SHIFT RADIX SYSTEMS FOR GAUSSIAN INTEGERS AND PETHŐ'S LOUDSPEAKER

HORST BRUNOTTE, PETER KIRSCHENHOFER, AND JÖRG M. THUSWALDNER

Dedicated to Professors Kálmán Győry, Attila Pethő, János Pintz, and András Sárközy on the occasion of their round birthdays

Abstract. Recently, Akiyama *et al.* introduced so-called shift radix systems. These simple dynamical systems form a common generalization of several well-known notions of number systems like beta numeration and canonical number systems. In the present paper we generalize shift radix systems as follows: for $(r_1,\dots,r_d)\in\mathbb{C}^d$ we study mappings $\mathbb{Z}[\mathrm{i}]^d\to\mathbb{Z}[\mathrm{i}]^d$ given by

$$(x_1,\dots,x_d)\mapsto(x_2,\dots,x_d,-\lfloor r_1x_1+\cdots+r_dx_d\rfloor),$$

where for $x\in\mathbb{C}$ we set $\lfloor x\rfloor=\lfloor\Re x\rfloor+\mathrm{i}\lfloor\Im x\rfloor$. We study basic dynamical properties of this class of mappings and relate them to known notions of number systems.

1. Introduction

In 2005 Akiyama *et al.* [1] defined *shift radix systems.* Before recalling the definition of these objects we define the *floor function* which assigns to each $y\in\mathbb{R}$ the largest integer that is less than or equal to y and is denoted by $\lfloor y\rfloor$. The shift radix system related to a vector $\mathbf{r}\in\mathbb{R}^d$ is given by the function $\tau_{\mathbf{r}}:\mathbb{Z}^d\to\mathbb{Z}^d$, $\mathbf{x}=(x_1,\dots,x_d)\mapsto(x_2,\dots,x_d,-\lfloor\mathbf{rx}\rfloor)$, where $\mathbf{rx}$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$. Shift radix systems have many interesting dynamical properties and are related to several notions of numeration (see [1] for details). In the recent years, they have been investigated extensively.

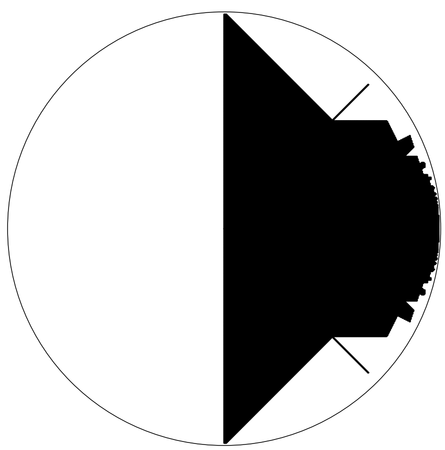

Figure 1. An approximation of Pethő's Loudspeaker $\mathcal{G}_1^{(0)}$.

The aim of the present note is to study a variant of shift radix systems for Gaussian integers. To this matter, the floor function is extended to $x\in\mathbb{C}$ by the *complex floor function* which is

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defined by

$$\lfloor x \rfloor := \lfloor \Re x \rfloor + \mathrm{i} \lfloor \Im x \rfloor,$$

i.e., by applying the floor function to the real and imaginary part of x separately. With the help of this function we define *Gaussian shift radix systems* as follows. Let $\mathbf{r} \in \mathbb{C}^d$ be given. With $\mathbf{r}$ we associate the mapping $\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \to \mathbb{Z}[\mathrm{i}]^d$ which is given by

$$\mathbf{x} = (x_1, \dots, x_d) \mapsto (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor).$$

This mapping is called the *Gaussian shift radix system* (GSRS) associated with $\mathbf{r}$. We say that $\gamma_{\mathbf{r}}$ has the *finiteness property* if all orbits of $\gamma_{\mathbf{r}}$ end up in zero, *i.e.*, if for each $\mathbf{x} \in \mathbb{Z}[\mathrm{i}]$ there exists a positive integer k such that the kth iterate of $\gamma_{\mathbf{r}}$ applied to $\mathbf{x}$ satisfies $\gamma_{\mathbf{r}}^k(\mathbf{x}) = \mathbf{0}$.

As for classical shift radix systems, GSRS have relations to number systems defined in rings of algebraic integers. In particular, we will see that Gaussian numeration systems in the sense of Jacob and Reveilles [6] are special cases of GSRS. Moreover, the symmetric number systems in imaginary quadratic fields studied in Kátai [8] are strongly related to them.

In the present paper we will study basic properties of $\gamma_{\mathbf{r}}$. We will discuss their relation to numeration and give first descriptions of the parameters $\mathbf{r}$ that give rise to the finiteness property. We describe an algorithm that allows to decide whether certain small parameter regions admit the finiteness property. Especially the case $d = 1$ will be studied in some detail. This case deserves special interest. Indeed, for classical shift radix systems the case $d = 1$ can easily be treated while in case $d = 2$ the according problems become already very hard and are not completely solved up to now. The one-dimensional case of GSRS seems to be of an intermediate level of difficulty on the one side and reveals new interesting properties on the other side.

In analogy to classical shift radix systems, the following sets will be of importance in our investigations:

$$\begin{aligned} \mathcal{G}_d^{(0)} &:= \{\mathbf{r} \in \mathbb{C}^d \ : \ \gamma_{\mathbf{r}} \text{ has the finiteness property}\} \quad \text{and} \\ \mathcal{G}_d &:= \{\mathbf{r} \in \mathbb{C}^d \ : \ \text{each orbit of } \gamma_{\mathbf{r}} \text{ is ultimately periodic}\}. \end{aligned}$$

The fact that the case $d = 1$ is already of interest in the context of GSRS is illustrated by Figure 1 which shows an approximation of the set $\mathcal{G}_1^{(0)}$ (observe its irregular structure on the right hand side). Because of its shape and in honor of Attila Pethő we call this set *Pethő's Loudspeaker*.

2. Orbits of $\gamma_{\mathbf{r}}$

Let $\mathbf{r} \in \mathbb{C}^d$ be given and consider the mapping $\gamma_{\mathbf{r}}$. If we take $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}[\mathrm{i}]^d$ as a starting point then, according to the definition of $\gamma_{\mathbf{r}}$, we have

$$\gamma_{\mathbf{r}}((x_1, \dots, x_d)) = (x_2, \dots, x_{d+1}) \iff \begin{cases} 0 \le \Re\left(\left(\sum_{j=1}^d r_j x_j\right) + x_{d+1}\right) < 1 \quad \text{and} \\ 0 \le \Im\left(\left(\sum_{j=1}^d r_j x_j\right) + x_{d+1}\right) < 1. \end{cases} \tag{2.1}$$

Thus, calculating $\gamma_{\mathbf{r}}$ for a given value $\mathbf{x}$ amounts to solving a finite system of linear inequalities.

Let $\mathbf{x}_1 = (x_1, \dots, x_d)$ be contained in a cycle π of $\gamma_{\mathbf{r}}$. This means that there is a positive integer p such that $\gamma_{\mathbf{r}}^p(\mathbf{x}_1) = \mathbf{x}_1$, *i.e.*, $\mathbf{x}_1$ is $\gamma_{\mathbf{r}}$-*periodic*. Thus, there exist $\mathbf{x}_2, \dots, \mathbf{x}_p$ such that

$$\mathbf{x}_1 \xrightarrow{\gamma_{\mathbf{r}}} \mathbf{x}_2 \xrightarrow{\gamma_{\mathbf{r}}} \cdots \xrightarrow{\gamma_{\mathbf{r}}} \mathbf{x}_p \xrightarrow{\gamma_{\mathbf{r}}} \mathbf{x}_1 \tag{2.2}$$

where each arrow indicates an application of the mapping $\gamma_{\mathbf{r}}$. According to the definition of $\gamma_{\mathbf{r}}$ there exist $x_{d+1}, \dots, x_{d+p-1} \in \mathbb{Z}[\mathrm{i}]$ such that

$$\mathbf{x}_\ell = (x_\ell, \dots, x_{\ell+d-1}) \qquad (1 \le \ell \le p).$$

Moreover, the fact that (2.2) is a cycle implies that $x_{p+\ell} = x_\ell$ for $\ell \in \{1, \dots, d-1\}$. Thus, the cycle in (2.2) is completely characterized by the sequence $x_1, \dots, x_p$. Therefore we write this cycle as

$$\pi = \begin{cases} (x_1, \dots, x_d), & \text{if } p \le d, \\ (x_1, \dots, x_d) x_{d+1}, \dots, x_p, & \text{otherwise.} \end{cases} \tag{2.3}$$

The cycle $\mathbf{0} \xrightarrow{\gamma_{\mathbf{r}}} \mathbf{0}$ occurs for each $\mathbf{r} \in \mathbb{C}^d$. It is called the *trivial cycle*. Each other cycle is called *nontrivial*.

The set $\mathcal{G}_d^{(0)}$ can be constructed starting from the set $\mathcal{G}_d$ by removing all points $\mathbf{r}$ that correspond to some non trivial cycle π. For this reason, as in the case of classical SRS we define

$$\mathbf{P}(\pi) = \{\mathbf{r} \in \mathcal{G}_d \ : \ \pi \text{ occurs as a cycle for the mapping } \gamma_{\mathbf{r}}\}.$$

We will now show that, as in the classical case, the sets $\mathbf{P}(\pi)$ are polyhedra. In particular, let[1] $\pi = (x_1, \ldots, x_d)x_{d+1}, \ldots, x_p$ be a given cycle. According to (2.1) a parameter $\mathbf{r}$ is contained in $\mathbf{P}(\pi)$ if and only if

$$0 \le \Re\left(\left(\sum_{j=1}^{d} r_j x_{\ell-1+j}\right) + x_{\ell+d}\right) < 1 \quad \text{and} \quad 0 \le \Im\left(\left(\sum_{j=1}^{d} r_j x_{\ell-1+j}\right) + x_{\ell+d}\right) < 1 \tag{2.4}$$

holds for all $\ell \in \{1, \ldots, p\}$ (here we set $x_{p+\ell} = x_\ell$ for $\ell \in \{1, \ldots, d\}$). Since $\mathbf{P}(\pi)$ is defined by the linear inequalities in (2.4) we see that it is a (half-open and possibly degenerated) polyhedron.

3. Fundamental properties of GSRS

For a matrix $M \in \mathbb{C}^{d\times d}$ we denote its spectral radius by $\rho(M)$. We need the following result.

Lemma 3.1. *Let $d \in \mathbb{N}$, $f : \mathbb{C}^d \to \mathbb{C}^d$ a bounded function and $M \in \mathbb{C}^{d\times d}$ with $\rho(M) > 1$. Furthermore, let*

$$F(\mathbf{x}) = M\mathbf{x}^T + f(\mathbf{x})$$

for $\mathbf{x} \in \mathbb{C}^d$. Then there exists some $\mathbf{x} \in \mathbb{C}^d$ such that the sequence $(F^n(\mathbf{x}))_{n\in\mathbb{N}}$ given by the iterates of F is not ultimately periodic.

Proof. The proof of [1, Lemma 4.1] can easily be adapted. □

Remark 3.2. Note that Lemma 3.1 was established by Gilbert [5, Proposition 3] for real diagonalizable matrices. In particular, Gilbert showed this statement for a bounded function $f : \mathbb{R}^d \to \mathbb{R}^d$ and $F : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\mathbf{x} \mapsto M\mathbf{x}^T + f(\mathbf{x})$ where M is a real diagonalizable $d \times d$ matrix.

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$ let

$$R_{\mathbf{r}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix} \in \mathbb{C}^{d\times d}. \tag{3.1}$$

Lemma 3.3. *Let $\mathbf{r} \in \mathbb{C}^d$. The n-th iterate $\gamma_{\mathbf{r}}^n(\mathbf{x})$ of the vector $\mathbf{x} \in \mathbb{Z}[\mathrm{i}]^d$ is*

$$\gamma_{\mathbf{r}}^n(\mathbf{x})^T = R_{\mathbf{r}}^n \mathbf{x}^T + \sum_{k=1}^{n} R_{\mathbf{r}}^{n-k} \mathbf{v}_k^T \tag{3.2}$$

with vectors $\mathbf{v}_k = (0, \ldots, 0, \{\mathbf{r}\gamma_{\mathbf{r}}^{k-1}(\mathbf{x})\}) \in \mathbb{C}^d$ and $\{z\} := z - \lfloor z \rfloor$ for $z \in \mathbb{C}$.

Proof. Using induction this is an immediate consequence of the definitions. □

If $\gamma_{\mathbf{r}}$ satisfies the finiteness property then each $\mathbf{x} \in \mathbb{Z}[\mathrm{i}]^d$ admits a representation of the form

$$\mathbf{x}^T = \sum_{k=1}^{n} R_{\mathbf{r}}^{-k}(-\mathbf{v}_k^T) = \sum_{k=0}^{n-1} R_{\mathbf{r}}^{-k}(-R_{\mathbf{r}}^{-1}\mathbf{v}_{k+1}^T) = \sum_{k=0}^{n-1} R_{\mathbf{r}}^{-k}(-R_{\mathbf{r}}^{-1}(0, \ldots, 0, \{\mathbf{r}\gamma_{\mathbf{r}}^k(\mathbf{x})\})^T). \tag{3.3}$$

This is reminiscent of a radix representation with base $R_{\mathbf{r}}^{-1}$ and digits $-R_{\mathbf{r}}^{-1}(0, \ldots, 0, \{\mathbf{r}\gamma_{\mathbf{r}}^k(\mathbf{x})\})^T$. We will come back to this interpretation in Section 5.

In the next proposition we formulate some results on $\mathcal{G}_d$.

[1] For a general cycle π we always use the second alternative of the notation introduced in (2.3). This should cause no confusion.

Proposition 3.4. *The following assertions hold.*

(i) *Let* $\mathbf{r} \in \mathbb{C}^d$. *If the spectral radius* $\rho(R_{\mathbf{r}})$ *of* $R_{\mathbf{r}}$ *is less than* 1 *then* $\mathbf{r} \in \mathcal{G}_d$ *and the set of* $\gamma_{\mathbf{r}}$*-periodic elements is finite. More precisely, for every norm* $||\cdot||$ *on* $\mathbb{C}^d$ *there exists a constant* $c \in \mathbb{R}$ *such that* $\|\mathbf{a}\| \le c$ *for every* $\gamma_{\mathbf{r}}$*-periodic* $\mathbf{a} \in \mathbb{Z}[\mathrm{i}]^d$.

(ii) *We have* $\mathcal{E}_d \subseteq \mathcal{G}_d \subseteq \mathrm{cl}(\mathcal{E}_d)$ *where* cl *denotes the topological closure and* $\mathcal{E}_d$ *is the set of all parameters* $\mathbf{r} \in \mathbb{C}^d$ *satisfying* $\rho(R_{\mathbf{r}}) < 1$.

(iii) *The boundary of* $\mathcal{G}_d$ *is given by* $\partial\mathcal{G}_d = \{\mathbf{r} \in \mathbb{C}^d \ : \ \rho(R_{\mathbf{r}}) = 1\}$.

Proof. (i) For $\mathbf{r} = \mathbf{0}$ the assertion is trivially true. Thus we may assume that $0 < \rho(R_{\mathbf{r}}) < 1$. In this case we may choose $\tilde{\rho} \in (\rho(R_{\mathbf{r}}), 1)$ and construct a norm $||\cdot||_{\tilde{\rho}}$ on $\mathbb{C}^d$ with the property

$$||R_{\mathbf{r}}\mathbf{x}^T||_{\tilde{\rho}} \le \tilde{\rho}||\mathbf{x}||_{\tilde{\rho}}$$

(see *e.g.* [9, formula (3.2)]). Using Lemma 3.3, the proof of the first part of [1, Lemma 4.2] shows

$$\|\gamma_{\mathbf{r}}^k(\mathbf{x})\|_{\tilde{\rho}} \le \tilde{\rho}^k\|\mathbf{x}\|_{\tilde{\rho}} + \frac{1}{1-\tilde{\rho}}.$$

for $k \in \mathbb{N}$, hence, there is some $k > 0$ such that

$$\|\gamma_{\mathbf{r}}^k(\mathbf{x})\|_{\tilde{\rho}} \le \frac{1}{1-\tilde{\rho}} + 1. \tag{3.4}$$

In view of the equivalence of norms on $\mathbb{C}^d$ the proof can easily be completed.

(ii) The first inclusion follows from (i) while the second one is an immediate consequence of Lemma 3.3 (with $n = 1$) and Lemma 3.1.

(iii) This follows in the same way as [1, Lemma 4.3]. Just take complex instead of real polynomials. □

Because of this result we will concentrate on contracting polynomials. More precisely, we see that the set $\mathcal{G}_d$ is intimately related to the set of all parameters $\mathbf{r}$ whose accompanying matrix $R_{\mathbf{r}}$ (see (3.1)) has spectral radius less than 1. Looking at the characteristic polynomial of $R_{\mathbf{r}}$ these parameters are given by the *Schur-Cohn region*, *i.e.*, the set $\mathcal{E}_d$ defined in Proposition 3.4 (ii) can be written as

$$\mathcal{E}_d = \{(r_1, \dots, r_d) \in \mathbb{C}^d \ : \ \text{all roots of } X^d + r_d X^{d-1} + \cdots + r_1 \text{ are inside the unit circle}\}.$$

This region has been characterized by Schur [12] as follows.

Proposition 3.5 ([12, Satz XVII])**.** *The zeros of the polynomial*

$$X^d + r_d X^{d-1} + \cdots + r_2 X + r_1 \in \mathbb{C}[X]$$

are all contained in the unit disk if and only if

$$\det\begin{pmatrix} 1 & 0 & \dots & 0 & r_1 & r_2 & \dots & r_{\nu+1} \\ r_d & 1 & \dots & 0 & 0 & r_1 & \dots & r_\nu \\ \vdots & & \ddots & & \vdots & & \ddots & \vdots \\ r_{d-\nu+1} & r_{d-\nu+2} & \cdots & 1 & 0 & 0 & \cdots & r_1 \\ \overline{r}_1 & 0 & \dots & 0 & 1 & \overline{r}_d & \dots & \overline{r}_{d-\nu+1} \\ \overline{r}_2 & \overline{r}_1 & \dots & 0 & 0 & 1 & \dots & \overline{r}_{d-\nu+2} \\ \vdots & & \ddots & & \vdots & & \ddots & \vdots \\ \overline{r}_{\nu+1} & \overline{r}_\nu & \cdots & \overline{r}_1 & 0 & 0 & \cdots & 1 \end{pmatrix} > 0 \qquad (0 \le \nu \le d-1).$$

In particular, in this case $|r_k| < \binom{d}{k-1}$ *holds for each* $k \in \{1, \dots, d\}$.

Obviously, $\mathcal{E}_1 = \{r \in \mathbb{C} \;:\; |r| < 1\}$. Moreover, one checks that

$$\mathcal{E}_2 = \{(r_1, r_2) \in \mathbb{C}^2 \;:\; |r_1| < 1 \text{ and } (1-|r_1|^2)^2 + 2\Re(\overline{r}_1 r_2^2) > (1+|r_1|^2)|r_2|^2\}.$$

Now we turn to results on $\mathcal{G}_d^{(0)}$. We will need the following notation. For $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{C}^d$ let $\overline{\mathbf{s}} = (\overline{s}_1, \dots, \overline{s}_d)$ be the vector containing the complex conjugates of the entries of $\mathbf{s}$.

We start with the following symmetry lemma.

Lemma 3.6. *Let $\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{G}_d$. Then $\mathbf{r} \in \mathcal{G}_d^{(0)}$ if and only if $\overline{\mathbf{r}} \in \mathcal{G}_d^{(0)}$.*

Proof. Let $\mathbf{x}_1 = (x_1, \dots, x_d)$ and $\mathbf{x}_2 = (x_2, \dots, x_{d+1})$ and assume that $\gamma_{\mathbf{r}}(\mathbf{x}_1) = \mathbf{x}_2$ then, according to (2.1), we have

$$0 \le \Re\left(\left(\sum_{j=1}^d r_j x_j\right) + x_{d+1}\right) < 1 \quad \text{and} \quad 0 \le \Im\left(\left(\sum_{j=1}^d r_j x_j\right) + x_{d+1}\right) < 1. \tag{3.5}$$

If we replace r_j $(1 \le j \le d)$ and x_k $(1 \le k \le d+1)$ in (3.5) by $\overline{r}_j$ and $\mathrm{i}\overline{x}_k$, respectively, the first chain of inequalities in (3.5) becomes the second one and vice versa. Indeed,

$$\begin{aligned}
\Re\left(\left(\sum_{j=1}^d \overline{r}_j \mathrm{i}\overline{x}_j\right) + \mathrm{i}\overline{x}_{d+1}\right) &= \Re\left(\left(\sum_{j=1}^d (\Re r_j - \mathrm{i}\Im r_j)(\Im x_j + \mathrm{i}\Re x_j)\right) + \Im x_{d+1} + \mathrm{i}\Re x_{d+1}\right) \\
&= \left(\sum_{j=1}^d (\Re r_j \Im x_j + \Im r_j \Re x_j)\right) + \Im x_{d+1} \\
&= \Im\left(\left(\sum_{j=1}^d r_j x_j\right) + x_{d+1}\right)
\end{aligned}$$

shows that the first chain becomes the second one by this replacement. The fact that the second one transforms to the first one is shown likewise. Thus

$$\gamma_{\mathbf{r}}(\mathbf{x}_1) = \mathbf{x}_2 \quad \Longleftrightarrow \quad \gamma_{\overline{\mathbf{r}}}(\mathrm{i}\overline{\mathbf{x}}_1) = \mathrm{i}\overline{\mathbf{x}}_2.$$

In particular, the cylce $(x_1, \dots, x_d)x_{d+1}, \dots, x_p$ is a nontrivial cycle for $\gamma_{\mathbf{r}}$ if and only if the cycle $(\mathrm{i}\overline{x}_1, \dots, \mathrm{i}\overline{x}_d)\mathrm{i}\overline{x}_{d+1}, \dots, \mathrm{i}\overline{x}_p$ is nontrivial for $\gamma_{\overline{\mathbf{r}}}$. Thus $\gamma_{\mathbf{r}}$ admits a nontrivial cycle if and only if $\gamma_{\overline{\mathbf{r}}}$ does. This implies the lemma. $\square$

Remark 3.7. Figure 1 shows that $\mathbf{r} \in \mathcal{G}_d^{(0)}$ does not imply that $\Im\mathbf{r}$ belongs to $\mathcal{D}_d^{(0)}$ (see [1, p. 211]).

We use Lemma 3.6 for a result on $\mathcal{G}_1^{(0)}$.

Lemma 3.8. *If $r \in \mathcal{G}_1^{(0)}$ then $\Re r \ge 0$.*

Proof. In view of Lemma 3.6 it suffices to exhibit a nontrivial period of γ_r for each $r = s + \mathrm{i}t$ with $-1 < s < 0$ and $0 \le t < 1$. Since for these r we have $\lfloor r \rfloor = -1$ we get $\gamma_r(1) = -\lfloor r \rfloor = 1$. Thus $1 \to 1$ is a nontrivial period and the lemma is proved. $\square$

4. Algorithms

In this section we present algorithms that allow to exhibit points and small regions in $\mathcal{G}_d$ which belong to $\mathcal{G}_d^{(0)}$. We start with an efficient algorithm to decide whether $\gamma_{\mathbf{r}}$ has the finiteness property for a given vector $\mathbf{r} \in \mathbb{C}^d$. It is an obvious generalization of the analogous result in [1, Theorem 5.1]. For convenience we denote by $\mathbf{e}_1, \dots, \mathbf{e}_d$ the canonical basis vectors of $\mathbb{C}^d$ considered as a vector space over $\mathbb{C}$. Furthermore, we use the notations

$$S_1 f(\mathbf{x}) = f(\mathbf{x}), \quad S_2 f(\mathbf{x}) = -f(-\mathbf{x}), \quad S_3 f(\mathbf{x}) = \overline{f(\overline{\mathbf{x}})}, \quad S_4 f(\mathbf{x}) = -\overline{f(-\overline{\mathbf{x}})}. \tag{4.1}$$

Theorem 4.1. *Let $\mathbf{r} \in \mathbb{C}^d$ and denote by $\mathcal{Z}_{\mathbf{r}}$ the set of elements in $\mathbb{Z}[\mathrm{i}]^d$ whose orbits of $\gamma_{\mathbf{r}}$ end up in zero. Suppose that there exists a subset V of $\mathcal{Z}_{\mathbf{r}}$ satisfying the following two properties.*

(i) *V contains the $4d$-element set*

$$V_1 = \{\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_d, \pm\mathrm{i}\mathbf{e}_1, \dots, \pm\mathrm{i}\mathbf{e}_d\}. \tag{4.2}$$

(ii) *For each $\mathbf{a} \in V$ the elements $S_\ell\gamma_{\mathbf{r}}(\mathbf{a})$ $(1 \le \ell \le 4)$ belong to V.*

Then $\mathbf{r} \in \mathcal{G}_d^{(0)}$.

Proof. We follow the proofs of [1, Theorem 5.1] and [13, Theorem 2.6] and observe that

$$\lfloor x+y \rfloor \in \left\{ \lfloor x \rfloor + \lfloor y \rfloor\,,\ \lfloor x \rfloor - \lfloor -y \rfloor\,,\ \lfloor x \rfloor + \overline{\lfloor \overline{y} \rfloor},\ \lfloor x \rfloor - \overline{\lfloor -\overline{y} \rfloor} \right\} = \{\lfloor x \rfloor + S_\ell(\lfloor y \rfloor)\ :\ 1 \le \ell \le 4\}$$

for $x, y \in \mathbb{C}$. Using the definition of $\gamma_{\mathbf{r}}$ we deduce

$$\gamma_{\mathbf{r}}(\mathbf{a}+\mathbf{b}) \in \{\gamma_{\mathbf{r}}(\mathbf{a}) + S_\ell\gamma_{\mathbf{r}}(\mathbf{b})\ :\ 1 \le \ell \le 4\}$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}[\mathrm{i}]^d$. Therefore, by (ii) for all $\mathbf{a} \in \mathbb{Z}[\mathrm{i}]^d$ and $\mathbf{b} \in V$ there is some $\mathbf{c} \in V$ such that

$$\gamma_{\mathbf{r}}(\mathbf{a}+\mathbf{b}) = \gamma_{\mathbf{r}}(\mathbf{a}) + \mathbf{c}.$$

Using induction, for every $n \in \mathbb{N}$ we find some $\mathbf{c} \in V$ with

$$\gamma_{\mathbf{r}}^n(\mathbf{a}+\mathbf{b}) = \gamma_{\mathbf{r}}^n(\mathbf{a}) + \mathbf{c}.$$

Now let $\mathbf{a} \in \mathcal{Z}_{\mathbf{r}}$. Then there exists $n \in \mathbb{N}$ with $\gamma_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$ and, hence,

$$\gamma_{\mathbf{r}}^n(\mathbf{a}+\mathbf{b}) \in V,$$

and we deduce $\mathbf{a}+\mathbf{b} \in \mathcal{Z}_{\mathbf{r}}$. By (i) the proof can now easily be completed inductively. $\square$

Let us assume that all roots of the polynomial

$$X^d + r_d X^{d-1} + r_{d-1} X^{d-2} + \cdots + r_1 \in \mathbb{C}[X]$$

lie inside the open unit disk, *i.e.*, $\rho(R_{\mathbf{r}}) < 1$ with $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{C}^d$. Then similarly as explained in [1, p. 223 f.] our results provide an efficient algorithm to determine whether $\mathbf{r}$ belongs to $\mathcal{G}_d^{(0)}$ or not: define inductively

$$V_{n+1} = V_n \cup \{S_\ell\gamma_{\mathbf{r}}(\mathbf{a})\ :\ \mathbf{a} \in V_n,\ 1 \le \ell \le 4\}$$

with V_1 given by (4.2). Set $m_n := \max\{||\mathbf{v}||\ :\ \mathbf{v} \in V_n\}$. Observe that, choosing $\tilde\rho$ as in the proof of Proposition 3.4 (i), we get

$$\max\{||S_\ell\gamma_{\mathbf{r}}(\mathbf{v})||\ :\ \mathbf{v} \in V_n,\ 1 \le \ell \le 4\} \le \tilde\rho m_n + \frac{1}{1-\tilde\rho}. \tag{4.3}$$

Thus $m_{n+1} \le \max\{m_n, \tilde\rho m_n + \frac{1}{1-\tilde\rho}\}$ which implies that the sequences (m_n) and (V_n) are uniformly bounded. Thus there must be some n such that $V_{n+1} \subseteq V_n$. Now, if $V_n \subset \mathcal{Z}_{\mathbf{r}}$ then we know $\mathbf{r} \in \mathcal{G}_d^{(0)}$ by Theorem 4.1, otherwise V_n contains a nonzero $\gamma_{\mathbf{r}}$-periodic element, and we conclude $\mathbf{r} \notin \mathcal{G}_d^{(0)}$.

This algorithm was used to construct the approximation of $\mathcal{G}_1^{(0)}$ in Figure 1. According to Lemmas 3.6 and 3.8 it suffices to consider only parameters whose real and imaginary parts are greater than or equal to zero. By (2.4) each explicitly given nontrivial period π cuts out a polygon form $\mathcal{G}_1$. For instance, the period

$$\pi:\ 1-i \to -1 \to 1+i \to 1-i$$

cuts out the set

$$\mathbf{P}(\pi) = \{r = s+it \in \mathbb{C}\ :\ s^2+t^2 \le 1,\ 1-s \le t < 2-s,\ s < t < 1+s\}$$

from $\mathcal{G}_1$. Thus this set has empty intersection with $\mathcal{G}_1^{(0)}$.

Analogously as in [1, Section 5] we generalize Theorem 4.1 and provide an efficient method to determine a small subregion of $\mathcal{G}_d^{(0)}$ contained in a given convex polyhedron inside the interior of $\mathcal{G}_d$.

Theorem 4.2. *Let $\mathbf{r}_1,\dots,\mathbf{r}_k$ be points of $\mathcal{G}_d$ and denote by $H=\operatorname{conv}(\mathbf{r}_1,\dots,\mathbf{r}_k)$ the convex hull of $\mathbf{r}_1,\dots,\mathbf{r}_k$. If H is contained in the interior of $\mathcal{G}_d$ and if the diameter of H is sufficiently small then there exists an algorithm to create a finite digraph (V,E) with vertices $V\subset\mathbb{Z}[\mathrm{i}]^d$ and edges $E\in V\times V$ with the following properties:*

(i) *$V\supset V_1$ with V_1 as defined in (4.2).*

(ii) *For each pair $(\mathbf{v},\mathbf{v}')\in V^2$ there exists an edge from $\mathbf{v}$ to $\mathbf{v}'$ if and only if there is some $\mathbf{h}\in H$ with*

$$\mathbf{v}'\in\{S_\ell\gamma_{\mathbf{h}}(\mathbf{v})\ :\ 1\le\ell\le4\},$$

where S_ℓ is defined as in (4.1).

(iii) *$H\cap\mathcal{G}_d^{(0)}=H\setminus\bigcup_{\pi\in C}\mathbf{P}(\pi)$, where C is the set of all nonzero simple cycles[2] of (V,E).*

Proof. Since $H\subset\operatorname{int}(\mathcal{G}_d)$, Proposition 3.4 implies the existence of a positive $\delta<1$ such that for each $\mathbf{h}\in H$ the spectral radius of the matrix $R_{\mathbf{h}}$ is less than δ. Then for all $\mathbf{h}'$ in a sufficiently small neighborhood of $\mathbf{h}$, the maps

$$\mathbf{z}\mapsto S_\ell\gamma_{\mathbf{h}'}(\mathbf{z})\qquad(1\le\ell\le4)$$

are contractive with respect to a suitable norm (see the proof of Proposition 3.4 (i)). Therefore, assuming that the diameter of H is sufficiently small we may choose a norm on $\mathbb{C}^d$ such that for all $\mathbf{h}\in H$ the matrix $R_{\mathbf{h}}$ is contractive with respect to this norm.

For $\mathbf{z}=(z_1,\dots,z_d)\in\mathbb{Z}[\mathrm{i}]^d$ we define

$$L_{\mathbf{z}}^{(\ell)}=\left\{\mathbf{w}\in\mathbb{Z}[\mathrm{i}]^d : H_{\mathbf{z}}^{(\ell)}(\mathbf{w})\ne\emptyset\right\}\quad\text{with}\quad H_{\mathbf{z}}^{(\ell)}(\mathbf{w})=\{\mathbf{h}\in H : S_\ell\gamma_{\mathbf{h}}(\mathbf{z})=\mathbf{w}\}\quad(1\le\ell\le4).$$

For convenience for $a,b\in\mathbb{C}$ we write $a\le b$ if $\Re a\le\Re b$ and $\Im a\le\Im b$. Moreover, we use the notation

$$\max\{a_j\in\mathbb{C}\ :\ 1\le j\le k\}=\max\{\Re a_j\in\mathbb{C}\ :\ 1\le j\le k\}+\mathrm{i}\max\{\Im a_j\in\mathbb{C}\ :\ 1\le j\le k\}.$$

Note that it is easy to check that $\mathbf{w}=(z_2,\dots,z_{d+1})\in L_{\mathbf{z}}^{(\ell)}$ implies

$$-M^{(\ell)}(-\mathbf{z})\le z_{d+1}\le M^{(\ell)}(\mathbf{z})$$

where we set

$$M^{(\ell)}(\mathbf{z})=\max\{S_\ell(-\lfloor\mathbf{r}_j\mathbf{z}\rfloor)\ :\ 1\le j\le k\}.$$

In other words, setting

$$K_{\mathbf{z}}^{(\ell)}=\{(z_2,\dots,z_d,z_{d+1})\ :\ z_{d+1}\in\mathbb{Z}[\mathrm{i}],\ -M^{(\ell)}(-\mathbf{z})\le z_{d+1}\le M^{(\ell)}(\mathbf{z})\}$$

we get $L_{\mathbf{z}}^{(\ell)}\subseteq K_{\mathbf{z}}^{(\ell)}$. Thus

$$L_{\mathbf{z}}^{(\ell)}=\{\mathbf{w}\in K_{\mathbf{z}}^{(\ell)}\ :\ H_{\mathbf{z}}^{(\ell)}(\mathbf{w})\ne\emptyset\}.$$

As $H_{\mathbf{z}}^{(\ell)}(\mathbf{w})$ is a polytope, the sets $L_{\mathbf{z}}^{(\ell)}$ are effectively computable.

For $n\ge1$ we define inductively

$$V_{n+1}=V_n\cup\{S_\ell\gamma_{\mathbf{h}}(\mathbf{a})\ :\ \mathbf{h}\in H,\ \mathbf{a}\in V_n,\ 1\le\ell\le4\}.$$

In the same way as in the remarks after the proof of Theorem 4.1 (see especially equation (4.3)) we get $V_{n+1}\subseteq V_n$ for some n. For this n we set $V:=V_n$.

Now we define the edges E by (ii). Then it is clear that (V,E) is a finite digraph with properties (i) and (ii).

We are left to show (iii). It is obvious that $H\setminus\bigcup_{\pi\in C}\mathbf{P}(\pi)\supset H\cap\mathcal{G}_d^{(0)}$. Let $\mathbf{h}\in H$. The construction after Theorem 4.1 delivers a subgraph of (V,E). If $\mathbf{h}\in H\setminus\mathcal{G}_d^{(0)}$ then by Theorem 4.1 there exists a nontrivial simple cycle π in the graph (V,E) and $\mathbf{h}\in P(\pi)$. □

This theorem gives an effective algorithm to obtain $H\cap\mathcal{G}_d^{(0)}$ for H being the convex hull of finitely many points lying in the interior of $\mathcal{G}_d$, see Algorithm 1. Observe that the number of simple cycles of a finite digraph is finite (see *e.g.* [7] for an algorithm that finds all simple cycles of a finite digraph).

[2]See [10, Definition 2.2.11] for the definition of a *simple* cycle.

Algorithm 1 Determination of $\mathcal{G}_d^{(0)} \cap \mathrm{conv}\{\mathbf{r}_1, \dots, \mathbf{r}_k\}$

Input: $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathcal{E}_d$, overflow $\in \mathbb{N}$.
Output: $G := \mathcal{G}_d^{(0)} \cap \mathrm{conv}\{\mathbf{r}_1, \dots, \mathbf{r}_k\}$ or "overflow"

```
  H ← conv{r_1, ..., r_k}
  V_1 ← {±e_1, ..., ±e_d, ±ie_1, ..., ±ie_d}
  n ← 1
  repeat
    W = ∅
    for z = (z_1, ..., z_d) ∈ V_n do
      for ℓ = 1, ..., 4 do
        M^(ℓ)(z) ← max{S_ℓ(−⌊r_j z⌋) : 1 ≤ j ≤ k};      {S_ℓ as in (4.1)}
        K_z^(ℓ) ← {(z_2, ..., z_d, z_{d+1}) : z_{d+1} ∈ ℤ[i], −M^(ℓ)(−z) ≤ z_{d+1} ≤ M^(ℓ)(z)}
        for w ∈ K_z^(ℓ) do
          H_z^(ℓ)(w) ← {h ∈ H : S_ℓ γ_h(z) = w}
        end for
        L_z^(ℓ) ← {w ∈ K_z^(ℓ) : H_z^(ℓ)(w) ≠ ∅}
      end for
      W ← W ∪ L_z^(1) ∪ L_z^(2) ∪ L_z^(3) ∪ L_z^(4)
    end for
    V_{n+1} ← V_n ∪ W
    n ← n + 1
  until V_n ⊆ V_{n−1} or n > overflow
  if V_n ⊆ V_{n−1} then
    V ← V_n
    G ← H
    for π in the set of simple cycles in (V, E) do
      G ← G \ P(π)
    end for
    return G
  else
    return "overflow"
  end if
```

In practice, to apply Theorem 4.2, there is no need to take care of the "sufficiently small" condition. Just choose a small convex hull inside the interior of $\mathcal{G}_d$ and try this algorithm to see whether the emerging process terminates or yields "overflow" in a certain prescribed time. If it terminates we get $H \cap \mathcal{G}_d^{(0)}$. If not, then we have to try again by a smaller convex hull.

5. Relations to Gaussian numeration systems

In the present section we show that our new notion of GSRS contains Gaussian numeration systems in the sense of Jacob and Reveilles [6] as special cases. First we recall the definition of these objects.

Let $\beta \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}$ be a Gaussian integer and set

$$\mathcal{C} = \left\{ c \in \mathbb{Z}[\mathrm{i}] \;:\; \left\lfloor \frac{c}{\beta} \right\rfloor = 0 \right\}. \tag{5.1}$$

The pair $(\beta, \mathcal{C})$ is called a *Gaussian numeration system* if each $x \in \mathbb{Z}[\mathrm{i}]$ can be written uniquely in the form

$$x = c_0 + c_1\beta + \cdots + c_n\beta^n \tag{5.2}$$

with $c_i \in \mathcal{C}$ and $c_n \neq 0$ for $n \neq 0$. Note that the digits c_i are unique because $\mathcal{C}$ is a complete set of representatives of cosets of $\mathbb{Z}[\mathrm{i}]/\beta\mathbb{Z}[\mathrm{i}]$. In [6, Theorem 3] the elements $\beta \in \mathbb{Z}[\mathrm{i}]$ that give rise to a Gaussian numeration system are characterized.

In the following lemma we show that for $d = 1$ certain GSRS mappings $\gamma_{\mathbf{r}}$ can be used to calculate the digits in (5.2).

Proposition 5.1. *Let $\beta \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}$ be given and define $\mathcal{C}$ as in* (5.1)*. Then $(\beta, \mathcal{C})$ is a Gaussian numeration system if and only if $-1/\beta \in \mathcal{G}_1^{(0)}$. In particular, the digits in the representation* (5.2) *of x are given by*

$$c_k = \beta \left\{ -\frac{1}{\beta} \gamma_{-1/\beta}^k(-x) \right\} \qquad (k \in \mathbb{N}). \tag{5.3}$$

Note that $c_k = 0$ for $k > n$.

Proof. Let $-\frac{1}{\beta} \in \mathcal{G}_1^{(0)}$ and $x \in \mathbb{Z}[\mathrm{i}]$. Then there exists $n \in \mathbb{N}$ such that $\gamma_{-1/\beta}^{n+1}(-x) = 0$. In view of (3.3) we can write

$$x = \sum_{k=0}^{n} \beta^k \left(\beta \left\{ -\frac{1}{\beta} \gamma_{-1/\beta}^k(-x) \right\} \right) = \sum_{k=0}^{\infty} \beta^k \left(\beta \left\{ -\frac{1}{\beta} \gamma_{-1/\beta}^k(-x) \right\} \right). \tag{5.4}$$

Since $\left\lfloor \left\{ -\frac{1}{\beta} \gamma_{-1/\beta}^k(-x) \right\} \right\rfloor = 0$, the elements $\beta \left\{ -\frac{1}{\beta} \gamma_{-1/\beta}^k(-x) \right\}$ belong to $\mathcal{C}$. Thus, in view of the uniqueness of the digit representation (5.2), the expansion in (5.4) is the one in (5.2) which proves (5.3). Moreover, since x was arbitrary, this implies that $(\beta, \mathcal{C})$ is a Gaussian numeration system.

Now assume that $(\beta, \mathcal{C})$ is a Gaussian numeration system and $x \in \mathbb{Z}[\mathrm{i}]$. Then x has a representation of the form (5.2). We see by induction that $\gamma_{-1/\beta}^{(n)}(-x) = 0$. Indeed, note that

$$\begin{aligned} \gamma_{-1/\beta}(-x) &= \gamma_{-1/\beta}(-c_0 - c_1\beta - \cdots - c_n\beta^n) = -\left\lfloor \frac{c_0 + c_1\beta + \cdots + c_n\beta^n}{\beta} \right\rfloor \\ &= -\left\lfloor \frac{c_0}{\beta} \right\rfloor - \left\lfloor c_1 + \cdots + c_n\beta^{n-1} \right\rfloor = -c_1 - \cdots - c_n\beta^{n-1}. \end{aligned}$$

Since x was arbitrary this implies $-\frac{1}{\beta} \in \mathcal{G}_1^{(0)}$. □

Observe that $\mathcal{G}_d^{(0)}$ contains much more elements than those which are in correspondence with Gaussian numeration systems.

6. Perspectives

In the present paper we started the investigations on GSRS by proving some basic results. Many things remain to be done. In what follows, we list some further problems and possible research directions related to this new class of dynamical systems.

(i) Is it true that $(r_1, \ldots, r_d) \in \mathcal{G}_d^{(0)}$ implies that $\Re r_1 \geq 0$? Lemma 3.8 shows that this is true for $d = 1$. In case of shift radix systems the analogous question has been answered affirmatively in [3, Theorem 2.1]. In the case of GSRS the situation seems to be more difficult.

(ii) Is 1 a critical point of $\mathcal{G}_1^{(0)}$? If so, is it the only critical point? For a definition of a critical point see [1, Section 7]. This definition carries over to GSRS in a natural way.

(iii) What can we say about the topology of $\mathcal{G}_d^{(0)}$? For instance, are these sets connected or simply connected? These questions are already interesting for the case $d = 1$. In this case there is some hope to get a complete description of $\mathcal{G}_1^{(0)}$.

(iv) What can we say about the geometry of $\mathcal{G}_d^{(0)}$? For instance, Figure 1 indicates that $\mathcal{G}_1^{(0)}$ is starlike. What is the Hausdorff dimension of the boundary of $\mathcal{G}_d^{(0)}$?

(v) The interior of $\mathcal{G}_d$ is equal to the Schur-Cohn region $\mathcal{E}_d$. However, it is not clear which part of the boundary of $\mathcal{G}_d$ belongs to $\mathcal{G}_d$. For classical shift radix systems the analogous problem was studied for instance in [2, 11]. In the classical case it is solved completely only in dimension one.

(vi) Define and study GSRS tiles (see [4] for tiles related to shift radix systems). The example $r = \frac{2}{3}(1+\mathrm{i})$, $x = -1+\mathrm{i}$ shows that the preimage of γ_r can be empty, *i.e.*, that $\gamma_r^{-1}(x) = \emptyset$. As this cannot occur for classical shift radix systems, it would be interesting to study the effect of this property on the underlying tiles.

(vii) Lemma 3.3 and Proposition 5.1 indicate that GSRS are related to radix representations. What kinds of numeration are hidden behind the notion of GSRS? In Section 5 we established a relation to Gaussian numeration systems in the sense of [6]. This should be generalized to higher dimensions. In particular, we think that GSRS are related to number systems defined in the rings $\mathbb{Z}[\mathrm{i}]/p\mathbb{Z}[\mathrm{i}]$ where $p \in \mathbb{Z}[\mathrm{i}][X]$ (see [1, Section 3] for the relation between shift radix systems and canonical number systems). Moreover, it would be interesting to find an analogue to beta numeration related to GSRS. Here we suggest to study the complex beta transformation $T_\beta : \mathbb{C}/\mathbb{Z}[\mathrm{i}] \to \mathbb{C}/\mathbb{Z}[\mathrm{i}]$ defined by $x \mapsto \{\beta x\}$ for each $\beta \in \mathbb{C}$ (see [1, Section 2] for shift radix systems and beta expansions).

(viii) Define and study generalizations of shift radix systems for other orders $\mathbb{Z}[\alpha]$. Moreover, analogously to Surer [13] one can define and study "ε-GSRS". It seems that for $\varepsilon = 1/2$ these generalized GSRS contain the number systems in imaginary quadratic fields studied by Kátai [8].

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References

[1] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. I*, Acta Math. Hungar., 108 (2005), pp. 207–238.

[2] S. Akiyama, H. Brunotte, A. Pethő, and W. Steiner, *Periodicity of certain piecewise affine planar maps*, Tsukuba J. Math., 32 (2008), pp. 197–251.

[3] S. Akiyama, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. III*, Osaka J. Math., 45 (2008), pp. 347–374.

[4] V. Berthé, A. Siegel, W. Steiner, P. Surer, and J. Thuswaldner, *Fractal tiles associated with shift radix systems*, Adv. Math., 226 (2011), pp. 139–175.

[5] W. J. Gilbert, *Radix representations of quadratic fields*, J. Math. Anal. Appl., 83 (1981), pp. 264–274.

[6] M.-A. Jacob and J.-P. Reveilles, *Gaussian numeration systems*, Actes du colloque de Géométrie Discrète DGCI, (1995).

[7] D. B. Johnson, *Finding all the elementary circuits of a directed graph*, SIAM J. Comput., 4 (1975), pp. 77–84.

[8] I. Kátai, *Number systems in imaginary quadratic fields*, Ann. Univ. Sci. Budapest. Sect. Comput., 14 (1994), pp. 91–103. Festschrift for the 50th birthday of Karl-Heinz Indlekofer.

[9] J. C. Lagarias and Y. Wang, *Self-affine tiles in* $\mathbf{R}^n$, Adv. Math., 121 (1996), pp. 21–49.

[10] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[11] J. Lowenstein, S. Hatjispyros, and F. Vivaldi, *Quasi-periodicity, global stability and scaling in a model of Hamiltonian round-off*, Chaos, 7 (1997), pp. 49–66.

[12] I. Schur, *Über Potenzreihen, die im Inneren des Einheitskreises beschränkt sind II*, J. reine angew. Math., 148 (1918), pp. 122–145.

[13] P. Surer, *ε-shift radix systems and radix representations with shifted digit sets*, Publ. Math. Debrecen, 74 (2009), pp. 19–43.

Haus-Endt-Strasse 88, D-40593 Düsseldorf, GERMANY
E-mail address: brunoth@web.de

Chair of Mathematics and Statistics, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
E-mail address: Peter.Kirschenhofer@unileoben.ac.at

Chair of Mathematics and Statistics, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
E-mail address: Joerg.Thuswaldner@unileoben.ac.at