are the only d such that $k = \mathbb{Q}(\sqrt{d})$ has maximal order that is UG (see *e.g.* [3]). Moreover, this maximal order is equal to $\mathbb{Z}[\varepsilon]$ where ε is the fundamental unit satisfying $\varepsilon > 1$.

In the case of $d = a^2 + 1$ we know that $\varepsilon = a + \sqrt{a^2 + 1} > 1$ is the fundamental unit and moreover we know $\varepsilon' = -\varepsilon^{-1} = a - \sqrt{a^2 + 1}$. Next we note that $2a < \varepsilon < 2a + 1$ which yields

$w_1 = a$. Thus, with r_1 as in Proposition 3.1 we have $|r_1| = |\varepsilon'| < \frac{1}{2}$. Therefore, applying this proposition in the quadratic case yields that each $\alpha \in \mathbb{Z}[\varepsilon]$ has a representation of the form

$$\alpha = \sum_{i=n}^{m} a_i \varepsilon^i$$

with $|a_i| \leq w_1$ and $m, n \in \mathbb{Z}$. Thus, using Lemma 4.1 we get $\omega(k) = w_1 = a$ in this case

For $d = a^2 - 1$ and $a \geq 2$ we have $\varepsilon = a + \sqrt{a^2 - 1}$, hence, $2a - 1 < \varepsilon < 2a$ and the result follows analogously.

For $d = a^2 - 4$ and $a > 3$ we have $\varepsilon = \frac{a+\sqrt{a^2-4}}{2}$, hence, $a - 1 < \varepsilon < a$ and the result follows analogously.

For $d = a^2 + 4$ and $a > 1$ we have $\varepsilon = \frac{a+\sqrt{a^2+4}}{2}$, hence, $a < \varepsilon < a + 1$ and the result follows analogously.

Note that for $d = 5$ Proposition 3.1 is not applicable, but already Jacobson [11] treated that case. One could also apply Theorem 2.1 to the case $d = 5$ and would obtain $\mathcal{B}_\varepsilon = \{0, \pm\varepsilon^{-2}\}$. But, ε^{-2} does not contribute to the sum (2.1), hence we have proved Theorem 2.7 also in this exceptional case.

5.2. **Cubic fields.** The finiteness property for general C is easy to see. First, note that for the unique fundamental unit $\varepsilon > 1$ of the field k we have the inequality $1 < \varepsilon < 2C + 1$ (cf. Theorem 2.1). By the following inequality due to Artin we have (see [2] or [6])

$$|D_k| < 4\varepsilon^3 + 24 < 4(2C+1)^3 + 24, \tag{5.1}$$

where D_k denotes the discriminant of k. On the other hand we know that for fixed degree there are only finitely many fields with bounded discriminant. Hence, the first part of Theorem 2.8 is proved.

Since Tichy and Ziegler [20] we know that the maximal order $\mathfrak{o}$ of a cubic field k with signature $(1, 1)$ is UG if and only if $\mathfrak{o} = \mathbb{Z}[\varepsilon]$ and we can apply Proposition 3.1 if $(r_1, r_2) \in D$. Therefore let us estimate the quantities r_1 and r_2:

$$r_1 = \frac{\pm 1}{\varepsilon} \quad \text{and} \quad |r_2| = |\varepsilon' + \bar{\varepsilon}'| \leq \left| \frac{2}{\sqrt{|\varepsilon|}} \right|.$$

In particular, Proposition 3.1 is applicable if

$$\frac{1}{2} - |r_1| = \frac{1}{2} - \frac{1}{|\varepsilon|} > \left| \frac{2}{\sqrt{|\varepsilon|}} \right| = |r_2|,$$

where ε' and $\bar{\varepsilon}'$ are the conjugates of ε. Note that the inequality above is true for

$$\varepsilon > (2 + \sqrt{6})^2 \sim 19.798.$$

So we are left with fields k that have fundamental unit $1 < \varepsilon < 19.8$, *i.e.* these fields satisfy $|D_k| < 31074$ because of Artin's inequality (5.1). Looking up in a table (*e.g.* [18] provides such a list as additional package "nftables") we can find all such cubic fields. In fact 200 fields are left. Next, we want to consider only those fields whose maximal orders are UG. But maximal orders of cubic fields of signature $(1, 1)$ are UG if and only if $\{1, \varepsilon, \varepsilon^2\}$ is an integral basis, *i.e.* if and only if the discriminant of the order $\mathbb{Z}[\varepsilon]$ is the field discriminant (cf. [20]). Computing these discriminants we are left with 170 fields whose maximal orders are generated by units. Computing the set $\mathcal{B}_\varepsilon \setminus \{0\}$ of critical points, we see that in many cases the critical points are of the form $\pm\varepsilon^{-1}, \pm 2\varepsilon^{-1}$ or $\pm\varepsilon^{-2}$, which do not contribute to the sum in (2.1). Therefore we compute for each field sets of the type

$$\mathcal{D}_\varepsilon(n, w) := \left\{ \alpha = \sum_{i=1}^{n} a_i \varepsilon^{-i} \; : \; a_i \in \mathbb{Z}, |a_i| \leq w \right\}.$$

We see that if the critical points are contained in a set of the form $\mathcal{D}_\varepsilon(n, w_1)$ for some integer n and w_1 is the quantity defined in Theorem 2.1, then we have $\omega(k) = w_1$. In our case it is sufficient to put $n=8$ to treat all cases. Indeed $n=4$ is sufficient for most cases, and the exceptions are the (unique) cubic fields with discriminant -31 and -23.

Remark 5.3. Let us consider the cubic field $\mathbb{Q}(\alpha)$ where α is a root of x^3-2x-3. Then we find $r_1 \sim 0.1067$ and $r_2 \sim 0.6288$ but $(r_1,r_2)\notin D$ and hence Proposition 3.1 is not applicable, but Theorem 2.1 still yields the desired result.

5.3. **Pure Cubic fields.** Tichy and Ziegler [20] proved that the only pure cubic fields whose maximal orders are UG are of the form $k=\mathbb{Q}(\sqrt[3]{d})$, where d is square-free, $d\not\equiv \pm 1 \mod 9$ and $d=a^3\pm 1$ for some integer a or $d=28$. Due to Theorem 2.8 we only have to compute $\lceil \frac{\varepsilon-1}{2}\rceil$. We distinguish between the case $d=a^3-1$ and $d=a^3+1$. Furthermore we consider the case $d=28$ separately.

In the case $d=a^3+1$ we use the inequality $a<\sqrt[3]{a^3+1}<a+\frac{1}{3a^2}$ and obtain

$$3a^2<\varepsilon=a^2+a\sqrt[3]{a^3+1}+\sqrt[3]{(a^3+1)^2}<3a^2+\frac{1}{3a^2}+\frac{2}{3a}+\frac{1}{9a^4}<3a^2+1,$$

provided $a>1$. Thus $\omega(k)=w_1=\frac{3a^2}{2}$ if a is even and $\omega(k)=w_1=\frac{3a^2+1}{2}$ if a is odd. In the case of $a=1$ we have $\varepsilon=1+\sqrt[3]{2}+\sqrt[3]{4}\sim 3.84$ and therefore we also obtain in this case $\omega(k)=w_1=1=\frac{3a^2+1}{2}$.

In the case $d=a^3-1$ we use the inequality $a>\sqrt[3]{a^3-1}>a-\frac{2}{3a^2}$ and obtain

$$3a^2>\varepsilon=a^2+a\sqrt[3]{a^3-1}+\sqrt[3]{(a^3-1)^2}>3a^2-\frac{2}{3a}-\frac{4}{3a^2}>3a^2-1$$

provided $a\geq 2$. Thus $\omega(k)=w_1=\frac{3a^2}{2}$ if a is even and $\omega(k)=w_1=\frac{3a^2-1}{2}$ if a is odd.

In the case $d=28$ we obtain $\varepsilon=\frac{5}{3}+\frac{2}{3}\sqrt[3]{28}+\frac{1}{6}\sqrt[3]{28}\sim 5.23$, *i.e.* we have $\omega(k)=3$.

6. Proof of Theorem 2.10

Our first aim is to prove by induction the following lemma which plays the same part in the proof of Theorem 2.10 as Lemma 4.1 did in the proof of Theorem 2.1.

Lemma 6.1. *Let the assumptions and notations of Theorem 2.10 be in force. Let $\alpha\in\mathfrak{o}$, then we can find $\beta\in\mathfrak{o}$ with $|\beta|\leq\frac{1}{2}$ such that*

$$\alpha=\beta+a_0+\sum_{j=1}^{m}\varepsilon^j(a_j+a_j'\varepsilon'^{e_j}), \tag{6.1}$$

where $m,a_0,\dots,a_m,a_1',\dots,a_m',e_1,\dots,e_m$ are integers with $|a_0|\leq w$, $|a_j|\leq w-1$, $|a_j'|\leq 1$ and $0\leq e_j\leq B$ for $j=1,\dots,m$, where

$$B=\left\lfloor-\frac{\log\varepsilon-\log 2-\log w}{\log|\varepsilon'|}\right\rfloor+1.$$

Proof. Let us write

$$S_l:=w\frac{\varepsilon^{l+1}-1}{\varepsilon-1}.$$

Obviously the lemma is true in the case $|\alpha|\leq w=S_0$.

Now, assume that every $|\alpha|\leq S_l$ can be written in the form (6.1). In view of induction on l we want to show that for each $S_l<|\alpha|\leq S_{l+1}$ we can find integers a,a' and E with $|a|\leq w-1$, $|a'|\leq 1$ and $E\geq 0$ such that

$$|\alpha-\varepsilon^{l+1}(a+a'\varepsilon'^E)|<S_l.$$

Without loss of generality we may assume that $\alpha>0$. Moreover, we choose a such that either $|\alpha-\varepsilon^{l+1}a|<S_l$ or $S_l<\alpha-\varepsilon^{l+1}a\leq S_k+\varepsilon^{l+1}$. In the first case we are done (choose $a'=0$ and $E=0$). Therefore we may assume $S_l<\tilde\alpha\leq S_l+\varepsilon^{l+1}$, with $\tilde\alpha=\alpha-\varepsilon^{l+1}a$.

For the next step in our proof we choose $|a'|=1$ and we want to find an integer $E\geq 0$ such that $|\tilde\alpha-a'\varepsilon^{l+1}(\varepsilon')^E|<S_l$ and $0<a'\varepsilon^{l+1}(\varepsilon')^E$, *i.e.* we want to find $E\geq 0$ such that

$$\tilde\alpha-S_l<\varepsilon^{l+1}|\varepsilon'|^E<\tilde\alpha+S_l\leq\varepsilon^{l+1}|\varepsilon'|^{E-1}. \tag{6.2}$$

If the choice $E=0$ satisfies inequality (6.2) we are done. Hence, we assume that the choice $E=0$ is not suitable, *i.e.* we assume that

$$\tilde{\alpha}+S_l<\varepsilon^{l+1}.$$

If we can prove that for some E satisfying $\tilde{\alpha}+S_l\leq\varepsilon^{l+1}|\varepsilon'|^{E-1}$ also $\tilde{\alpha}-S_l<\varepsilon^{l+1}|\varepsilon'|^{E}$ holds, we are done. Indeed, since $\varepsilon^{l+1}|\varepsilon'|^E\to 0$ as $E\to\infty$ we must have for some $E\geq 0$ the desired inequality (6.2).

Therefore we are left to prove

$$\varepsilon^{l+1}|\varepsilon'|^E>|\varepsilon'|(S_l+\tilde{\alpha})=\tilde{\alpha}|\varepsilon'|+|\varepsilon'|\cdot\overbrace{w\frac{\varepsilon^{l+1}-1}{\varepsilon-1}}^{=S_l}\geq\tilde{\alpha}-S_l.$$

Rewriting the right inequality we obtain

$$1\geq\frac{1-|\varepsilon'|}{1+|\varepsilon'|}\cdot\frac{\tilde{\alpha}}{S_l}. \tag{6.3}$$

Let us consider the term $\frac{\tilde{\alpha}}{S_l}$. By our assumption that $\tilde{\alpha}\leq S_l+\varepsilon^{l+1}$ we get

$$\frac{\tilde{\alpha}}{S_l}\leq\frac{S_l+\varepsilon^{l+1}}{S_l}=1+\frac{\varepsilon^{l+1}(\varepsilon-1)}{w(\varepsilon^{l+1}-1)}\leq 1+\frac{\varepsilon}{w}.$$

Therefore (6.5) is satisfied if

$$1\geq\frac{1-|\varepsilon'|}{1+|\varepsilon'|}\left(1+\frac{\varepsilon}{w}\right).$$

Solving this inequality for w yields

$$w\geq\varepsilon\cdot\frac{1-|\varepsilon'|}{2|\varepsilon'|}$$

which is true by the choice of w in Theorem 2.10. Hence there exists an E that satisfies (6.2) and therefore we have proved our claim concerning the representation (6.1).

Since we know that an E exists we finally show that we can choose $E\leq B$. In any case we can choose $E\leq B$ for some B satisfying

$$\varepsilon^{l+1}|\varepsilon'|^B<2S_l<\tilde{\alpha}+S_l.$$

In order to keep B small we choose B such that $\varepsilon^{l+1}|\varepsilon'|^{B-1}>2S_l$. In particular this yields $|\varepsilon'|^{B-1}>\frac{2w}{\varepsilon}$, *i.e.* we obtain $B=\lfloor-\frac{\log\varepsilon-\log 2-\log w}{\log|\varepsilon'|}\rfloor+1$. □

Now, we are able to prove Theorem 2.10. Remember that the order of the conjugates of ε was chosen such that $\sigma\varepsilon=\varepsilon'=\varepsilon^{(2)}$ and $\sigma^{-1}\varepsilon=\varepsilon^{(r)}$. In particular, this choice yields $\varepsilon'^{(r)}=\sigma^{-1}(\sigma\varepsilon)=\varepsilon$. We also assume that w has been chosen according to Theorem 2.10. Given a constant $\delta>0$ we may assume $|\alpha^{(i)}|<\delta$ for $i=2,\dots,n$. This can be achieved if we multiply α by ε^N with N sufficiently large (cf. Section 4). Now we apply Lemma 6.1 to α and find a representation of α of the form (6.1) for some β defined by (6.1). The rest of this section is devoted to the proof that indeed $\beta\in\mathcal{B}_{\varepsilon,\sigma}$, which immediately yields Theorem 2.10.

Note that Lemma 6.1 yields $|\beta|<1/2$ and we are left to consider the conjugates of β. Let us estimate $\beta^{(i)}$ for $2\leq i\leq r+s$, but $i\neq r$. We obtain according to (6.1) and the assumptions on α

$$|\beta^{(i)}|\leq\delta+|a_0|+\sum_{j=1}^{m}|\varepsilon^{(i)}|^j(w-1+|\varepsilon'^{(i)}|^{e_j})\leq\delta+w\sum_{j=0}^{\infty}|\varepsilon^{(i)}|^j=\delta+\frac{w}{1-|\varepsilon^{(i)}|},$$

since $|\varepsilon'^{(i)}|^{e_j}\leq 1$ for $i\neq r$.

In the case of $i = r$ we have by definition $\varepsilon'^{(r)} = \varepsilon$ and we therefore obtain according to representation (6.1)

$$|\beta^{(r)}| \leq \delta + \sum_{j=0}^{N} |\varepsilon^{(r)}|^j (w - 1 + |\varepsilon|^{e_j}) \leq \delta + w + \sum_{j=1}^{\infty} |\varepsilon^{(r)}|^j (w - 1 + |\varepsilon|^B)$$
$$= \delta + w + \frac{(w - 1 + |\varepsilon|^B)|\varepsilon^{(r)}|}{1 - |\varepsilon^{(r)}|}.$$

Because of the discreteness of $\mathfrak{o}$ we deduce as in the proof of Theorem 2.1 that

$$\beta \in T\left(\frac{1}{2}, \frac{w}{1 - |\varepsilon^{(2)}|}, \dots, \frac{w}{1 - |\varepsilon^{(r-1)}|}, w + \frac{(w - 1 + |\varepsilon|^B))|)|\varepsilon^{(r)}|}{1 - |\varepsilon^{(r)}|}, \frac{w}{1 - |\varepsilon^{(r+1)}|}, \dots, \frac{w}{1 - |\varepsilon^{(r+s)}|}\right),$$

hence $\beta \in \mathcal{B}_{\varepsilon,\sigma}$ as claimed.

7. Simplest cubic fields

This section is devoted to the additive properties of the unit structure of Shank's simplest cubic fields in order to demonstrate how Theorems 2.1 and 2.10 work. In view of our purposes these fields provide interesting instances.

First, let us remind some facts about simplest cubic fields (see [19] and for the last statement see [7]):

Lemma 7.1. *Let α be a root of the polynomial $f_a = X^3 - (a-1)X^2 - (a+2)X - 1$. Then we have:*

1. *The polynomials f_a are irreducible for all $a \in \mathbb{Z}$. Moreover all roots of f_a are real.*
2. *The number fields $K = \mathbb{Q}(\alpha)$ are cyclic Galois extensions of degree three of $\mathbb{Q}$ for all $a \in \mathbb{Z}$.*
3. *The roots of f_a are permuted by the map $\alpha \mapsto -1 - \frac{1}{\alpha}$.*
4. *Any two of $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$ form a fundamental system of units of the order $\mathbb{Z}[\alpha]$.*
5. *The asymptotic expansions of the conjugates of α are*
$$\begin{aligned}\alpha^{(1)} &= a + \frac{2}{a} - \frac{1}{a^2} - \frac{3}{a^3} + \frac{5}{a^4} + \frac{10\theta_1}{a^5},\\ \alpha^{(2)} &= -1 - \frac{1}{a} + \frac{2}{a^3} - \frac{1}{a^4} + \frac{8\theta_2}{a^5},\\ \alpha^{(3)} &= -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} + \frac{18\theta_3}{a^5},\end{aligned}$$
valid for all $a \geq 8$, with $|\theta_i| < 1$ for $i = 1, 2, 3$.

The family of cubic fields $\mathbb{Q}(\alpha)$ in Lemma 7.1 is called the family of *Shank's simplest cubic fields.* Note that $X^3 f_a(1/X) = f_{-a-1}(X)$ and therefore the fields $\mathbb{Q}(\alpha)$ and orders $\mathbb{Z}[\alpha]$ are isomorphic for a and $-a-1$. Therefore one may consider only the case $a \geq 0$. We start with computing the quantity w in Theorem 2.10.

Lemma 7.2. *For all $a \in \mathbb{Z}$ we can choose the parameters in Theorem 2.10 such that we obtain $w = 1$.*

Proof. We choose $\varepsilon = |1/\alpha^{(3)}|$ and $\varepsilon' = |1/\alpha^{(2)}|$. Moreover, we note that $\alpha^{(3)} = -\frac{1}{\alpha^{(1)}+1}$ and $\alpha^{(2)} = -1 - \frac{1}{\alpha^{(1)}}$. This yields

$$w = \left\lfloor \varepsilon \cdot \frac{1 - |\varepsilon'|}{2|\varepsilon'|} \right\rfloor + 1 = \left\lfloor -\frac{\alpha^{(2)}}{2} \right\rfloor + 1 = 1,$$

hence we deduce $w = 1$ provided $|\alpha^{(2)}| < 2$, which is true by Lemma 7.2 at least for $a \geq 8$. In the case of $0 \leq a < 8$ we obtain by direct numeric computations that indeed $|\alpha^{(2)}| < 2$ for all $a \geq 0$. □

This immediately leads to the following conjecture.

Conjecture 7.3. *Let α be a root of $X^3-(a-1)X^2-(a+2)X-1$. Then for all $a \in \mathbb{Z}$ we have $\omega(\mathbb{Z}[\alpha])=1$.*

In order to prove this conjecture we are left to compute the finite sets $\mathcal{B}_{\varepsilon,\sigma}$ and consider their elements. Unfortunately these sets lie in a "parallelotop" (we think of $\mathbb{Z}[\alpha]$ to be embedded into the Minkowski space) with volume growing exponential with a. Hence, with our methods we are limited to verify Conjecture 7.3 only for small values of a, *i.e.* $a \leq 4$. First, let us note that for $a=0,1$ Theorem 2.1 is sufficient to verify Conjecture 7.3.

Indeed, let us choose $\varepsilon = |\frac{1}{\alpha^{(3)}}|$. In the case of $a=0$ we obtain

$$\mathcal{B}_\varepsilon = \{0, \pm\alpha^{(3)}, \pm\alpha^{(3)}(\alpha^{(3)}+1), \pm(1+\alpha^{(3)})^2\}.$$

Since $\alpha^{(1)} = -\frac{\alpha^{(3)}+1}{\alpha^{(3)}}$ we get

$$\mathcal{B}_\varepsilon = \{0, \pm\alpha^{(3)}, \pm\alpha^{(1)}(\alpha^{(3)})^2, \pm(\alpha^{(1)})^2(\alpha^{(3)})^2\}$$

and no element of $\mathcal{B}_\varepsilon$ is of the form $\pm(\alpha^{(3)})^j$ with $j \leq 0$ and contributes to the sum in (2.1). In the case of $a=1$ we obtain $\mathcal{B}_\varepsilon = \{0, \pm\alpha^{(3)}\}$ and this case is also proved.

Now let us consider the cases $a=2$, $a=3$ and $a=4$. As in the proof of Lemma 7.2 we choose $\varepsilon = -\frac{1}{\alpha^{(3)}}$ and $\varepsilon' = -\frac{1}{\alpha^{(2)}}$. In the case of $a=2$ we have $|\mathcal{B}_{\varepsilon,\sigma}|=23$, in the case of $a=3$ we have $|\mathcal{B}_{\varepsilon,\sigma}|=187$ and in the case of $a=4$ we have $|\mathcal{B}_{\varepsilon,\sigma}|=1195$. Let us call a unit admissible if it is of the form $\pm(\alpha^{(3)})^j(\alpha^{(2)})^k$ such that not both j and k are negative nor $k=0$ and $j \leq 0$. In the case of $a=2$ we compute all admissible units such that $|j|,|k| \leq 2$ and call this set of units U_2. Now we compute every sum of 3 distinct units $\in U_2$ (these are 3276 sums) and call this set of sums S_2. Comparing the sets $\mathcal{B}_{\varepsilon,\sigma}$ and S_2 we obtain $S_2 \supset \mathcal{B}_{\varepsilon,\sigma}$, hence $\omega(\mathbb{Z}[\alpha])=1$.

The case $a=3$ runs analogously. In this case we compute all admissible units such that $|j|,|k| \leq 3$ and call this set U_3. Now computing every sum of 4 distinct units $\in U_3$ (these are 316251 sums) we obtain the set S_3 and comparing the sets $\mathcal{B}_{\varepsilon,\sigma}$ and S_3 yields $S_3 \supset \mathcal{B}_{\varepsilon,\sigma}$. Again we deduce $\omega(\mathbb{Z}[\alpha])=1$.

In the case of $a=4$ we proceed similar as in the cases $a=2$ and $a=3$. In particular, we compute all admissible units such that $|j| \leq 4$ and $|k| \leq 3$ and denote this set by U_4. Now the set S_4 containing all sums consisting of 5 distinct units $\in U_4$ covers again the set of critical points $\mathcal{B}_{\varepsilon,\sigma} \setminus \{0\}$ (note that S_4 contains 10424128 elements), hence we have verified Conjecture 7.3 in this case too.

Now let us consider the case $a=5$. It is hopeless to try to proof the conjecture in this case by "brute-force" as in the cases $a=2$, $a=3$ and $a=4$. But, we can apply to each element of $\mathcal{B}_{\varepsilon,\sigma}$ the following algorithm:

Let $\beta \in \mathcal{B}_{\varepsilon,\sigma}$ and let U_5 be the set of admissible units with $|j|,|k| \leq 5$. Then we construct the (finite) sequence $\beta_0 = \beta$ and $\beta_n = \beta_{n-1} - u_{n-1}$, where u_{n-1} is chosen such that $a_1^2+a_2^2+a_3^2$ is minimal, where $a_1+a_2\alpha+a_3\alpha^2 = \beta_{n-1}-u_{n-1}$ and $u_{n-1} \in U_5 \setminus \{u_0,\dots,u_{n-2}\}$. If no u_{n-1} exists such that $a_1^2+a_2^2+a_3^2 \leq b_1^2+b_2^2+b_3^2$ with $\beta_{n-1} = b_1+b_2\alpha+b_3\alpha^2$ we terminate the process and return the value β_{n-1}.

Note that $|\mathcal{B}_{\varepsilon,\sigma}|=69265$ and so this reduction algorithm can be applied to each element in a reasonable amount of time (some hours on a common work station). However, applying the reduction algorithm to each element of $\mathcal{B}_{\varepsilon,\sigma}$ we obtain the set $\tilde{B}$ of 71 "reduced" elements. On this set we apply our brute force strategy as in the cases $a=2$, $a=3$ and $a=4$. In particular, we take the set $\tilde{U}_5$ which consists of all units of the form $(\alpha^{(1)})^j(\alpha^{(2)})^k$ with $|j|,|k| \leq 3$ (not necessarily admissible) and let $\tilde{S}_5$ be the set of all distinct sums of 4 units $\in \tilde{U}_5$. Then we see that $\tilde{B} \setminus \tilde{S}_5 = \emptyset$. Therefore we have shown that each element is a sum of units, where each unit occurs at most two times, *i.e.* the unit sum height is at most 2.

The computations described above lead us to the following theorem:

Theorem 7.4. *Let α be a root of $X^3-(a-1)X^2-(a+2)X-1$, then for $a=0,1,2,3,4,6,13,55$ we have $\omega(\mathbb{Z}[\alpha])=1$. For $a=5$ we have $\omega(\mathbb{Z}[\alpha]) \leq 2$.*

Proof. The cases $a = 0, 1, 2, 3, 4, 5$ are due to the computations above. The cases $a = 6, 13, 55$ are due to Belcher's theorem [6], which states that a UG order in which the Diophantine equation $u_1 + u_2 = 2$ has a non-trivial solution has unit sum height 1, and the following computational result:

The unit equation $u_1 + u_2 = 2$ has a non-trivial solution over $\mathbb{Z}[\alpha]$ only for $a = 0, 1, 4, 6, 13, 55$ in the range $0 \leq a \leq 1000$.

This computational result was established by using the computer algebra system MAGMA [8]. $\square$

Remark 7.5. Note that in case of $a = 2$, $a = 3$ Theorem 2.1 would yield $\omega(\mathbb{Z}[\alpha]) \leq 2$ and in case of $a = 4$ or $a = 5$ we would obtain $\omega(\mathbb{Z}[\alpha]) \leq 3$.

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