ON A FAMILY OF THREE TERM NONLINEAR INTEGER RECURRENCES

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Abstract. In the present paper we study sequences defined by the recurrence relation

$$a_{n+3} = -\left\lfloor a_n + \lambda^2 a_{n+1} + \lambda^2 a_{n+2} \right\rfloor$$

for $n \geq 0$, where $\lambda = \frac{1+\sqrt{5}}{2}$ the golden ratio. These sequences are related to shift radix systems as well as to β-expansions with respect to Salem numbers.

1. Introduction

In [1] the following notion of *shift radix system* was introduced. Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ as follows: if $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ then we let

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1 a_1 + \cdots + r_d a_d$ is the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$.

If iterates of τ always end up at zero, i.e., if

$$\text{for all } \mathbf{a} \in \mathbb{Z}^d \text{ there exists } k > 0 \text{ with } \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \tag{1.1}$$

holds, we will call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short). For simplicity, we write $0 = (0, \dots, 0)$.

In the present paper we are interested in parameters $\mathbf{r}$ such that $\tau_{\mathbf{r}}$ is eventually periodic for each starting value. Let $\mathcal{D}_d$ be the set of all d-dimensional vectors $\mathbf{r}$ having this property. Moreover, let

$$\mathcal{E}_d := \left\{ (r_1, \dots, r_d) \in \mathbb{R}^d \,|\, X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < 1 \right\}. \tag{1.2}$$

It is not hard to see (cf. [1, Section 4]) that

$$\mathcal{E}_d \subseteq \mathcal{D}_d \subseteq \overline{\mathcal{E}_d}.$$

Thus the problem that remains is to find out for which parameters $\mathbf{r} \in \partial\mathcal{E}_d$ do we have $\mathbf{r} \in \mathcal{D}_d$. This question was addressed for the two-dimensional case in [3, Section 2]. It is easy to see that $\mathcal{D}_2$ is (apart from its boundary) an isosceles triangle. In [3, Section 2] $\partial\mathcal{D}_2$ was characterized for two sides of this triangle. To characterize which points of the third side of this triangle belong to $\mathcal{D}_2$ turned out to be a hard problem. In particular, what remains to be proved is

$$\{(1, y) \mid |y| < 2\} \subset \mathcal{D}_2.$$

Using the definition of $\tau_{\mathbf{r}}$ this reads as follows. Let $|\lambda| < 2$ and let $(a_n)_{n=1}^{\infty}$ be a sequence of integers which satisfies

$$0 \leq a_n + \lambda a_{n+1} + a_{n+2} < 1 \qquad (n \in \mathbb{N}).$$

Then $(a_n)_{n=1}^{\infty}$ is periodic.

Results concerning this conjecture are contained in [2]. Especially, in that paper it is shown that the conjecture is true for $\lambda = \frac{1+\sqrt{5}}{2}$.

In the present paper we want to start with the consideration of $\partial\mathcal{D}_3$. Up to now nothing is known about the periodicity properties of $\tau_{\mathbf{r}}$ with $\mathbf{r} \in \partial\mathcal{D}_3$. Here we want to study the behavior

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of certain sequences related to a point in $\mathbf{r} \in \partial\mathcal{D}_3$. In particular, we study the behavior of the sequences defined by the initial values $a_0 = a_1 = 1, a_2 \in \mathbb{N}$ and

$$a_{n+3} = -\lfloor a_n + \lambda^2 a_{n+1} + \lambda^2 a_{n+2} \rfloor$$

for $n \geq 0$, where $\lambda = \frac{1+\sqrt{5}}{2}$ the golden ratio. Equivalently a_{n+3} is the unique integer solution of the inequality

$$0 \leq a_n + \lambda^2 a_{n+1} + \lambda^2 a_{n+2} + a_{n+3} < 1. \tag{1.3}$$

Our aim is to characterize for which values of a_2 the sequence is periodic and for which it becomes divergent. We will show (see Theorems 4.1, 5.1, 5.2 and 5.3) that the behavior of the sequence depends on the starting digits of the Zeckendorf representation of a_2.

Notice that the characteristic polynomial of the linearized sequence

$$\bar{a}_{n+3} = -\bar{a}_n - \lambda^2 \bar{a}_{n+1} - \lambda^2 \bar{a}_{n+2}$$

is $x^3 + \lambda^2 x^2 + \lambda^2 x + 1 = (x+1)(x^2 + \lambda x + 1)$, whose zeros are roots of unity, thus $\{\bar{a}_n\}$ is periodic.

2. Preliminaries

To prove our result we need some well-known properties of λ and the Fibonacci sequence (cf. [4]). The number λ is a zero of the polynomial $x^2 - x - 1$. Its algebraic conjugate, which is the other zero of $x^2 - x - 1$ will be denoted by $\lambda' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\lambda} = 1 - \lambda$.

$$\lambda^2 = \lambda + 1 = \frac{3+\sqrt{5}}{2}, \quad \lambda^3 = 2\lambda + 1 = 2 + \sqrt{5}, \quad \lambda^4 = 3\lambda + 2 = \frac{7+3\sqrt{5}}{2},$$
$$\lambda'^2 = \lambda' + 1 = \frac{3-\sqrt{5}}{2}, \quad \lambda'^3 = 2\lambda' + 1 = 2 - \sqrt{5}, \quad \lambda'^4 = 3\lambda' + 2 = \frac{7-3\sqrt{5}}{2}.$$

We also need the following relations, which are very easy to prove

$$\sum_{j=0}^{\infty} \lambda'^{2j} = \frac{1}{1-\lambda'^2} = -\frac{1}{\lambda'} = \lambda, \tag{2.1}$$

$$\sum_{j=0}^{\infty} \lambda'^{2j+1} = \frac{\lambda'}{1-\lambda'^2} = -1, \tag{2.2}$$

$$-1 < \sum_{j=0}^{\infty} a_{2,j} \lambda'^j < \lambda, \tag{2.3}$$

where in the third relation $a_{2,j} \in \{0,1\}$.

The members of the Fibonacci sequence are defined by the initial terms $F_1 = F_2 = 1$ and by the recursion $F_{n+2} = F_{n+1} + F_n, n \geq 0$. It is well known (cf. [5]) that any positive integer, e.g. a_2 has a unique representation

$$a_2 = \sum_{j=2}^{\infty} a_{2,j} F_j \tag{2.4}$$

such that $a_{2,j} \in \{0,1\}, a_{2,j} a_{2,j+1} = 0, j \geq 2$ and $a_{2,j} = 0$ for all but finitely many indices j. This is called the Zeckendorf representation.

If $n \geq 2$ then it is easy to show, that

$$F_n \lambda^2 = F_{n+2} - \lambda'^n. \tag{2.5}$$

We now study a family of sequences, which is periodic and helps to prove the periodicity of other sequences. For $x \in \mathbb{R}$ let $\|x\|$ denote the distance of x to the nearest integer. We also need the nearest integer to x, which is $\lfloor x + \frac{1}{2} \rfloor$.

Lemma 2.1. *Let $k \geq 3$ and $f_0 = f_1 = 0$ and $f_2 = \sum_{j \geq k} f_{2,j} F_j$, where $f_{2,j} \in \{0,1\}, f_{2,j} f_{2,j+1} = 0$. Let $\tilde{f}_0 = \tilde{f}_1 = \tilde{f}_2 = 0$ and for $h \geq 0$*

$$\tilde{f}_{h+3} = f_h + (f_{h+1} + f_{h+2})\lambda^2 \quad \text{and} \quad f_{h+3} = -\left\lfloor \tilde{f}_{h+3} + \frac{1}{2} \right\rfloor.$$

Then the sequence $\{f_h\}$ is periodic with minimal period length 10 and its members and upper bound for $\|\tilde{f}_h\|$ are given in Table 1.

h mod 10	*0*	*1*	*2*	*3*	*4*
f_h	*0*	*0*	$\sum f_{2,j}F_j$	$-\sum f_{2,j}F_{j+2}$	$\sum f_{2,j}F_{j+3}$
$\|\tilde{f}_h\| \leq$	$\|\lambda'\|^k$	$\|\lambda'\|^{k-1}$	*0*	$\|\lambda'\|^{k-1}$	$\|\lambda'\|^k$
h	*5*	*6*	*7*	*8*	*9*
f_h	$-2\sum f_{2,j}F_{j+2}$	$2\sum f_{2,j}F_{j+2}$	$-\sum f_{2,j}F_{j+3}$	$\sum f_{2,j}F_{j+2}$	$-\sum f_{2,j}F_j$
$\|\tilde{f}_h\| \leq$	$\|\lambda'\|^k$	$\|\lambda'\|^{k-1}$	*0*	$\|\lambda'\|^{k-1}$	$\|\lambda'\|^k$

Table 1. Behavior of f_h

Proof. We prove the statement only for $h = 5$, because the other cases are similar. By using the properties of the Fibonacci numbers and (2.5) we get

$$\begin{aligned}
\tilde{f}_5 &= f_2 + (f_3 + f_4)\lambda^2 \\
&= \sum_{j \geq k} f_{2,j} F_j + \left(-\sum_{j \geq k} f_{2,j} F_{j+2} + \sum_{j \geq k} f_{2,j} F_{j+3} \right) \lambda^2 \\
&= \sum_{j \geq k} f_{2,j} F_j + \sum_{j \geq k} f_{2,j} \left(F_{j+3} - F_{j+2} \right) \lambda^2 \\
&= \sum_{j \geq k} f_{2,j} F_j + \sum_{j \geq k} f_{2,j} F_{j+3} - \sum_{j \geq k} f_{2,j} \lambda'^{j+1} \\
&= 2\sum_{j \geq k} f_{2,j} F_{j+2} - \sum_{j \geq k} f_{2,j} \lambda'^{j+1}.
\end{aligned}$$

We can estimate the second summand by using (2.3), which gives

$$|\sum_{j \geq k} f_{2,j} \lambda'^{j+1}| < |\lambda \lambda'^{k+1}| = |\lambda'^k|.$$

As $k \geq 3$ we have $|\lambda'^k| < \frac{1}{2}$, thus

$$\|\tilde{f}_5\| < |\lambda'^k|$$

implies

$$f_5 = -2\sum_{j \geq k} f_{2,j} F_{j+2}.$$

□

To the sequence (a_n) satisfying (1.3) we associate the sequence $(\hat{a}_n)$ by the rule

$$\hat{a}_n = a_{n-3} + \lambda^2 a_{n-2} + \lambda^2 a_{n-1} + a_n, \quad n \in \mathbb{Z}.$$

We have $0 \leq \hat{a}_n < 1$ for any $n \in \mathbb{Z}$. The following lemma will be used later frequently.

Lemma 2.2. *Let (a_n) be a sequence, which satisfies (1.3). Assume that there exists a $k \geq 3$ such that*

$$|\lambda'|^{k-1} \leq \hat{a}_n < 1 - |\lambda'|^{k-1}$$

holds for any $n \in \mathbb{Z}$. Then the sequence $(a_n + f_n)$, where (f_n) is defined in Lemma 2.1, satisfies (1.3) too. Moreover, if (a_n) is periodic with period length p, then $(a_n + f_n)$ is periodic too with period length lcm$(p, 10)$.

Proof. We have

$$\begin{aligned} a_{n-3}+f_{n-3}+(a_{n-2}+f_{n-2}+a_{n-1}+f_{n-1})\lambda^2+a_n+f_n &= \hat{a}_n+\tilde{f}_n-\left\lfloor \tilde{f}_n+\frac{1}{2}\right\rfloor \\ &= \hat{a}_n \pm \|\tilde{f}_n\|. \end{aligned}$$

By Lemma 2.1 $\|\tilde{f}_n\| < |\lambda'|^{k-1}$, thus the lemma is proved. □

3. The first six terms of the sequence

Set $a_0 = a_1 = 1, a_2 \in \mathbb{N}$. Starting from the Zeckendorf representation (2.4) of a_2 we establish the Zeckendorf representation of $a_3, \ldots, a_6$.

To find a_3 we have to find the integer part of

$$\begin{aligned} S &= a_0+(a_1+a_2)\lambda^2 \\ &= 1+\left(1+\sum_{j=2}^{\infty} a_{2,j}F_j\right)\lambda^2 \\ &= 1+\lambda^2+\sum_{j=2}^{\infty} a_{2,j}F_{j+2}-\sum_{j=2}^{\infty} a_{2,j}\lambda'^j \\ &= 3+\sum_{j=2}^{\infty} a_{2,j}F_{j+2}+\lambda-1-\sum_{j=2}^{\infty} a_{2,j}\lambda'^j. \end{aligned}$$

Using (2.1) and (2.2) we obtain

$$0 \le \lambda-1-\sum_{j=1}^{\infty}\lambda'^{2j} < \lambda-1-\sum_{j=2}^{\infty} a_{2,j}\lambda'^j < \lambda-1-\sum_{j=1}^{\infty}\lambda'^{2j+1} = \lambda+\lambda' = 1.$$

Thus

$$a_3 = -3-\sum_{j=2}^{\infty} a_{2,j}F_{j+2}.$$

If $a_{2,2} = a_{2,3} = 0$ then we have the stronger estimate

$$\lambda'^2 = \lambda-1-\lambda'^4\lambda < \lambda-1-\sum_{j=4}^{\infty} a_{2,j}\lambda'^j < \lambda-1+\lambda'^4 = 2\lambda'^2. \tag{3.1}$$

Using a_1, a_2, a_3 we will compute a_4.

$$\begin{aligned} S &= a_1+(a_2+a_3)\lambda^2 \\ &= 1+\left(\sum_{j=2}^{\infty} a_{2,j}F_j-3-\sum_{j=2}^{\infty} a_{2,j}F_{j+2}\right)\lambda^2 \\ &= 1-3\lambda^2+\sum_{j=2}^{\infty} a_{2,j}(F_j-F_{j+2})\lambda^2 \\ &= \frac{-7-3\sqrt{5}}{2}-\sum_{j=2}^{\infty} a_{2,j}F_{j+1}\lambda^2 \\ &= -7-\sum_{j=2}^{\infty} a_{2,j}F_{j+3}+\lambda'^4+\sum_{j=2}^{\infty} a_{2,j}\lambda'^{j+1}. \end{aligned}$$

To obtain a_4 we have to analyze the summand

$$R = \lambda'^4+\sum_{j=2}^{\infty} a_{2,j}\lambda'^{j+1}.$$

We distinguish two cases.
Case I. $a_{2,2} = 0$. Using again (2.1) and (2.2) we obtain

$$0 = \lambda'^4 + \lambda'^5\lambda < R < \lambda'^4 + \lambda'^4\lambda = \lambda'^2. \tag{3.2}$$

Thus we have $\lfloor S \rfloor = -7 - \sum_{j=2}^{\infty} a_{2,j}F_{j+3}$. Notice that if $a_{2,3} = 0$ holds as well, then

$$0 = \lambda'^4 + \lambda'^5\lambda < R < \lambda'^4 + \lambda'^6\lambda = -\lambda'^3. \tag{3.3}$$

We need this stronger estimate later.
Case II. $a_{2,2} = 1$. In this case $a_{2,3} = 0$, by the property of the Zeckendorf representation and we obtain

$$-1 < \lambda'^3 = \lambda'^4 + \lambda'^3\lambda < R < \lambda'^4 + \lambda'^3 + \lambda'^6\lambda = 0, \tag{3.4}$$

hence we obtain $\lfloor S \rfloor = -8 - \sum_{j=2}^{\infty} a_{2,j}F_{j+3}$.

Summarizing our result $a_4 = a_{4,c} + \sum_{j=2}^{\infty} a_{2,j}F_{j+3}$, where

$$a_{4,c} = \begin{cases} 7, & \text{if} \quad a_{2,2} = 0 \\ 8, & \text{if} \quad a_{2,2} = 1. \end{cases}$$

Now we turn to establish a_5. Let

$$\begin{aligned} S &= a_2 + (a_3 + a_4)\lambda^2 \\ &= \sum_{j=2}^{\infty} a_{2,j}F_j + \left(-3 + a_{4,c} + \sum_{j=2}^{\infty} a_{2,j}(F_{j+3} - F_{j+2}) \right) \lambda^2 \\ &= (-3 + a_{4,c})\lambda^2 + \sum_{j=2}^{\infty} a_{2,j}F_j + \sum_{j=2}^{\infty} F_{j+1}\lambda^2 \\ &= (-3 + a_{4,c})\lambda^2 + \sum_{j=2}^{\infty} a_{2,j}F_j + \sum_{j=2}^{\infty} F_{j+3} - \sum_{j=2}^{\infty} a_{2,j}\lambda'^{j+1} \\ &= 2\sum_{j=2}^{\infty} a_{2,j}F_{j+2} + R, \end{aligned}$$

where

$$R = \begin{cases} 4\lambda^2 - \lambda'^4 \sum_{j=0}^{\infty} a_{2,j+3}\lambda'^j, & \text{if} \quad a_{2,2} = 0 \\ 5\lambda^2 - \lambda'^3 - \lambda'^5 \sum_{j=0}^{\infty} a_{2,j+4}\lambda'^j, & \text{if} \quad a_{2,2} = 1. \end{cases}$$

We estimate R by using (2.3) and get

$$10 < 8 + \sqrt{5} = 4\lambda^2 + \lambda'^3 < R < 4\lambda^2 + \lambda'^4 = 9 + \lambda < 11, \tag{3.5}$$

i.e. $\lfloor R \rfloor = 10$ in case $a_{2,2} = 0$.

If $a_{2,2} = 1$ then we get similarly

$$13 < 2 + 5\sqrt{5} = 5\lambda^2 - \lambda'^3 - \lambda'^4 < R < 5\lambda^2 - \lambda'^3 + \lambda'^5 = 11 + \sqrt{5} < 14, \tag{3.6}$$

i.e. $\lfloor R \rfloor = 13$. Thus we have $a_5 = a_{5,c} - 2\sum_{j=2}^{\infty} a_{2,j}F_{j+2}$, where

$$a_{5,c} = \begin{cases} -10, & \text{if} \quad a_{2,2} = 0 \\ -13, & \text{if} \quad a_{2,2} = 1. \end{cases}$$

Finally we compute a_6. Let

$$\begin{aligned} S &= a_3 + (a_4 + a_5)\lambda^2 \\ &= -3 - \sum_{j=2}^{\infty} a_{2,j} F_{j+2} + \left(a_{4,c} + a_{5,c} + \sum_{j=2}^{\infty} a_{2,j} (F_{j+3} - 2F_{j+2}) \right) \lambda^2 \\ &= -3 + (a_{4,c} + a_{5,c})\lambda^2 - \sum_{j=2}^{\infty} a_{2,j} \left(F_{j+2} + F_j \lambda^2 \right) \\ &= -3 + (a_{4,c} + a_{5,c})\lambda^2 - 2 \sum_{j=2}^{\infty} a_{2,j} F_{j+2} + \sum_{j=2}^{\infty} a_{2,j} \lambda'^j \\ &= -2 \sum_{j=2}^{\infty} a_{2,j} F_{j+2} + R, \end{aligned}$$

where

$$R = \begin{cases} -3 - 3\lambda^2 + \lambda'^3 \sum_{j=0}^{\infty} a_{2,j+3} \lambda'^j, & \text{if} \quad a_{2,2} = 0 \\ -3 - 5\lambda^2 + \lambda'^2 + \lambda'^4 \sum_{j=0}^{\infty} a_{2,j+4} \lambda'^j, & \text{if} \quad a_{2,2} = 1. \end{cases}$$

We estimate R by using (2.3) and get

$$-16 < -9 - \lambda^4 = -3 - 5\lambda^2 + \lambda'^2 - \lambda'^4 < R < -3 - 5\lambda^2 + \lambda'^2 - \lambda'^3 = -11 - 2\sqrt{5} < -15, \tag{3.7}$$

i.e. $\lfloor R \rfloor = -16$ in the case $a_{2,2} = 1$.

If $a_{2,2} = 0$ then

$$R + 11 = \lambda'^4 + \lambda'^3 \sum_{j=0}^{\infty} a_{2,j+3} \lambda'^j.$$

It follows from (2.3) that $|R + 11| < 1$. Moreover, if $a_{2,3} = 0$ then

$$0 < \lambda'^4 - \lambda'^4 \leq R + 11 = \lambda'^4 + \lambda'^4 \sum_{j=0}^{\infty} a_{2,j+4} \lambda'^j \leq \lambda'^4 - \lambda'^3 = \lambda'^2, \tag{3.8}$$

i.e. $\lfloor R \rfloor = -11$. However, if $a_{2,3} = 1$ then $a_{2,4} = 0$ by the property of the Zeckendorf expansion and we get

$$R + 11 = \lambda'^4 + \lambda'^3 + \lambda'^5 \sum_{j=0}^{\infty} a_{2,j+5} \lambda'^j < \lambda'^4 + \lambda'^3 - \lambda'^5 = 0,$$

i.e. $\lfloor R \rfloor = -12$ in this case. Thus we have $a_6 = a_{6,c} + 2 \sum_{j=2}^{\infty} a_{2,j} F_{j+2}$, where

$$a_{6,c} = \begin{cases} 11, & \text{if} \quad a_{2,2} = a_{2,3} = 0 \\ 12, & \text{if} \quad a_{2,2} = 0, a_{2,3} = 1 \\ 16, & \text{if} \quad a_{2,2} = 1. \end{cases}$$

4. The divergent case, $a_{2,2} = a_{2,3} = 0$

We continue our investigation first with the case $a_{2,2} = a_{2,3} = 0$ and prove that then the sequence is divergent. Notice that now $a_6 = -a_5 + 1$. The next member of the sequence is defined by

$$a_7 = -\left\lfloor a_4 + (a_5 + a_6)\lambda^2 \right\rfloor = -\left\lfloor a_4 + \lambda^2 \right\rfloor = -a_4 - 2.$$

We prove that $a_8 = -a_3 + 2$. To see this we examine the number

$$\begin{aligned} S &= a_5 + (a_6 + a_7)\lambda^2 - a_3 + 2 \\ &= -a_6 + 1 + (-a_5 + 1 - a_4 - 2)\lambda^2 - a_3 + 2 \\ &= -(a_3 + (a_4 + a_5)\lambda^2 + a_6) + 3 - \lambda^2 \end{aligned}$$

Using (3.8) we conclude

$$0 = 3 - \lambda^2 - \lambda'^2 < S < 3 - \lambda^2 = \lambda'^2 < 1,$$

which proves our claim.

To prove $a_9 = -a_2$ we consider

$$\begin{aligned} S &= a_6 + (a_7 + a_8)\lambda^2 - a_2 \\ &= -a_5 + 1 + (-a_4 - 2 - a_3 + 2)\lambda^2 - a_2 \\ &= -(a_2 + (a_3 + a_4)\lambda^2 + a_5) + 1. \end{aligned}$$

We know that $0 \le a_2 + (a_3 + a_4)\lambda^2 + a_5 < 1$, but it cannot be 0, because otherwise $a_2 = -a_5$, which is absurd.

Now we prove that $a_{10} = -4 = -a_1 - 3 = a_0 - 5$. We have

$$\begin{aligned} S &= a_7 + (a_8 + a_9)\lambda^2 - a_1 - 3 \\ &= -a_4 - 2 + (-a_3 + 2 - a_2)\lambda^2 - a_1 - 3 \\ &= -(a_1 + (a_2 + a_3)\lambda^2 + a_4) - 5 + 2\lambda^2. \end{aligned}$$

Using (3.3) we get $S < -5 + 2\lambda^2 = -2 + \sqrt{5} < 1$.

$$0 = -5 + 2\lambda^2 + \lambda'^3 < S < -5 + 2\lambda^2 = -2 + \sqrt{5} < 1.$$

We prove that $a_{11} = 6 = -a_0 + 7 = a_1 + 5$. Indeed

$$\begin{aligned} S &= a_8 + (a_9 + a_{10})\lambda^2 - a_0 + 7 \\ &= -a_3 + 2 + (-a_2 - a_1 - 3)\lambda^2 - a_0 + 7 \\ &= -(a_0 + (a_1 + a_2)\lambda^2 + a_3) + 9 - 3\lambda^2. \end{aligned}$$

Using (3.1) we get

$$0 < \lambda'^2 = 9 - 3\lambda^2 - 2\lambda'^2 < S < 9 - 3\lambda^2 - \lambda'^2 = 2\lambda'^2 < 1$$

and our claim is proved.

Next we prove that $a_{12} = a_2 - 5$. Let

$$\begin{aligned} S &= a_9 + (a_{10} + a_{11})\lambda^2 + a_2 - 5 \\ &= -a_2 + 2\lambda^2 + a_2 - 5 \\ &= 2\lambda^2 - 5 = -2 + \sqrt{5} \end{aligned}$$

and our claim is proved.

Now we can formulate the first theorem.

Theorem 4.1. *If $a_0 = a_1 = 1$ and $a_2 = \sum_{j=4}^{\infty} a_{2,j} F_j$, where $a_{2,j} \in \{0, 1\}, a_{2,j} a_{2,j+1} = 0$, then*

$$\begin{aligned} a_{10n+2k} &= a_{2k} - 5n \\ a_{10n+2k+1} &= a_{2k+1} + 5n \end{aligned}$$

holds for any $n \in \mathbb{Z}$ and $k = 0, \ldots, 4$. Thus the sequence is divergent.

Proof. We proved the statement for a_{10}, a_{11} and a_{12}. Assume that it holds for three consecutive terms $a_{10n+2k}, a_{10n+2k+1}, a_{10n+2k+2}$. Then

$$\begin{aligned} a_{10n+2k+3} &= -\lfloor a_{10n+2k} + (a_{10n+2k+1} + a_{10n+2k+2})\lambda^2 \rfloor \\ &= -\lfloor a_{2k} - 5n + (a_{2k+1} + 5n + a_{2k+2} - 5n)\lambda^2 \rfloor \\ &= -\lfloor a_{2k} + (a_{2k+1} + a_{2k+2})\lambda^2 - 5n \rfloor \\ &= a_{2k+3} + 5n. \end{aligned}$$

We can now finish similarly the induction in the case where the index of the first term is even. Finally, the recursive procedure (1.3) is symmetric, thus the induction works for negative indices too. □

5. Periodic sequences, $a_{2,2} = 1$ or $a_{2,2} = 0, a_{2,3} = 1$.

In this section we prove that the sequences with $a_{2,2} = 1$ or $a_{2,2} = 0, a_{2,3} = 1$ are periodic.

5.1. **The case $a_{2,2}=1, a_{2,3}=a_{2,4}=0$.** We prove the following theorem.

Theorem 5.1. *Let the sequence (a_n) be defined by the starting values $a_0=a_1=1$ and $a_2=F_2+\sum_{j\geq 5} a_{2,j}F_j$, where $a_{2,j}\in\{0,1\}, a_{2,j}a_{2,j+1}=0$. Then (a_n) is periodic with minimal period length* 30.

Proof. Consider first the special case $a_0'=a_1'=a_2'=F_2=1$. Let $\hat{a}_n'=a_{n-3}'+(a_{n-2}'+a_{n-1}')\lambda^2+a_n', n\in\mathbb{Z}$. A simple computation shows that (a_n') is periodic with minimal period length 30. Moreover

$$-\lambda'^3\leq \hat{a}_n'\leq 1-\lambda'^6$$

hold for all $n\in\mathbb{Z}$. Thus, if $a_2=F_2+\sum_{j\geq 7} a_{2,j}F_j$, then by Lemma 2.2 for the sequence (a_n) the assertion holds. The sequences with starting values $a_0'=a_1'=1, a_2'=F_2+F_5+\sum_{j\geq 7}a_{2,j}F_j$ and $a_0''=a_1''=1, a_2''=F_2+F_6+\sum_{j\geq 8}a_{2,j}F_j$ require a more careful analysis. Considering the auxiliary sequence with starting values $b_0'=b_1'=1, b_2'=F_2+F_5$ a simple computation shows that

$$\lambda'^4\leq \hat{b}_n'\leq 1-\lambda'^6$$

holds except when $n=4,19$. (In the exceptional cases ;$\hat{b}_n'=1+\lambda'^7$.) It follows from Section 3 that $a_n'=b_n'+f_n$ holds for $0\leq n\leq 6$, especially

$$\begin{aligned} a_1' &= 1, \quad a_2'=F_2+F_5+\sum_{j\geq 7}a_{2,j}F_j, \quad a_3'=-3-\left(F_4+F_7+\sum_{j\geq 7}a_{2,j}F_{j+2}\right),\\ a_4' &= 8+\left(F_5+F_8+\sum_{j\geq 7}a_{2,j}F_{j+3}\right). \end{aligned}$$

Thus

$$0\leq \hat{a}_4'=a_1'+(a_2'+a_3')\lambda^2+a_4'=9-3\lambda^2+\lambda'^3+\lambda'^6+\sum_{j\geq 7}a_{2,j}\lambda'^{j+1}<1+\lambda'^7. \tag{5.1}$$

By the above inequality, which holds for $n\leq 18$ and by Lemma 2.2 we have $a_n'=b_n'+f_n$ for $0\leq n\leq 18$, i.e.

$$\begin{aligned} a_{16}' &= 15+2\left(F_4+F_7+\sum_{j\geq 7}a_{2,j}F_{j+2}\right), \quad a_{17}'=-14-\left(F_5+F_8+\sum_{j\geq 7}a_{2,j}F_{j+3}\right),\\ a_{18}' &= 11+\left(F_4+F_7+\sum_{j\geq 7}a_{2,j}F_{j+2}\right). \end{aligned}$$

We have

$$b_{19}'+f_{19}=-6-\left(F_2+F_5+\sum_{j\geq 7}a_{2,j}F_j\right).$$

Using these data we obtain

$$\begin{aligned} a_{16}'+(a_{17}'+a_{18}')\lambda^2+b_{19}'+f_{19} &= 9-3\lambda^2+\lambda'^3+\lambda'^6+\sum_{j\geq 7}a_{2,j}\lambda'^{j+1}\\ &= \hat{a}_4', \end{aligned}$$

i.e. $a_{19}'=b_{19}'+f_{19}$. Using this and Lemma 2.2 we have $a_n'=b_n'+f_n$ for $20\leq n\leq 30$, consequently for $0\leq n\leq 30$.

The proof for the sequence (a_n'') is similar, therefore we omit it. □

5.2. **The case $a_{2,2} = 0, a_{2,3} = 1, a_{2,4} = 0$.** Now we turn to the second kind of periodic sequences.

Theorem 5.2. *Let the sequence (a_n) be defined by the starting values $a_0 = a_1 = 1$ and $a_2 = F_3 + \sum_{j\geq 5} a_{2,j}F_j$, where $a_{2,j} \in \{0,1\}, a_{2,j}a_{2,j+1} = 0$. Then (a_n) is periodic with minimal period length* 30.

Proof. The proof is analogous to the proof of Theorem 5.1, therefore we omit it. □

5.3. **The case $a_{2,2} = 1, a_{2,3} = 0, a_{2,4} = 1$.** To complete the study of the sequences with starting values $a_0 = a_1 = 1$, a_2 a positive integer, it remains to consider the case in the title. Now we fill this gap.

Theorem 5.3. *Let the sequence (a_n) be defined by the starting values $a_0 = a_1 = 1$ and $a_2 = F_2 + F_4 + \sum_{j\geq 6} a_{2,j}F_j$, where $a_{2,j} \in \{0,1\}, a_{2,j}a_{2,j+1} = 0$. Then (a_n) is periodic with minimal period length* 70.

Proof. Considering again the auxiliary sequence with initial values $a'_0 = a'_1 = 1, a'_2 = F_2 + F_4 = 4$ we see that (a'_n) is periodic with length 70 and

$$0 < -\lambda'^7 \leq \hat{a}'_n < 1 + \lambda'^5,$$

which implies that the assertion is true if we assume $j \geq 8$ in the definition of a_2. The remaining cases can be proved by a more careful analysis like in the proof of Theorem 5.1. □

At the end of this note we mention that the same methods also apply for the investigation of sequences starting with other values than $a_0 = a_1 = 1$. However, the higher the values of a_0 and a_1 are, the more cases have to be distinguished according to the Zeckendorf representation of a_2. Computer experiments suggest that the length of the occurring periods is not bounded.

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