On a variant of the Kakeya problem in $\mathbb{R}$

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Abstract. We prove that the sharp lower bounds of the Minkowski and Hausdorff dimensions of circular Kakeya sets in $\mathbb{R}$ are $1/2$ and 0 respectively.

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1. Introduction

A circular Kakeya set in $\mathbb{R}^d$ is a closed set $E \subset \mathbb{R}^d$ of measure zero that contains a $(d-1)$-sphere of every radius in a nondegenerate interval of $\mathbb{R}$. Besicovitch and Rado [1] and Kinney [8] have constructed such sets in $\mathbb{R}^2$. The Besicovitch-Rado construction also works in $\mathbb{R}^d$ for $d \geq 3$. See [9] for an exposition of this construction. Similar to the original Kakeya problem on the minimal dimension of a measure zero set containing a unit segment in every direction [4], a problem is to determine the minimal dimension of circular Kakeya sets, called the circular Kakeya problem in this paper. Kolasa and Wolff [9] show that circular Kakeya sets in $\mathbb{R}^d$ with $d \geq 3$ has Hausdorff dimension d while a circular Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension at least $11/6$. The last result has been improved to full dimension by Wolff in [11].

We observe that in contrary to the original Kakeya problem, the circular Kakeya problem make sense for $d = 1$ and apparently has not been answered. Let $\mathbb{S}^d$ be a d-dimensional sphere and $\mathbb{S}_r^d$ denotes a d-dimensional sphere of radius r. If the center m of the sphere is relevant, we write $\mathbb{S}_r^d(m)$. Notice that a sphere $\mathbb{S}_r^0$ is a set of two points of distance $2r$ apart. Hence a circular Kakeya set in $\mathbb{R}$ is a closed set E of measure zero with its distance set $D(E) = \{|x-y| : x, y \in E\}$ containing an interval (or equivalently, its difference set $\Delta(E) = E - E$ containing an interval).

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One can ask whether circular Kakeya sets in $\mathbb{R}$ exists, and if so determine their minimal Minkowski and Hausdorff dimensions. Research on this problem has appeared in the form of difference sets. It is well-known that that middle-third Cantor set C has difference set $\Delta(C) = [-1, 1]$ (see e.g. [7, p.87]). Therefore C is a circular Kakeya set in $\mathbb{R}$ of Hausdorff and Minkowski dimension $\log 2/\log 3$. This implies that the minimal dimension of such sets in $\mathbb{R}$ is less than one, different from the $\mathbb{R}^d$ cases for $d \geq 2$. There are conditions on independent copies of certain random Cantor sets [2, 3, 6] and dynamically generated Cantor sets [10] that results in their difference set containing an interval. However, apparently the minimal dimension of circular Kakeya sets in $\mathbb{R}$ has not been discussed. Our main results are as follows.

Theorem 1.1. *Let E be a circular Kakeya set in $\mathbb{R}$ such that its Minkowski dimension $\dim_B E$ exists. Then $\dim_B E \geq 1/2$. This lower bound is sharp.*

Theorem 1.2. *There is a circular Kakeya set F in $\mathbb{R}$ with Hausdorff dimension $\dim_H F = 0$.*

We prove Theorem 1.1 in Section 2, proving the sharpness of the lower bound by explicitly constructing a circular Kakeya set of Minkowski dimension 1/2. Theorem 1.2 is deduced from a classical result in fractal geometry in Section 3.

2. The Minkowski dimension of circular Kakeya sets

Proof of Theorem 1.1. We first show that $\dim_B E \geq \frac{1}{2}$. Since the surjective $f : E \times E \to D(E)$ given by $f(x, y) := |x - y|$ is Lipschitz, a basic property of Minkowski dimension (see e.g. [5, p.44]) implies that

$$\dim_B E + \dim_B E = \dim_B (E \times E) \geq \dim_B D(E) = 1,$$

giving 1/2 as a lower bound of the Minkowski dimension.

We prove the sharpness of this bound by explicitly constructing a circular Kakeya set in $\mathbb{R}$ with Minkowski dimension 1/2 using the Besicovitch-Rado construction. We will construct a closed set E of measure zero containing an S_r^0 for every $r \in [1/4, 1/2]$.

Let $P = 1/2$. Start with $E_1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, which contains a sphere $\mathbb{S}_r^0(P)$ of every radius $r \in [\frac{1}{4}, \frac{1}{2}]$. Divide it into two bunches of circles, the outer bunch $[0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$ and the inner bunch $[\frac{1}{8}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{7}{8}]$, and shift the later to the left by $\frac{1}{8}$. The resulting set is $E_2 = [0, \frac{1}{8}] \cup [\frac{5}{8}, \frac{3}{4}] \cup [\frac{7}{8}, 1]$, still containing a circle of every radius in $[\frac{1}{4}, \frac{1}{2}]$. In the next step, divide each of the two bunches into the inner and outer halves, and shift each of the the inner halves to the right by $\frac{1}{16}$ and get $E_3 = [0, \frac{1}{16}] \cup [\frac{1}{8}, \frac{3}{16}] \cup [\frac{11}{16}, \frac{3}{4}] \cup [\frac{15}{16}, 1]$. Repeat this process, shifting to the left and right alternatingly, shifting a suitable half of E_n by $2^{-(n+2)}$ to get E_{n+1}, $n = 1, 2, \ldots$. Precisely, E_n consists of intervals of length $2^{-(n+1)}$. For $n = 2k - 1$, $k = 1, 2, \ldots$, E_{2k-1} has 2^{k-1} intervals to the left of P and a same number of intervals to the right, namely

$$2^{-2k}\left[\sum_{i=1}^{k-1}\varepsilon_i 2^{2i-1}, \sum_{i=1}^{k-1}\varepsilon_i 2^{2i-1}+1\right] \quad \text{and} \tag{2.1}$$

$$2^{-2k}\left[2^{2k}-\sum_{i=1}^{k-1}\varepsilon_i 2^{2i}-1, 2^{2k}-\sum_{i=1}^{k-1}\varepsilon_i 2^{2i}\right] \tag{2.2}$$

respectively, where $\varepsilon_i \in \{0,1\}$ for $i = 1,\ldots,k-1$, and the empty sum for $k = 1$ is interpreted as 0. For $n = 2k$, $k = 1,2,\ldots$, E_{2k} has 2^{k-1} intervals to the left of P and 2^k to the right, namely

$$2^{-(2k+1)}\left[\sum_{i=1}^{k-1}\varepsilon_i 2^{2i}, \sum_{i=1}^{k-1}\varepsilon_i 2^{2i}+1\right] \quad \text{and}$$

$$2^{-(2k+1)}\left[2^{2k+1}-\sum_{i=1}^{k}\varepsilon_i 2^{2i-1}-1, 2^{2k+1}-\sum_{i=1}^{k}\varepsilon_i 2^{2i-1}\right]$$

respectively, where the $\varepsilon_i \in \{0,1\}$. This can be proved by induction on n. Notice that $E_{n+2} \subset E_n \cup E_{n+1}$ and each E_n contains an $\mathbb{S}^0_r$ of every radius $r \in [\frac{1}{4}, \frac{1}{2}]$.

Let E be the limit set of $\{E_n\}$, by definition the set consisting of all points p for which there is a sequence $\{p_n\}$ converging to it, $p_n \in E_n$. From a simple limit argument, E contains an $\mathbb{S}^0_r$ for every $r \in [\frac{1}{4}, \frac{1}{2}]$. To see that E has measure 0, construct a sequence of coverings of E as follows. As $E_{n+2} \subset E_n \cup E_{n+1}$, $E \subset E_n \cup E_{n+1}$ for every n and it suffices to construct coverings $\mathcal{C}_n$ for $E_n \cup E_{n+1}$. For $n = 2k-1$, let $\mathcal{C}_{2k-1}$ consists of $3 \cdot 2^{k-1}$ intervals of length $2^{-(n+1)}$, including those making up E_{2k-1} in (2.1)-(2.2) and in addition the intervals of the same length (for simplicity) immediately to the left of those in (2.2), namely,

$$2^{-2k}\left[2^{2k}-\sum_{i=1}^{k-1}\varepsilon_i 2^{2i}-2, 2^{2k}-\sum_{i=1}^{k-1}\varepsilon_i 2^{2i}-1\right] \tag{2.3}$$

with ε_i as specified above. The size of $\mathcal{C}_{2k-1}$ is $(3 \cdot 2^{k-1})(2^{-2k})$, which gets arbitrarily small as k increases. Hence E is of measure 0.

It remains to show that the Minkowski dimension $\dim_B E$ is 1/2. We estimate the upper and lower Minkowski dimensions $\overline{\dim}_B E$ and $\underline{\dim}_B E$ as follows (see e.g. [5, p.43, e.g. 3.3]). Let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which can cover E. Recall from the last paragraph that $E \subset E_{2k-1} \cup E_{2k}$ can be covered by $3 \cdot 2^{k-1}$ intervals of length 2^{-2k}. Then for $\delta \in (2^{-2k}, 2^{-2k+2}]$, $k = 1,2,3,\ldots$, $N_\delta(E) \leq 3 \cdot 2^{k-1}$. Notice that we already get an upper estimate of $N_\delta(E)$ for every $\delta \in (0,1]$ without considering the n even cases. Hence

$$\overline{\dim}_B E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \leq \limsup_{k \to \infty} \frac{\log(3 \cdot 2^{k-1})}{-\log 2^{-2k+2}} = \frac{1}{2}.$$

Again for odd $n = 2k - 1$, consider the collection of intervals contained in those in (2.1), (2.2) and (2.3) but of half their sizes, namely

$$2^{-2k}\left[\sum_{i=1}^{k-1}\varepsilon_i 2^{2i-1}, \sum_{i=1}^{k-1}\varepsilon_i 2^{2i-1} + \frac{1}{2}\right], \tag{2.4}$$

$$2^{-2k}\left[2^{2k} - \sum_{i=1}^{k-1}\varepsilon_i 2^{2i} - \frac{1}{2}, 2^{2k} - \sum_{i=1}^{k-1}\varepsilon_i 2^{2i}\right] \quad \text{and} \tag{2.5}$$

$$2^{-2k}\left[2^{2k} - \sum_{i=1}^{k-1}\varepsilon_i 2^{2i} - \frac{3}{2}, 2^{2k} - \sum_{i=1}^{k-1}\varepsilon_i 2^{2i} - 1\right]. \tag{2.6}$$

Notice that there are $3 \cdot 2^{k-1}$ of them, any two of them are at least 2^{-2k-1} apart and each of them contains at least a point in E: the left end points of those in (2.4) and the right end points of those in (2.5) and (2.6) are in E. It follows that for $\delta \in [2^{-2k-3}, 2^{-2k-1})$, a set of diameter at most δ can intersect at most one of these intervals. Therefore $N_\delta(E) \geq 3 \cdot 2^{k-1}$. Hence

$$\underline{\dim}_B E = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log \delta} \geq \lim_{k\to\infty} \frac{\log(3\cdot 2^{k-1})}{-\log 2^{-2k-3}} = \frac{1}{2}.$$

Hence $\text{diam}_B E = 1/2$. The theorem is proved. □

3. The Hausdorff dimension of circular sets

Proof of Theorem 1.2. Recall that there are sets $A, B \subset \mathbb{R}$ with $\dim_H A = \dim_H B = 0$ such that $A - B = (0,1)$ (see for example [5, p. 97]). We include the details for completeness. Precisely, let $\{m_k\}_{k=0}^{\infty}$ be a rapidly increasing sequence of integers, to be specified precisely soon, with $m_0 = 0$. Let A be the set of numbers in $(0,1)$ with its r-th decimal place equals 0 whenever $m_k + 1 \leq r \leq m_{k+1}$, k even. Let B be the *negative* of a similar set with k odd. For $k > 0$ even, let $j_k = (m_2 - m_1) + \cdots + (m_k - m_{k-1})$ and for k odd, let $j_k = m_1 + (m_3 - m_2) + \cdots + (m_k - m_{k-1})$. Then each of A and B can be covered by 10^{j_k} intervals of length $10^{-m_{k+1}}$. Now choose m_k increasing so fast that

$$\lim_{k\to\infty} \log 10^{j_k} / -\log 10^{-m_{k+1}} = \lim_{k\to\infty} j_k/m_{k+1} = 0.$$

Hence $\dim_H A \leq \underline{\dim}_B A = 0$ and $\dim_H B \leq \underline{\dim}_B B = 0$. Also, $A - B$ contains $(0,1)$.

Let $F = A \cup B$. Then $\dim_H F = 0$ and $F - F$ contains $(0,1)$, proving Theorem 1.2. □

Remark 3.1. This does not contradict Theorem 1.1 as the box dimension of F is not well-defined. This fact can also be deduced from the two theorems.

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