A NUMBER SYSTEM WITH BASE $-\frac{3}{2}$

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ABSTRACT. In the present article we explore a way to represent numbers with respect to the base $-\frac{3}{2}$ using the set of digits $\{0, 1, 2\}$. Although this number system shares several properties with the classical decimal system, it shows some remarkable differences and reveals interesting new features. For instance, it is related to the field of 2-adic numbers, and to a self-affine "fractal" set that gives rise to a tiling of a non-Euclidean space. Moreover, it has relations to Mahler's $\frac{3}{2}$-problem and to the Josephus problem.

1. INTRODUCTION

A number system is, intuitively, a way of representing a certain set of numbers in a consistent manner, using strings of some given digits in relation to a base. The aim of this article is to define and explore a number system that has base $-\frac{3}{2}$ and uses the set of digits $\{0, 1, 2\}$. We chose this example because it has many beautiful properties that link different areas of mathematics, and nevertheless can be studied in a way that is accessible to a broad readership.

We first give some insight into the more general context in which this number system can naturally be embedded. Then, proceeding in analogy to the decimal system, we start our study of the $(-\frac{3}{2})$-number system by investigating expansions of integers. After that we move on to representations of real numbers, and realize that the real line is somehow "too small" for our number system. We introduce the space $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$, where $\mathbb{Q}_2$ is the field of 2-adic numbers, and show how it naturally arises as a representation space. We relate to our number system the set $\mathcal{F}$ depicted in Figure 1, which has a self-affine structure, like many of the famous fractal sets. We show that the set $\mathcal{F}$ has nice topological properties and prove that it induces a tiling of the space $\mathbb{K}$ by translations. We will see that this tiling property is strongly related to the existence and uniqueness of expansions with respect to the base $-\frac{3}{2}$ in $\mathbb{K}$. At the end of the article we give some applications of the $(-\frac{3}{2})$-number system in number theory.

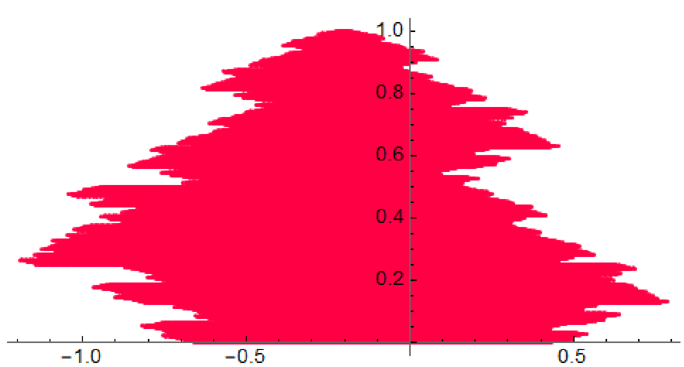

FIGURE 1. The tile $\mathcal{F}$ related to the number system with base $-\frac{3}{2}$.

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2. General context

Over time, many generalizations of the classical number systems came to the fore. They have applications in various areas of mathematics and computer science. To cite some examples, back in 1885 Grünwald [6] studied number systems with negative integers as bases. In 1936 Kempner [9] and later also Rényi [18] proposed expansions with respect to nonintegral real bases (see also [20]). Knuth [11] introduced complex bases and, as we will illustrate below, he dealt with their relations to fractal sets. Gordon [5] discussed the relevance of number systems with varying digit sets in cryptography and in [14] binary and hexadecimal expansions were used in this context. Number systems with rational base similar to ours were introduced by Akiyama et al. [1] and generalizations of these have been studied in [3] and [19] (there are other ways to define a number system with base $-\frac{3}{2}$; see for example the one using the digit set $\{0,1\}$ in [2]).

Number systems with integer bases can be generalized pretty well by using algebraic integers as bases. Recall that an *algebraic integer* is a complex number that is a zero of a monic polynomial with integer coefficients. Consider for example $-1+i$, which is a zero of x^2+2x+2. Every complex number can be expressed in this base using the digits 0 and 1, and associated to this number system is a famous fractal set known as *Knuth's Twin Dragon*, depicted in Figure 2 (see [12, p. 206]). It is defined as the set of "fractional parts" of this number system, that is, the set of points $x = \sum_{k=1}^{\infty}(-1+i)^{-k}d_{-k}$ with $d_{-k} \in \{0,1\}$. This set plays the same role as the interval $[0,1]$ does in the decimal system, and it gives a tiling of the complex plane.

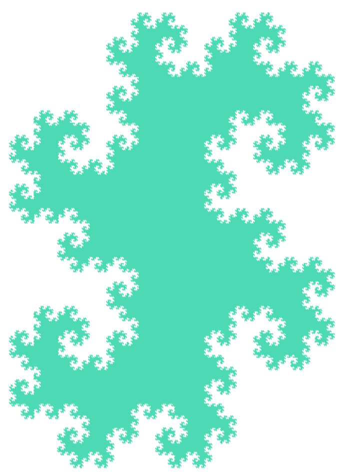

Figure 2. Knuth's Twin Dragon, a set related to a number system with base $-1+i$.

The minimal polynomial of $-\frac{3}{2}$ is $2x+3$, which is not monic; hence $-\frac{3}{2}$ is not an algebraic integer. This is a relevant feature because it means that, serving as a base of a number system, $-\frac{3}{2}$ is of a different nature than integers such as 10 and 2. Steiner and Thuswaldner [22] considered number systems whose base is an algebraic number, not necessarily an algebraic integer. They introduced the so-called *rational self-affine tiles*, defined in terms of these number systems, and proved the existence of lattice tilings arising from them, generalizing important work by Lagarias and Wang [13]. The set $\mathcal{F}$ from Figure 1 is an example of a rational self-affine tile.

When the base under consideration is not an algebraic integer, rational self-affine tiles live in subrings of certain adèle rings, which are important objects in algebraic number theory. In particular, the representation space $\mathbb{K}$ introduced here is an example of such a subring. The reason why we consider $\mathbb{K}$ instead of $\mathbb{R}$ is that $-\frac{3}{2}$ acts in some sense like an algebraic integer in $\mathbb{K}$, because its denominator is "multiplied away" in the 2-adic factor of $\mathbb{K}$. We will explain this in more detail in Section 5.

Another generalization of the $(-\frac{3}{2})$-number system is furnished by *shift radix systems*, or SRS for short. These are simple dynamical systems that provide a unified notion for several types of number systems, and admit an interesting tiling theory (see [10]). In particular, Knuth's Twin Dragon from Figure 2 is an example of a *shift radix system tile*. In our $(-\frac{3}{2})$-example, the SRS tile will just be a line — a line which will turn out to be of considerable interest. We will see in Section 3 that the $(-\frac{3}{2})$-number system relates to the SRS $\tau_{\mathbf{r}}$ with parameter $\mathbf{r} = \frac{2}{3}$. In Section 8 we come back to SRS and provide applications of our number system to Mahler's $\frac{3}{2}$-problem and the Josephus problem, and pose a question related to its sum-of-digits function.

3. Expansions of the integers — and more

In the decimal system, each integer can be expanded without using digits after the decimal point. In this section, we wish to define and explore such "integer expansions" in the number system with *base* $-\frac{3}{2}$ and *digit set* $\mathcal{D} = \{0, 1, 2\}$, which we denote as $(-\frac{3}{2}, \mathcal{D})$. The fact that $-\frac{3}{2}$ is not an integer will entail new properties. The reason for the choice of a negative base is that there will be no need for a minus sign to represent negative numbers and that the tiling theory will be nicer than it would be for a positive base.

To get a feeling for this number system, we first deal with the integers and follow the ideas of [1]. Given $N \in \mathbb{Z}$, we want to find an expansion of the form

$$N = \tfrac{1}{2}\Big(d_k \left(-\tfrac{3}{2}\right)^k + d_{k-1}\left(-\tfrac{3}{2}\right)^{k-1} + \cdots + d_1\left(-\tfrac{3}{2}\right) + d_0\Big) \qquad (d_i \in \mathcal{D}) \tag{1}$$

for some $k \in \mathbb{N}$. If $k \geqslant 1$ we will always assume that $d_k \neq 0$. This avoids padded zero digits on the left which would disturb us later, when we focus on the issue of unique expansions. The factor $\frac{1}{2}$ at the beginning of the expansion in (1) is just there for convenience (and to be consistent with [1]) and will not be crucial. To compute an expansion of the form (1), we use the following algorithm. Write $2N = -3N_1 + d_0$ with $d_0 \in \mathcal{D}$ and $N_1 \in \mathbb{Z}$. Since N is given, d_0 has to be the unique digit satisfying $d_0 \equiv 2N \pmod 3$, so this equation has a unique solution N_1. More generally, we set $N_0 = N$ and recursively define the integers N_{i+1} for $i \geqslant 0$ by the equations

$$2N_i = -3N_{i+1} + d_i \tag{2}$$

with $d_i \in \mathcal{D}$. One can easily see by induction on i that this yields

$$N = \left(-\tfrac{3}{2}\right)^{i+1} N_{i+1} + \tfrac{d_i}{2}\left(-\tfrac{3}{2}\right)^i + \cdots + \tfrac{d_1}{2}\left(-\tfrac{3}{2}\right) + \tfrac{d_0}{2}. \tag{3}$$

If we can prove that $N_k = 0$ for k large enough, our algorithm gives the desired representation (1) for each $N \in \mathbb{Z}$. And this is indeed our first result.

Proposition 1. *Each $N \in \mathbb{Z}$ can be represented in the form* (1) *in a unique way.*

Proof. By (3), it suffices to show that, for each $N = N_0 \in \mathbb{Z}$, the sequence $(N_i)_{i\geqslant 0}$ produced by the recurrence (2) is eventually zero. We have $N_{i+1} = -\frac{2}{3}N_i + \frac{1}{3}d_i$ with $d_i \in \mathcal{D}$; hence $|N_{i+1}| \leqslant \frac{2}{3}|N_i| + \frac{2}{3}$ and therefore $|N_{i+1}| < |N_i|$ holds for each $|N_i| \geqslant 3$. This implies that there is an $i \in \mathbb{N}$ with $|N_i| \leqslant 2$.

Direct calculation shows that if $N_i = -2$, then $N_{i+1} = 2$, $N_{i+2} = -1$, $N_{i+3} = 1$, $N_{i+4} = 0$, and $N_{i+5} = 0$. Thus for each $N = N_0 \in \mathbb{Z}$ there is $k_0 \in \mathbb{N}$ with $N_k = 0$ for all $k \geqslant k_0$, and N has an expansion of the form (1).

Concerning uniqueness of the expansion, we just note that each digit d_i in (1) has to lie in a prescribed residue class modulo 3, and hence is uniquely determined. ☐

We write

$$(d_k \dots d_0)_{-3/2} := \tfrac{1}{2}\sum_{i=0}^{k} d_i \left(-\tfrac{3}{2}\right)^i \qquad (k \in \mathbb{N},\ d_i \in \mathcal{D};\ d_k \neq 0 \text{ whenever } k \geqslant 1)$$

and call $(d_k \dots d_0)_{-3/2}$ an *integer $(-\frac{3}{2})$-expansion*. We proved in Proposition 1 that each $N \in \mathbb{Z}$ has a unique integer $(-\frac{3}{2})$-expansion. For instance, $-3 = (2110)_{-3/2}$ and $4 = (21122)_{-3/2}$.

As a next step, we characterize the set

$$\tfrac{1}{2}\mathcal{D}[-\tfrac{3}{2}] = \left\{(d_k \dots d_0)_{-3/2} \mid k \in \mathbb{N},\ d_i \in \mathcal{D};\ d_k \neq 0 \text{ whenever } k \geqslant 1\right\}$$

of all real numbers with an integer $(-\frac{3}{2})$-expansion (see also [19, Example 3.3]).

Theorem 2. *The set of all numbers having an integer* $(-\frac{3}{2})$*-expansion is* $\mathbb{Z}[\frac{1}{2}]$*. Here, as usual, we set* $\mathbb{Z}[\frac{1}{2}] = \{a2^{-\ell} \mid a \in \mathbb{Z}, \ell \in \mathbb{N}\}$.

Proof. We have to show that $\frac{1}{2}\mathcal{D}[-\frac{3}{2}] = \mathbb{Z}[\frac{1}{2}]$. The inclusion $\frac{1}{2}\mathcal{D}[-\frac{3}{2}] \subset \mathbb{Z}[\frac{1}{2}]$ is trivial. For the reverse inclusion, let $N_0 \in \mathbb{Z}[\frac{1}{2}]$ be arbitrary. There exist $a \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ such that $N_0 = a2^{-\ell}$. We need to find an expansion of N_0 of the form (1) for some $k \in \mathbb{N}$. We define the same recurrence as in (2) but forcing $N_i \in \mathbb{Z}[\frac{1}{2}]$ for $i \geqslant 0$, and proceed to show that $N_k = 0$ for some $k \in \mathbb{N}$.

Since N_1 is given by $2N_0 = -3N_1 + d_0$, where $N_1 \in \mathbb{Z}[\frac{1}{2}]$ and $d_0 \in \mathcal{D}$, we have

$$-3N_1 = 2N_0 - d_0 = \frac{a}{2^{\ell-1}} - d_0 = \frac{a - 2^{\ell-1}d_0}{2^{\ell-1}}. \tag{4}$$

To guarantee that $N_1 \in \mathbb{Z}[\frac{1}{2}]$, the numerator of this fraction has to be divisible by 3, namely, we need to choose d_0 in a way that $2^{\ell-1}d_0 \equiv a \pmod 3$. As the inverse of $2^{\ell-1}$ in $\mathbb{Z}/3\mathbb{Z}$ is $2^{\ell-1}$, we get $d_0 \equiv 2^{\ell-1}a \pmod 3$ and N_1 is uniquely defined by (4).

Iterating (2) yields that $N_i = a_i 2^{-\ell+i}$ for some $a_i \in \mathbb{Z}$ for $i \in \{0, \dots, \ell\}$. After ℓ steps, we get $N_\ell \in \mathbb{Z}$, and we are in the case covered by Proposition 1. This implies that there is $k_0 \in \mathbb{N}$ such that $N_k = 0$ for $k \geqslant k_0$ and thus $N_0 \in \frac{1}{2}\mathcal{D}[-\frac{3}{2}]$. □

By residue class considerations one can show that each $z \in \mathbb{Z}[\frac{1}{2}]$ has a *unique* integer $(-\frac{3}{2})$-expansion. For instance, $-\frac{3}{8} = (120)_{-3/2}$ and $\frac{7}{8} = (111)_{-3/2}$.

Remark. We mentioned in the introduction that the number system $(-\frac{3}{2}, \mathcal{D})$ is related to the shift radix system $\tau_{2/3}$. For each parameter $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$, a shift radix system (SRS) $\tau_\mathbf{r}$ is defined by

$$\tau_\mathbf{r} : \mathbb{Z}^d \to \mathbb{Z}^d, \qquad (x_1, \dots, x_d) \mapsto (x_2, \dots, x_d, -\lfloor r_1x_1 + \cdots + r_dx_d \rfloor),$$

where $\lfloor y \rfloor = \max\{n \in \mathbb{Z} : n \leq y\}$ is the *floor function*. Therefore, $\tau_{2/3} : \mathbb{Z} \to \mathbb{Z}$ is given by $\tau_{2/3}(N) = -\lfloor \frac{2}{3}N \rfloor$. One can show quite easily that the digits d_i of the representation (1) can be obtained in terms of iterations of this SRS, in particular, $d_i = 3\{\frac{2}{3}\tau_{2/3}^i(N)\}$, where $\{y\} := y - \lfloor y \rfloor$ denotes the fractional part of a real (see [3, Lemma 5.5 and Section 5.4]).

4. Expansions with fractional part: The reals

We proceed to study $(-\frac{3}{2})$-expansions for arbitrary reals, allowing negative powers of the base. We consider the matter of uniqueness and motivate the subsequent construction of a tiling.

A desirable property of a number system is that almost all numbers (in a measure-theoretic sense) can be expanded in a unique way. For example, although the decimal expansion is not always unique (e.g., the number 4 can also be written as 3.9999...), the set of numbers admitting more than one decimal expansion is very small in the sense that it has Lebesgue measure zero. This fact is reflected by the following *tiling property*. Consider the set of *fractional parts* in the decimal number system, that is, the set of numbers that can be expanded using only negative powers of 10: this set equals the unit interval $[0, 1]$. On the other hand, the set of numbers whose expansion uses only nonnegative powers of 10, that is, numbers with an *integer expansion*, is equal to $\mathbb{Z}$ (if we permit the use of the minus sign). Consider the collection

$$\{[0, 1] + z \mid z \in \mathbb{Z}\}. \tag{5}$$

This collection covers $\mathbb{R}$, and the only overlaps occur in the boundary points of the intervals, which form a measure zero set. We thus say that $\{[0, 1] + z \mid z \in \mathbb{Z}\}$ forms a *tiling* of the real line. Here $[0, 1]$ is the *central tile* and $\mathbb{Z}$ is the *translation set*. This tiling property is a geometric interpretation of the fact that almost all real numbers admit a unique expansion in the decimal system, and it works the same for any other q-ary number system ($q \in \mathbb{Z}$; $|q| \geqslant 2$). However, we will see that for a real number the representation in the number system $(-\frac{3}{2}, \mathcal{D})$ is a priori never unique. Let

$$(d_k \dots d_0.d_{-1}d_{-2}\dots)_{-3/2} := \frac{1}{2}\sum_{i=-\infty}^{k} d_i \left(-\tfrac{3}{2}\right)^i \qquad (k \in \mathbb{N},\ d_i \in \mathcal{D}) \tag{6}$$

and assume that there are no padded zeros to the left of the decimal point "." (i.e., that $d_k \neq 0$ whenever $k \geq 1$). Consider

$$\Omega = \{(0.d_{-1}d_{-2}\ldots)_{-3/2} \mid d_{-i} \in \mathcal{D}\}, \tag{7}$$

the set of *fractional parts* in $(-\frac{3}{2}, \mathcal{D})$. One can prove that $\Omega = \big[-\frac{6}{5}, \frac{4}{5}\big]$ by using the fact that $-\frac{3}{2}\Omega = \Omega \cup \big(\Omega + \frac{1}{2}\big) \cup (\Omega + 1)$. (We revisit this idea later on in Section 6.)

Each decomposition of a real number x as the sum of an element of $\mathbb{Z}[\frac{1}{2}]$ and an element of Ω leads to an expansion of x of the form (6), by Theorem 2 and the definition of Ω. Because the collection $\{\Omega + z \mid z \in \mathbb{Z}[\frac{1}{2}]\}$ covers the real line, each real number can be written as such a sum, and hence admits an expansion of the form (6). But since each $x \in \mathbb{R}$ is contained in multiple elements of the collection $\{\Omega + z \mid z \in \mathbb{Z}[\frac{1}{2}]\}$ (in fact, in infinitely many), it admits multiple expansions of the form (6). For example, $\frac{4}{5} = (0.020202\ldots)_{-3/2} = (2.11111\ldots)_{-3/2}$.

Different translations of Ω by elements of $\mathbb{Z}[\frac{1}{2}]$ overlap in sets of positive measure; in other words, we do not have the desired tiling property. This results in expansions that are not unique. In the subsequent sections, we will find a way to "embed" the collection $\{\Omega + z \mid z \in \mathbb{Z}[\frac{1}{2}]\}$ in a suitable space where it will give rise to a tiling.

5. The representation space

The real line seems to be "too small" for the collection $\big\{\big[-\frac{6}{5}, \frac{4}{5}\big] + z \mid z \in \mathbb{Z}[\frac{1}{2}]\big\}$, so we wish to enlarge the space $\mathbb{R}$ in order to mend the issue with the overlaps.[1] Indeed, our next goal is to define a new space, called $\mathbb{K}$, in which the number system $(-\frac{3}{2}, \mathcal{D})$ induces a tiling in a natural way. The idea behind this is as follows: the overlaps occur because the three digits $\{0,1,2\}$ are "too many" for a base whose modulus is $\frac{3}{2}$. Such a base would need one and a half digits, which is of course not doable. What causes all the problems is the denominator 2. Roughly speaking, this denominator piles up powers of 2 which are responsible for the overlaps. It turns out that these overlaps can be "unfolded" by adding a 2*-adic factor* to our representation space. The strategy of enlarging the representation space by p-adic factors that we are about to present was used in the setting of substitution dynamical systems, e.g., by Siegel [21] and in a much more general framework than ours in [22].

We begin by introducing the 2-adic numbers; for more on this topic we refer the reader to [7]. Consider a nonzero rational number y and write $y = 2^{\ell}\frac{p}{q}$ where $\ell \in \mathbb{Z}$ and both p and q, are odd. The 2-adic absolute value in $\mathbb{Q}$ is defined by

$$|y|_2 = \begin{cases} 2^{-\ell}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0, \end{cases}$$

and the 2-adic distance between two rationals x and y is given by $|x-y|_2$. Two points are close under this metric if their difference is divisible by a large positive power of 2.

We define $\mathbb{Q}_2$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_2$. The space $\mathbb{Q}_2$ is a field called the *field of* 2*-adic numbers*. Every nonzero $y \in \mathbb{Q}_2$ can be written uniquely as a series

$$y = \sum_{i=\ell}^{\infty} c_i 2^i \qquad (\ell \in \mathbb{Z},\ c_i \in \{0,1\},\ c_\ell \neq 0).$$

This series converges in $\mathbb{Q}_2$ because large powers of two have small 2-adic absolute value. Indeed, we have $|y|_2 = 2^{-\ell}$.

We define our *representation space* as $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$, with the additive group structure given by componentwise addition. Moreover, $\mathbb{Z}[\frac{1}{2}]$ acts on $\mathbb{K}$ by multiplication, more precisely, if $\alpha \in \mathbb{Z}[\frac{1}{2}]$ and $(x_1, x_2) \in \mathbb{K}$, then $\alpha \cdot (x_1, x_2) = (\alpha x_1, \alpha x_2)$.

For every $(x_1, x_2), (y_1, y_2) \in \mathbb{K}$ define

$$\mathbf{d}((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|_2\}.$$

Then $\mathbf{d}$ is a metric on $\mathbb{K}$. Intuitively, two points in $\mathbb{K}$ are far apart if either their real components are far apart or their 2-adic components are far apart.

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[1] Another way, which we do not pursue here, would be to restrict the "admissible" digit strings; see [1].

We define the embedding

$$\varphi : \mathbb{Q} \to \mathbb{K}, \quad z \mapsto (z, z).$$

Consider the image of $\mathbb{Z}[\frac{1}{2}]$ under φ. Despite both coordinates of $\varphi(z)$ being the same, the set $\varphi(\mathbb{Z}[\frac{1}{2}])$ does not lie on a "diagonal" as it would if it were embedded in $\mathbb{R}^2$. Indeed, points of $\mathbb{Z}[\frac{1}{2}]$ that are close in $\mathbb{R}$ are far apart in the 2-adic distance. In particular, we will show that the points of $\varphi(\mathbb{Z}[\frac{1}{2}])$ form a *lattice*.

A subset Λ of $\mathbb{K}$ is a *lattice* if it satisfies the three following conditions.

(1) Λ is a group.
(2) Λ is uniformly discrete, meaning there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}$ contains at most one point of Λ.
(3) Λ is relatively dense, meaning there exists $R > 0$ such that every closed ball of radius R in $\mathbb{K}$ contains at least one point of Λ.

Lemma 3. *$\varphi(\mathbb{Z}[\frac{1}{2}])$ is a lattice in $\mathbb{K}$.*

Proof. (1) The fact that $\varphi(\mathbb{Z}[\frac{1}{2}])$ is a group follows from the additive group structure of $\mathbb{Z}[\frac{1}{2}]$ because φ is a group homomorphism.

(2) To get uniform discreteness of $\varphi(\mathbb{Z}[\frac{1}{2}])$, we show first that $\mathbf{d}(\varphi(z), \varphi(0)) \geqslant 1$ holds for every nonzero $z \in \mathbb{Z}[\frac{1}{2}]$. Recall that $\mathbf{d}(\varphi(z), \varphi(0)) = \max\{|z|, |z|_2\}$. If $|z| < 1$, there exist $a, \ell \in \mathbb{Z}$ with a odd and $\ell \geq 1$ such that $z = a2^{-\ell}$, so $|z|_2 = 2^\ell > 1$. Because of the group structure, this implies that the distance between any two elements of $\varphi(\mathbb{Z}[\frac{1}{2}])$ is at least one; hence $\varphi(\mathbb{Z}[\frac{1}{2}])$ is uniformly discrete.

(3) For the relative denseness, consider an arbitrary element $(x_1, x_2) \in \mathbb{K}$. We claim that there exists $z \in \mathbb{Z}[\frac{1}{2}]$ such that $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$. Let $z_1 \in \mathbb{Z}$ be one of the integers being closest to x_1. If $x_2 = \sum_{i=\ell}^{\infty} c_i 2^i$ with $\ell \in \mathbb{Z}$ and $c_i \in \{0, 1\}$, then set $z_2 = \sum_{i=\ell}^{-1} c_i 2^i \in \mathbb{Z}[\frac{1}{2}]$ (note that $z_2 = 0$ if $\ell \geqslant 0$). Therefore,

$$\mathbf{d}((x_1, x_2), (z_1, z_2)) = \max\{|x_1 - z_1|, |x_2 - z_2|_2\} \leqslant 1.$$

Now we set $z = z_1 + z_2 \in \mathbb{Z}[\frac{1}{2}]$. Because z_1 is an integer, $|z_1|_2 \leqslant 1$, and since $z_2 \in [0, 1]$, we have $|z_2| \leqslant 1$. Thus $\mathbf{d}(\varphi(z), (z_1, z_2)) = \max\{|z_2|, |z_1|_2\} \leqslant 1$, and so $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$ by the triangle inequality; hence $\varphi(\mathbb{Z}[\frac{1}{2}])$ is relatively dense. $\square$

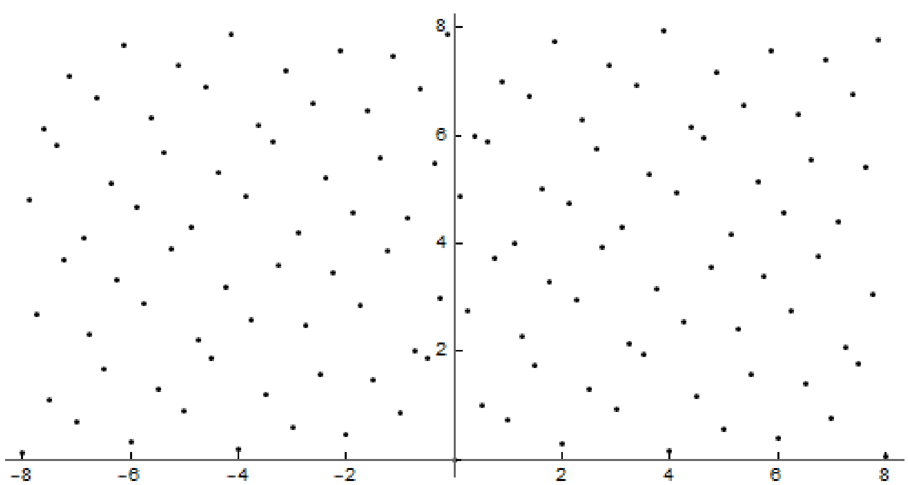

FIGURE 3. Representation of the lattice points $\varphi(\frac{j}{8})$ for $-64 \leqslant j \leqslant 64$ in $\mathbb{R}^2$.

Figure 3 illustrates some points of $\varphi(\mathbb{Z}[\frac{1}{2}])$. Drawing pictures in this setting is not straightforward: the space $\mathbb{K}$ is non-Euclidean, so we need to represent it in $\mathbb{R}^2$ while somehow maintaining the 2-adic nature of the second component. We do this in the following way: any point $y \in \mathbb{Q}_2$ can be written uniquely (up to padded zeros) as a series $y = \frac{1}{2}\sum_{i=\ell}^{\infty} c_i(-\frac{2}{3})^i$ with $\ell \in \mathbb{Z}$, $c_i \in \{0,1\}$, that converges in the 2-adic metric. (This is a 2-adic expansion, not a $(-\frac{3}{2})$-expansion!) We consider the mapping

$$\gamma : \mathbb{Q}_2 \to \mathbb{R}; \qquad \frac{1}{2}\sum_{i=\ell}^{\infty} c_i(-\tfrac{2}{3})^i \mapsto \sum_{i=\ell}^{\infty} c_i 2^{-i}, \tag{8}$$

which is well-defined since the sum on the right-hand side converges in $\mathbb{R}$. A point $(x_1, x_2) \in \mathbb{K}$ is now represented as $(x_1, \gamma(x_2)) \in \mathbb{R}^2$. The reason for the choice of this embedding is that, in $\mathbb{Q}_2$, multiplying by $-\frac{3}{2}$ has the same effect as dividing by 2 in $\mathbb{R}$. We note also that in [22] a very similar embedding was used to illustrate rational self-affine tiles.

The lattice $\varphi(\mathbb{Z}[\frac{1}{2}])$, which will play the role of the "integers" in $\mathbb{K}$, turns out to be a proper translation set for a tiling of $\mathbb{K}$ related to the number system $(-\frac{3}{2}, \mathcal{D})$.

In the real case, when defining a tiling we allowed overlaps as long as it was on a set of Lebesgue measure zero. In order to generalize this, we need to define a natural measure on $\mathbb{K}$. Such a *Haar measure* is a translation-invariant Borel measure that is finite for compact sets. It can be defined in spaces with a sufficiently "nice" structure. (More specifically, it is defined on locally compact topological groups.) A Haar measure is unique up to a scaling factor. The Lebesgue measure μ_∞ on $\mathbb{R}$ is a Haar measure.

Let $\mathbb{Z}_2 = \left\{\sum_{i=0}^{\infty} c_i 2^i \mid c_i \in \{0,1\}\right\} \subset \mathbb{Q}_2$ be the ring of 2-adic integers, and let μ_2 be the Haar measure on $\mathbb{Q}_2$ that satisfies $\mu_2(2^\ell \mathbb{Z}_2) = 2^{-\ell}$. Note that multiplying a set by large powers of 2 makes its measure μ_2 small.

Let $\mu = \mu_\infty \times \mu_2$ be the product measure of μ_∞ and μ_2 on $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$; that is, if $M_1 \subset \mathbb{R}$ and $M_2 \subset \mathbb{Q}_2$ are respectively measurable, then the sets of the form $M = M_1 \times M_2$ generate the σ-algebra of μ, and $\mu(M) := \mu_\infty(M_1)\mu_2(M_2)$. One can show that μ is a Haar measure on $\mathbb{K}$. For any measurable set $M = M_1 \times M_2 \subset \mathbb{K}$, we have $\mu_\infty(-\frac{3}{2}M_1) = \frac{3}{2}\mu_\infty(M_1)$ and $\mu_2(-\frac{3}{2}M_2) = 2\mu_2(M_2)$, which yields

$$\mu(-\tfrac{3}{2}M) = \mu_\infty(-\tfrac{3}{2}M_1)\mu_2(-\tfrac{3}{2}M_2) = 3\mu(M). \tag{9}$$

Thus multiplying any measurable set $M \subset \mathbb{K}$ by the base $-\frac{3}{2}$ enlarges the measure by the factor 3, which can be interpreted as having "enough space" for three digits. This is what we meant in Section 2 by saying that we "multiply away" the denominator of $-\frac{3}{2}$ by using the 2-adic factor.

6. The tile $\mathcal{F}$

In this section, we define a set $\mathcal{F} \subset \mathbb{K}$ that plays the same role for the number system $(-\frac{3}{2}, \mathcal{D})$ as the unit interval does for the decimal system. We explore some of its topological and measure-theoretic properties.

Recall that in (7) we defined the set Ω of fractional parts, consisting of elements of the form $(0.d_{-1}d_{-2}\ldots)_{-3/2}$. We now embed the digits in $\mathbb{K}$, obtaining the set

$$\mathcal{F} := \Big\{ \frac{1}{2}\sum_{i=1}^{\infty} \left(-\tfrac{3}{2}\right)^{-i} \varphi(d_{-i}) \mid d_{-i} \in \mathcal{D} \Big\}. \tag{10}$$

The set $\mathcal{F}$ is a compact subset of $\mathbb{K}$. Indeed, given any sequence in $\mathcal{F}$, we can use a Cantor diagonal argument to find a convergent subsequence.

Let $x \in \mathcal{F}$. If we multiply x by the base $-\frac{3}{2}$, we obtain $-\frac{3}{2}x \in \mathcal{F} + \frac{1}{2}\varphi(d_{-1})$ with $d_{-1} \in \{0,1,2\}$. (This can be interpreted as the analog of moving the decimal point one place to the right.) Thus $\mathcal{F}$ satisfies the *set equation*

$$-\tfrac{3}{2}\mathcal{F} = \mathcal{F} \cup \left(\mathcal{F} + \varphi(\tfrac{1}{2})\right) \cup \left(\mathcal{F} + \varphi(1)\right) \tag{11}$$

in $\mathbb{K}$, which can be written more compactly as $-\frac{3}{2}\mathcal{F} = \mathcal{F} + \frac{1}{2}\varphi(\mathcal{D})$. It turns out that this set equation completely characterizes $\mathcal{F}$. Note that (11) is equivalent to

$$\mathcal{F} = \left(-\tfrac{2}{3}\right)\mathcal{F} \cup \left(-\tfrac{2}{3}\right)\left(\mathcal{F} + \varphi(\tfrac{1}{2})\right) \cup \left(-\tfrac{2}{3}\right)\left(\mathcal{F} + \varphi(1)\right), \tag{12}$$

and multiplying by $-\frac{2}{3}$ is a uniform contraction in $\mathbb{K}$: it is a contraction in $\mathbb{R}$ because $|-\frac{2}{3}| < 1$ and also in $\mathbb{Q}_2$ because $|-\frac{2}{3}|_2 = \frac{1}{2} < 1$. Thus (12) states that $\mathcal{F}$ is equal to the union of three contracted copies of itself. Because of this contraction property we may apply Hutchinson's theorem (see [8]) which says that there exists a unique nonempty compact subset of $\mathbb{K}$ that satisfies the set equation (12). Thus $\mathcal{F}$ is uniquely defined as the nonempty compact set satisfying (12) (or, equivalently, (11)). The set $\mathcal{F}$ is an example of a *rational self-affine tile* in the sense of [22].

Remark. The term *fractal* has no formal mathematical definition. However, the set $\mathcal{F}$ has two features that we have in mind when informally talking about a "fractal" set: its self-affine structure and the locally intricate shape of its boundary. Many famous fractals, like the *Sierpiński triangle* and the *Cantor set*, are solutions of set equations of form similar to (11). Moreover, there exist many fractals in nature which show strong self-affinity, like the leaf of the *fern* and the beautiful *Romanesco broccoli*.

Since according to Theorem 2 the set $\mathbb{Z}[\frac{1}{2}]$ is the analog of $\mathbb{Z}$ in the number system $(-\frac{3}{2}, \mathcal{D})$, we define the analog of the collection in (5) by setting

$$\mathcal{C} = \{\mathcal{F} + \varphi(z) \mid z \in \mathbb{Z}[\tfrac{1}{2}]\}.$$

Then $\mathcal{C}$ is a collection of copies of $\mathcal{F}$ translated by elements of the lattice $\varphi(\mathbb{Z}[\frac{1}{2}])$. We will show in Theorem 6 that $\mathcal{C}$ is a *tiling* of $\mathbb{K}$, meaning that:

(1) $\mathcal{C}$ is a *covering* of $\mathbb{K}$, i.e., $\langle\mathcal{C}\rangle = \mathbb{K}$, where $\langle\mathcal{C}\rangle = \mathcal{F} + \varphi(\mathbb{Z}[\frac{1}{2}])$ is the union of the elements of $\mathcal{C}$.
(2) Almost every point in $\mathbb{K}$ (with respect to the measure μ) is contained in exactly one element of $\mathcal{C}$. In particular, the elements of $\mathcal{C}$ have disjoint interiors.

Figure 1 shows a representation of $\mathcal{F}$ in $\mathbb{R}^2$, again using the function γ from (8) to map $\mathbb{Q}_2$ to $\mathbb{R}$. Figure 4 shows a patch of $\mathcal{C}$; the translates of $\mathcal{F}$ appear to have different shapes, but this is

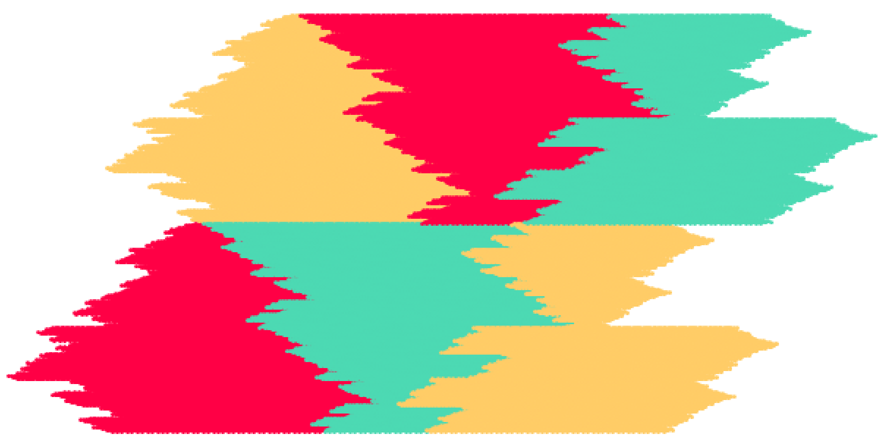

FIGURE 4. A patch of the tiling $\mathcal{C}$ of $\mathbb{K}$ by translates of $\mathcal{F}$.

due to the embedding of $\mathbb{K}$ in $\mathbb{R}^2$. The illustration indicates, however, that the different translates of $\mathcal{F}$ do not overlap other than along their boundaries. The bottom line of the tiles in Figure 1 is the real axis.

In order to define a tiling, it is necessary for our central tile $\mathcal{F}$ to have reasonably nice topological properties. In particular, we prove that $\mathcal{F}$ is the closure of its interior and that its boundary $\partial\mathcal{F}$ has measure zero. In a general setting, this result is contained in [22, Theorem 1].

Theorem 4. *$\mathcal{F}$ is the closure of its interior.*

Proof. We first prove that $\mathcal{C}$ is a covering of $\mathbb{K}$, i.e., $\langle\mathcal{C}\rangle = \mathbb{K}$, where $\langle\mathcal{C}\rangle$ is the union of the elements of $\mathcal{C}$. Applying the set equation (11), we obtain

$$-\tfrac{3}{2}\langle\mathcal{C}\rangle = -\tfrac{3}{2}\mathcal{F} - \tfrac{3}{2}\varphi(\mathbb{Z}[\tfrac{1}{2}]) = \mathcal{F} + \tfrac{1}{2}\varphi(\mathcal{D}) - \tfrac{3}{2}\varphi(\mathbb{Z}[\tfrac{1}{2}]).$$

Note that $\frac{1}{2}\mathcal{D} = \{0, \frac{1}{2}, 1\}$ is a complete set of representatives of residue classes of $\mathbb{Z}[\frac{1}{2}]/(-\frac{3}{2})\mathbb{Z}[\frac{1}{2}]$, so $\frac{1}{2}\varphi(\mathcal{D}) - \frac{3}{2}\varphi(\mathbb{Z}[\frac{1}{2}]) = \varphi(\mathbb{Z}[\frac{1}{2}])$. Thus $-\frac{3}{2}\langle\mathcal{C}\rangle = \langle\mathcal{C}\rangle$ and, a fortiori, for any $k \in \mathbb{N}$ we have $(-\frac{2}{3})^k\langle\mathcal{C}\rangle = \langle\mathcal{C}\rangle$. Recall that multiplying by $-\frac{2}{3}$ is a contraction in $\mathbb{K}$. We have shown in Lemma 3 that $\varphi(\mathbb{Z}[\frac{1}{2}])$ is a relatively dense set in $\mathbb{K}$, and therefore so is $\langle\mathcal{C}\rangle$, meaning there is some $R > 0$ for which every closed ball of radius R intersects $\langle\mathcal{C}\rangle$. But since $\langle\mathcal{C}\rangle$ is invariant under contractions by $(-\frac{2}{3})^k$, this implies that any ball of radius $(\frac{2}{3})^k R$ with $k \in \mathbb{N}$ intersects $\langle\mathcal{C}\rangle$; hence it is dense in $\mathbb{K}$.

Consider now an arbitrary point in $x \in \mathbb{K}$ and a bounded neighborhood V of x. Since $\varphi(\mathbb{Z}[\frac{1}{2}])$ is uniformly discrete and V is bounded, V intersects only a finite number of translates of $\mathcal{F}$, each of which is compact. Since $\langle\mathcal{C}\rangle$ is dense in $\mathbb{K}$, x cannot be at positive distance from all these translates of $\mathcal{F}$. Thus x is contained in some translate of $\mathcal{F}$ and, hence, $x \in \langle\mathcal{C}\rangle$. Since x was arbitrary, this implies that $\langle\mathcal{C}\rangle = \mathbb{K}$.

Next, we show that $\operatorname{int}\mathcal{F} \neq \varnothing$. Assume on the contrary that $\operatorname{int}\mathcal{F} = \varnothing$. Consider the sets $U_z := \mathbb{K} \setminus (\mathcal{F} + \varphi(z))$ with $z \in \mathbb{Z}[\frac{1}{2}])$. By assumption, U_z is dense in $\mathbb{K}$ for each $z \in \mathbb{Z}[\frac{1}{2}]$, and $\{U_z \mid z \in \mathbb{Z}[\frac{1}{2}]\}$ is a countable collection. Baire's theorem asserts that a countable intersection of dense sets is dense. But

$$\bigcap_{z\in\mathbb{Z}[\frac{1}{2}]} U_z = \mathbb{K} \setminus \bigcup_{z\in\mathbb{Z}[\frac{1}{2}]} \mathcal{F} + \varphi(z) = \mathbb{K} \setminus \langle\mathcal{C}\rangle = \varnothing,$$

which is clearly not dense. This contradiction yields $\operatorname{int}\mathcal{F} \neq \varnothing$.

We now prove the result. Iterating the set equation (11) for $k \in \mathbb{N}$ times yields

$$\mathcal{F} = \left(-\tfrac{2}{3}\right)^k \mathcal{F} + \tfrac{1}{2}\left(\left(-\tfrac{2}{3}\right)^k \varphi(\mathcal{D}) + \left(-\tfrac{2}{3}\right)^{k-1} \varphi(\mathcal{D}) + \cdots + \left(-\tfrac{2}{3}\right) \varphi(\mathcal{D})\right).$$

Setting

$$\mathcal{D}_k := \mathcal{D} + \left(-\tfrac{3}{2}\right)\mathcal{D} + \cdots + \left(-\tfrac{3}{2}\right)^{k-1}\mathcal{D}, \tag{13}$$

this becomes

$$\mathcal{F} = (-\tfrac{2}{3})^k(\mathcal{F} + \tfrac{1}{2}\varphi(\mathcal{D}_k)) \qquad (k \in \mathbb{N}), \tag{14}$$

which means we can write $\mathcal{F}$ as a finite union of arbitrarily small shrunken and translated copies of itself. We know that $\mathcal{F}$ has an inner point x, and therefore each copy of the form $(-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d))$, $d \in \mathcal{D}_k$, has an inner point. Thus for any $y \in \mathcal{F}$ and any $\varepsilon > 0$ we can choose $k \in \mathbb{N}$ and $d \in \mathcal{D}_k$ so that $\operatorname{diam}((-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d))) < \varepsilon$ and $y \in (-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d))$. Thus there is an inner point at distance less than ε from y. Since $y \in \mathcal{F}$ and $\varepsilon > 0$ were arbitrary, this proves the result. □

Theorem 5. *The boundary of $\mathcal{F}$ has measure zero.*

Proof. Let x be an inner point of $\mathcal{F}$ and $B_\varepsilon(x) \subset \mathcal{F}$ an open ball of radius $\varepsilon > 0$ centered at x. Because multiplication by $-\frac{2}{3}$ is a uniform contraction in $\mathbb{K}$, there is $k \in \mathbb{N}$ such that $\operatorname{diam}(-\frac{2}{3})^k\mathcal{F} < \varepsilon$. Thus by (14) there is $d_0 \in \mathcal{D}_k$ such that

$$\left(-\tfrac{2}{3}\right)^k \left(\mathcal{F} + \tfrac{1}{2}\varphi(d_0)\right) \subset B_\varepsilon(x) \subset \operatorname{int}\mathcal{F}.$$

Let $y \in \partial\big(\left(-\frac{2}{3}\right)^k \left(\mathcal{F} + \frac{1}{2}\varphi(d_0)\right)\big) \subset \operatorname{int}\mathcal{F}$. Since y is also an inner point of $\mathcal{F}$, and (14) exhibits $\mathcal{F}$ as a finite union of compact sets, y must lie in $(-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d))$ for some $d \in \mathcal{D}_k \setminus \{d_0\}$. Thus the boundary $\partial((-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d_0)))$ is covered at least twice by the collection $\{(-\frac{2}{3})^k(\mathcal{F} + \frac{1}{2}\varphi(d)) \mid d \in \mathcal{D}_k\}$. This entails that

$$\begin{aligned}\mu(\mathcal{F}) &= \mu\big(\bigcup_{d\in\mathcal{D}_k} \left(-\tfrac{2}{3}\right)^k \left(\mathcal{F} + \tfrac{1}{2}\varphi(d)\right)\big)\\ &\leqslant \sum_{d\in\mathcal{D}_k} \mu\big(\left(-\tfrac{2}{3}\right)^k \left(\mathcal{F} + \tfrac{1}{2}\varphi(d)\right)\big) - \mu\big(\partial\big(\left(-\tfrac{2}{3}\right)^k \left(\mathcal{F} + \tfrac{1}{2}\varphi(d_0)\right)\big)\big).\end{aligned}$$

Note that (as a Haar measure) μ is translation invariant, the cardinality of $\mathcal{D}_k$ is 3^k, and from (9) it follows that $\mu((-\frac{2}{3})^k\mathcal{F}) = 3^{-k}\mu(\mathcal{F})$. All this combined yields

$$\mu(\mathcal{F}) \leqslant \sum_{d\in\mathcal{D}_k} \mu\big(\big(-\tfrac{2}{3}\big)^k\mathcal{F}\big) - \mu\big(\partial\big(\big(-\tfrac{2}{3}\big)^k\mathcal{F}\big)\big) \leqslant \mu(\mathcal{F}) - \mu\big(\partial\big(\big(-\tfrac{2}{3}\big)^k\mathcal{F}\big)\big)$$

and therefore $\mu(\partial((-\frac{2}{3})^k\mathcal{F})) = 0$. This implies that $\mu(\partial\mathcal{F}) = 0$. □

7. The tiling

This section contains a tiling theorem for the $(-\frac{3}{2})$-number system. This result is contained in [22, Theorem 2] in a more general setting. For our special case, the proof is much simpler. We will also show that the tiling property relates to the uniqueness (almost everywhere) of expansions in the $(-\frac{3}{2})$-number system embedded in $\mathbb{K}$.

Theorem 6. *The collection $\mathcal{C} = \{\mathcal{F} + \varphi(z) \mid z \in \mathbb{Z}[\frac{1}{2}]\}$ forms a tiling of $\mathbb{K}$.*

Proof. We have shown in the proof of Theorem 4 that $\mathcal{C}$ is a covering of $\mathbb{K}$. It remains to show that almost every point of $\mathbb{K}$ is covered by exactly one element of the collection $\mathcal{C}$. Recall that for each $k \geqslant 1$, the sets $\frac{1}{2}\mathcal{D}_k$ (see (13)) consist of all the integer $(-\frac{3}{2})$-expansions with at most k digits. According to Theorem 2, the set $\mathbb{Z}[\frac{1}{2}]$ is the set of all integer $(-\frac{3}{2})$-expansions. This implies that $\mathbb{Z}[\frac{1}{2}] = \bigcup_{k\geqslant 1}\frac{1}{2}\mathcal{D}_k$ and, hence, $\mathbb{K} = \mathcal{F} + \varphi(\mathbb{Z}[\frac{1}{2}]) = \bigcup_{k\geqslant 1}\mathcal{F} + \frac{1}{2}\varphi(\mathcal{D}_k)$. Therefore, it suffices to prove that the collection $\{\mathcal{F} + \frac{1}{2}\varphi(d) \mid d \in \mathcal{D}_k\}$ has *essentially disjoint* elements for each $k \geqslant 1$, that is, if $d, d' \in \mathcal{D}_k$ are distinct then $\mu((\mathcal{F} + \frac{1}{2}\varphi(d)) \cap (\mathcal{F} + \frac{1}{2}\varphi(d'))) = 0$. Applying (14) we obtain

$$3^k\mu(\mathcal{F}) = \mu((-\tfrac{3}{2})^k\mathcal{F}) = \mu\Big(\bigcup_{d\in\mathcal{D}_k}\mathcal{F} + \tfrac{1}{2}\varphi(d)\Big) \leqslant \sum_{d\in\mathcal{D}_k}\mu(\mathcal{F} + \tfrac{1}{2}\varphi(d)) = 3^k\mu(\mathcal{F}).$$

This implies equality everywhere and, hence, different $\frac{1}{2}\varphi(\mathcal{D}_k)$-translates of $\mathcal{F}$ only overlap in sets of measure zero. Thus the same is true for different $\mathbb{Z}[\frac{1}{2}]$-translates of $\mathcal{F}$. So the tiles in $\mathcal{C}$ are essentially disjoint, and $\mathcal{C}$ is a tiling. □

Corollary 7. *Almost every point $x \in \mathbb{K}$ has a unique expansion of the form*

$$x = \tfrac{1}{2}\sum_{i=-\infty}^{k}\big(-\tfrac{3}{2}\big)^i\varphi(d_i) \qquad (k \in \mathbb{N},\ d_i \in \mathcal{D};\ d_k \neq 0 \text{ whenever } k \geqslant 1). \tag{15}$$

Proof. Let $x \in \mathbb{K}$ and suppose it has two different expansions

$$x = \tfrac{1}{2}\sum_{i=-\infty}^{k}\big(-\tfrac{3}{2}\big)^i\varphi(d_i) = \tfrac{1}{2}\sum_{i=-\infty}^{k}\big(-\tfrac{3}{2}\big)^i\varphi(d_i'),$$

where $d_k \neq 0$ for $k \geq 1$ and where we pad the second expansion with zeros if necessary. Let $m \leqslant k$ be the largest integer such that $d_m \neq d_m'$, and consider the point $(-\frac{3}{2})^{-m}x$. Recall that multiplying x by $(-\frac{3}{2})^{-m}$ is the analog of moving the decimal point m places to the left if m is positive and to the right if it is negative. Let $\omega := (d_k \dots d_{m+1}d_m)_{-3/2}$ and $\omega' := (d_k' \dots d_{m+1}'d_m')_{-3/2}$. Then $\omega, \omega' \in \mathbb{Z}[\frac{1}{2}]$ are distinct, and it follows from our assumption and the definition of the tile $\mathcal{F}$ that $(-\frac{3}{2})^{-m}x - \varphi(\omega), (-\frac{3}{2})^{-m}x - \varphi(\omega') \in \mathcal{F}$. Hence, we obtain

$$(-\tfrac{3}{2})^{-m}x \in (\mathcal{F} + \varphi(\omega)) \cap (\mathcal{F} + \varphi(\omega')).$$

As tiles only overlap on their boundaries, this implies that $x \in (-\frac{3}{2})^m\partial(\mathcal{F} + \varphi(\omega))$. Therefore, a point $x \in \mathbb{K}$ has two different expansions if and only if $x \in \mathit{\Gamma}$, where $\mathit{\Gamma} := \bigcup_{m\in\mathbb{Z}}(-\frac{3}{2})^m\partial(\mathcal{F} + \varphi(\mathbb{Z}[\frac{1}{2}]))$. Since $\mathbb{Z}[\frac{1}{2}]$ is countable, $\mathit{\Gamma}$ is a countable union of the sets $(-\frac{3}{2})^m\partial(\mathcal{F} + \varphi(z))$, $m \in \mathbb{Z}$, $z \in \mathbb{Z}[\frac{1}{2}]$, each of which has measure 0. Thus $\mu(\mathit{\Gamma}) = 0$, which gives the result. □

8. Applications

In this final section we relate our object of study to some problems in number theory. In this context, shift radix systems (SRS) will play a role again. In the remark at the end of Section 3 we showed how the expansion of integers in $(-\frac{3}{2}, \mathcal{D})$ relates to the map $\tau_{2/3} : \mathbb{Z} \to \mathbb{Z}$ given by $\tau_{2/3}(N) = -\lfloor \frac{2}{3} N \rfloor$, which is an example of an SRS. For each parameter $\mathbf{r}$, the mapping $\tau_{\mathbf{r}}$ can be used to define so-called *SRS tiles* in a quite easy way (see [10]). We do not give the definition of these tiles. Instead, we just mention that, according to [22, Proposition 6.15], for rational parameters $\mathbf{r} = r_1 \in \mathbb{Q}$, an SRS tile is just the intersection of a rational self-affine tile with the real line. In our particular example, the (central) SRS tile associated to the map $\tau_{2/3}$ is therefore given by

$$\mathcal{F} \cap \mathbb{R} := \{z \in \mathbb{R} \mid (z, 0) \in \mathcal{F}\},$$

which is a line. (Note that when writing $\mathcal{F} \cap \mathbb{R}$ we identify $\mathbb{R} \times \{0\}$ with $\mathbb{R}$.)

8.1. **Mahler's Problem.** Rational base number systems can be used to shed some light on *Mahler's* $\frac{3}{2}$*-problem* (see [15]), a well-known question in number theory concerning the distribution of powers of rationals modulo 1. It is formulated as follows: We say that $z \in \mathbb{R}$ is a Z*-number* if the sequence of fractional parts $\{z(\frac{3}{2})^n\}$, $n \geq 1$, is contained in the interval $[0, \frac{1}{2}]$. The (still open) question stated by Mahler is: Do Z-numbers exist? In [15], Mahler proves that the set of Z-numbers is at most countable.

More generally, given a rational number $\frac{p}{q}$ with $|\frac{p}{q}| > 1$ and a set $I \subsetneq [0, 1]$, one can ask for which $z \in \mathbb{R}$ the sequence $\{z(\frac{p}{q})^n\}$, $n \geq 1$, is *eventually contained* in I, meaning $\{z(\frac{p}{q})^n\} \in I$ for all sufficiently large n. This can be answered for particular sets I using rational base number systems (see [1]). Indeed, consider the number system $(\frac{3}{2}, \mathcal{D})$ with $\mathcal{D} = \{0, 1, 2\}$; it is defined analogously as $(-\frac{3}{2}, \mathcal{D})$. Associated to this number system is the tile $\mathcal{F}_{3/2}$, defined by replacing $-\frac{3}{2}$ by $\frac{3}{2}$ in the set equation (10). A k*-subtile* is an element of the kth iteration of the set equation of $\mathcal{F}_{3/2}$. Let

$$Z_{-3/2}(I) = \left\{z \in \mathbb{R} \;\mid\; \{z(-\tfrac{3}{2})^n\} \text{ is eventually contained in } I\right\}.$$

By [1, Theorem 49] the set $Z_{-3/2}([0, \frac{1}{3}] \cup [\frac{2}{3}, 1])$ is countably infinite and equals the set of all $z \in \mathbb{R}$ such that $(z, 0) \in \mathbb{K}$ has multiple $\frac{3}{2}$-expansions. (The "$-$" sign makes no difference because $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ (mod 1) is symmetric around 0.) With our theory we can give a geometric interpretation of this result! Indeed, analogously to Corollary 7, one can show that points with multiple $\frac{3}{2}$-expansions correspond to points on the real line that are located in more than one k*-subtile* of $\mathcal{F}_{3/2}$ for some $k \in \mathbb{N}$. Thus $Z_{-3/2}([0, \frac{1}{3}] \cup [\frac{2}{3}, 1])$ consists of all points in $\mathbb{R}$ in which two k-subtiles of $\mathcal{F}_{3/2}$ meet, and we arrive at the following result.

Corollary 8. *The sequence* $(\{z(-\frac{3}{2})^n\})_{n\geq 1}$ *stays eventually in* $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ *if and only if* $(z, 0) \in \mathbb{K}$ *is contained in more than one* k*-subtile of* $\mathcal{F}_{3/2}$ *for some* $k \in \mathbb{N}$.

In the language of SRS, these points $z \hookrightarrow (z, 0) \in \mathbb{K}$ are the intersections of k-subtiles of the SRS tile $\mathcal{F}_{3/2} \cap \mathbb{R}$ associated with $\tau_{-2/3}$ (which is again a line). We strongly believe that there is a relation between sets of the type $Z_{-3/2}(I)$, $I \subsetneq [0, 1]$, and the set of points on the real line on which two k-subtiles of our tile $\mathcal{F}$ meet (see Figure 5). Can this be clarified by combining our study with [1]?

8.2. **Josephus Problem.** This famous riddle goes back to the Jewish historian Flavius Josephus (see [4, Book 3, Chapter 8, Part 7]), and has a simple formulation: given two positive integers m and p, consider a group of m people standing in a circle, numbered clockwise from 1 to m. Starting at position 1, the first $p - 1$ people are skipped, and the pth person is executed. The procedure is repeated with the remaining people, starting with the next person, going clockwise and skipping $p - 1$ people, until only one person remains, who is freed. Where should a person be positioned in the initial circle in order to avoid execution? We denote the answer to this question by $J_p(m)$.

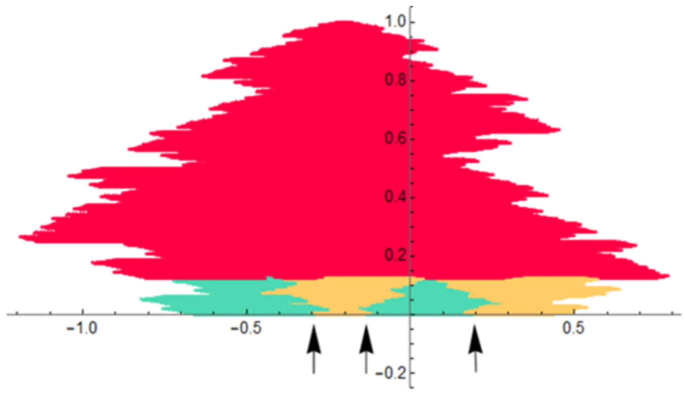

FIGURE 5. In green and orange are depicted the 3-subtiles of $\mathcal{F}$ that meet the real line. The three points corresponding to their intersection with each other on the real line are emphasized by arrows. How do these points relate to the distribution of $(\{z(-\frac{3}{2})^n\})_{n\geq 1}$?

The solution is not hard for $p=2$, that is, when every other person gets executed, and it is given explicitly by $J_2(m) = 1+2(m-2^{\lfloor \log_2(m)\rfloor})$. In the case $p=3$, according to [17], the solution is given by

$$J_3(m) = 3m+1-\left\lfloor K(3)\left(\frac{3}{2}\right)^{\left\lceil \log_{3/2}\frac{2m+1}{K(3)}\right\rceil}\right\rfloor,$$

where $\lceil y\rceil = \min\{n\in\mathbb{Z} : n\geq y\}$ is the *ceiling function*, and $K(3) = 1.62227\ldots$ is a constant. Interestingly, the value of $K(3)$ is equal to the length of the SRS tile $\mathcal{F}_{3/2}\cap\mathbb{R}$ of $\tau_{-2/3}$ (see [22, Section 2] and [1, Section 4.4 and Theorem 2]).

8.3. **Sum of Digits Function.** Since each $N\in\mathbb{Z}$ can be represented in the form (1), it makes sense to define the $(-\frac{3}{2})$*-sum-of-digits function* $S_{-3/2}(N) = d_0+\cdots+d_k$. The proofs in [16] suggest that $\mathcal{F}$ is useful in the study of $S_{-3/2}(N)$. It would be interesting to explore this further as there are many open questions. For example, can one prove that $S_{-3/2}(N)$ is equidistributed in residue classes mod m for $m\geq 2$?

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