BETA-CONTINUED FRACTIONS OVER LAURENT SERIES

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Abstract. The present paper is devoted to a new notion of continued fractions in the field of Laurent series over a finite field. The definition of this kind of continued fraction algorithm is based on a general notion of number systems. We will prove some ergodic properties and compute the Hausdorff dimensions of bounded type continued fraction sets.

1. Introduction

Let p be a prime, q a power of p and $\mathbb{F}_q$ the finite field with q elements. In the present paper, we introduce a new kind of continued fraction algorithm in $\mathbb{F}_q((X^{-1}))$, the field of formal Laurent series over $\mathbb{F}_q$. This algorithm is based on so called greedy expansions with respect to a base sequence $(\beta_i)_{i\in\mathbb{Z}}$ such that $\beta_i \in \mathbb{F}((X^{-1}))$ and $(\deg \beta_i)_{i\in\mathbb{Z}}$ is strictly increasing. Analogously to the case of continued fractions of real numbers, the new algorithm is coupled with a dynamical system on a certain subset of $\mathbb{F}_q((X^{-1}))$ and the underlying transformation turns out to be ergodic. Applying the ergodic theorem, we extend some metrical properties of the classical algorithm to the new situation. In particular, we obtain results on the average speed of convergence of our algorithm and the Hausdorff dimension of some exceptional sets. Most of our results are inspired by results of Berthé, Nakada and Wu (cf. [2, 11, 12]).

The paper is organized as follows. In Section 2, some necessary notations are introduced and the algorithm is described in detail. In Section 3, ergodicity is proved for the underlying transformation and results on the mean convergence of the algorithm are established. In Section 4, the Hausdorff dimensions of the exceptional sets will we determined. In Section 5, the set of series having a fixed given rate of convergence is studied.

2. Continued fraction expansions of Laurent series

Let $\mathbb{F}_q$ be a finite field of q elements, where q is a power of some prime number and let $\mathbb{F}_q[X]$ denote the ring of polynomials in X with coefficients in $\mathbb{F}_q$. Denote

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by $\mathbb{F}_q((X^{-1}))$ the field of formal Laurent series

$$f = \sum_{n=n_0}^{+\infty} a_n X^{-n} \quad a_n \in \mathbb{F}_q \quad \text{and} \quad n_0 \in \mathbb{Z}. \tag{2.1}$$

If $f \neq 0$ we may assume without loss of generality, that $a_{n_0} \neq 0$. We say that $\deg f = -n_0$ is the degree of f. The norm (or valuation) of f is given by

$$|f| = q^{\deg f}.$$

It is well known that $|\cdot|$ is a non-Archimedean valuation over $\mathbb{F}_q((X^{-1}))$ and $\mathbb{F}_q((X^{-1}))$ is complete with respect to the metric

$$\nu(f, g) = |f - g|.$$

For $a \in \mathbb{F}_q((X^{-1}))$ and $r > 0$, define

$$D(a, r) = \{\omega \in \mathbb{F}_q((X^{-1})) : \ |\omega - a| < r\} \quad \text{and}$$
$$\overline{D}(a, r) = \{\omega \in \mathbb{F}_q((X^{-1})) : \ |\omega - a| \leq r\}.$$

We shall use the notation $|\cdot|$ also for the diameter of a disc. Thus

$$|D(a, q^n)| = q^{n-1} \quad \text{and} \quad |\overline{D}(a, q^n)| = q^n \quad \text{for} \quad n \in \mathbb{Z}.$$

If f is of the form (2.1), let $[f]$ be the integral (polynomial) part of f in $\mathbb{F}_q[X]$ and

$$\{f\} = f - [f] = \sum_{n=1}^{+\infty} a_n X^{-n}$$

be the fractional part of f.

Let $\beta = (\beta_i)_{i\in\mathbb{Z}}$ with $\beta_i \in \mathbb{F}_q((X^{-1}))\backslash\{0\}$ such that $(\deg \beta_i)_{i\in\mathbb{Z}}$ is a strictly increasing sequence of integers. β is called base sequence. Let

$$\mathcal{S} = \{(s_i)_{-\infty<i\leq k} : k \in \mathbb{Z}, \ s_i \in \mathbb{F}_q[X], \ \deg s_i < \deg \beta_{i+1} - \deg \beta_i\} \tag{2.2}$$

be the set of admissible digit strings associated to the base sequence β.

Lemma 2.1. *Let $\beta = (\beta_i)_{i\in\mathbb{Z}}$ be a base sequence and $\mathcal{S}$ the associated set of admissible digit strings. Then each $f \in \mathbb{F}_q((X^{-1}))$ admits a unique representation of the form*

$$f = \sum_{-\infty<i\leq k} s_i \beta_i, (s_i)_{-\infty<i\leq k} \in \mathcal{S}. \tag{2.3}$$

Proof. The existence of a representation of the form (2.3) is guaranteed by performing the so called greedy algorithm. It remains to prove unicity. To this matter let $s := (s_i)_{-\infty<i\leq k} \in \mathcal{S}$ be an admissible digit string giving rise to a representation of a given element w. Assume that the representation (2.3) of w is not unique. Then there is an admissible digit string $s' := (s'_i)_{-\infty<i\leq k'} \in \mathcal{S}$ with

$s' \neq s$ satisfying $w = \sum_{-\infty < i \leq k'} s'_i \beta_i$. Subtracting this from (2.3) yields (assume w.l.o.g that $k \geq k'$ and set $s'_i = 0$ for $k' < i \leq k$)

$$0 = \sum_{-\infty < i \leq k} (s_i - s'_i)\beta_i. \tag{2.4}$$

As $s, s' \in \mathcal{S}$ we conclude that $s - s' \in \mathcal{S}$, where the subtraction is done componentwise. Since $s' \neq s$ there is a maximal $i_0 \in \mathbb{Z}$ with $s_{i_0} \neq s'_{i_0}$. Thus

$$\deg 0 = \deg \left(\sum_{-\infty < i \leq k} (s_i - s'_i)\beta_i \right) \geq \deg \beta_{i_0} > -\infty,$$

contradicting (2.4). □

The above lemma justifies that we call $(\beta, \mathcal{S})$ a *digit system*. Conversely, a Laurent series associated to a given string in the digit system $(\beta, \mathcal{S})$ is given by the evaluation map

$$\pi : \mathcal{S} \to \mathbb{F}_q((X^{-1})), \quad (s_i)_{-\infty < i \leq k} \mapsto \sum_{-\infty < i \leq k} s_i \beta_i.$$

When a representation ends in infinitely many zeros, it is said to be finite, and the final zeros are omitted. When all the s_i on the right hand side of the radix point are zeros, the representation is said to be an integer representation.

The set of all $f \in \mathbb{F}_q((X^{-1}))$ admitting an integer representation is called the set of β-integers. For $f \in \mathbb{F}_q((X^{-1}))$, we define the β-integer and the β-fractional part by

$$[f]_\beta = \pi(s_k \cdots s_0.)_\beta \quad \text{and} \quad \{f\}_\beta = \pi(.s_{-1}s_{-2}\cdots)_\beta,$$

respectively.

Now we are in a position to introduce our new algorithm, called β-continued fraction algorithm. The study of this algorithm is similar to the study of the usual continued fraction expansions. Let $\beta = (\beta_i)_{i \in \mathbb{Z}}$ be a base sequence and let

$$\begin{aligned}
\mathcal{H}'_0(\beta) &:= \{d\beta_0 \,:\, d \in \mathbb{F}_q[X],\, 0 < \deg d < \deg \beta_1 - \deg \beta_0\},\\
\mathcal{H}''_0(\beta) &:= \{d\beta_0 \,:\, d \in \mathbb{F}_q[X],\, \deg d < \deg \beta_1 - \deg \beta_0\},\\
\mathcal{H}_n(\beta) &:= \{d_0\beta_0 + \cdots + d_n\beta_n \,:\, d_i \in \mathbb{F}_q[X],\\
&\qquad\qquad \deg d_i < \deg \beta_{i+1} - \deg \beta_i, d_n \neq 0\} \quad (n \geq 1)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}(\beta) &:= \mathcal{H}'_0(\beta) \cup \bigcup_{n \geq 1} \mathcal{H}_n(\beta),\\
\mathcal{I}(\beta) &:= \mathcal{H}''_0(\beta) \cup \bigcup_{n \geq 1} \mathcal{H}_n(\beta).
\end{aligned}$$

Remark 2.2. Note that $|z| \geq |\beta_0|$ for all $z \in \mathcal{I}(\beta)$ and $|z| > |\beta_0|$ for all $z \in \mathcal{H}(\beta)$.

We start with the following easy result.

Lemma 2.3. *Let $\beta = (\beta_i)_{i\in\mathbb{Z}}$ be a base sequence and $f \in \mathbb{F}_q((X^{-1}))\backslash\mathcal{I}(\beta)$. Then*

$$\left|\frac{\beta_0^2}{\{f\}_\beta}\right| > |\beta_0|.$$

Proof. This is just another way to write $|\beta_0| > |\{f\}_\beta|$. The latter follows immediately from the definition of the β-fractional part $\{\cdot\}_\beta$. $\square$

Define the β-continued fraction expansion for f to be an expression of the form

$$f = a_0 + \cfrac{\beta_0^2}{a_1 + \cfrac{\cdots}{\cdots + \cfrac{\beta_0^2}{a_n + \cdots}}} = [a_0; a_1, a_2, \ldots]_\beta,$$

with $a_0 \in \mathcal{I}(\beta)$ and $a_i \in \mathcal{H}(\beta)$ for $i \geq 1$. Notice that for any $i \geq 1$, we have $\deg a_i > \deg \beta_0$.

We extend the standard simple continued fraction algorithm by defining the β-continued fraction algorithm. Given a formal power series f, we write $f_0 = f$ and let $a_0 = [f_0]_\beta$ denote the β-integer part of f_0. Thus we have $f = a_0 + \{f_0\}_\beta$ where $\{f_0\}_\beta$ denotes the β-fractional part of f_0. If $f \neq a_0$ we write f as

$$f = a_0 + \frac{\beta_0^2}{\beta_0^2/\{f_0\}_\beta}.$$

Next, let $f_1 = \beta_0^2/\{f_0\}_\beta$ and $a_1 = [f_1]_\beta$. If $f_1 \neq a_1$, we get

$$f = a_0 + \frac{\beta_0^2}{f_1} = a_0 + \cfrac{\beta_0^2}{a_1 + \cfrac{\beta_0^2}{\beta_0^2/\{f_1\}_\beta}}.$$

In general if $f_n \neq a_n$ we set $f_{n+1} = \beta_0^2/\{f_n\}_\beta$ and $a_{n+1} = [f_{n+1}]_\beta$. It is clear from Lemma 2.3, that $a_{n+1} \in \mathcal{H}(\beta)$. If $f_n = a_n$ holds for some n, the process terminates. If it goes on infinitely often we have

$$f = [a_0; a_1, a_2 \ldots]_\beta.$$

If $f \in \mathbb{F}_q((X^{-1}))$, then the sequence $(f_i)_{i\in\mathbb{N}}$ which define the β-continued fraction expansion of f is given by the transformation

$$\begin{aligned} T_\beta : \quad D(0, |\beta_0|) &\rightarrow D(0, |\beta_0|) \\ f \quad &\mapsto \begin{cases} \{\beta_0^2/f\}_\beta & \text{if } f \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $f = f_0$, then for each $i \in \mathbb{N}$, we obtain that

$$f_{i+1} = \frac{\beta_0^2}{T_\beta\left(\frac{\beta_0^2}{f_i}\right)} = \frac{\beta_0^2}{T_\beta^{i+1}\left(\frac{\beta_0^2}{f_0}\right)}.$$

It is clear that $T_\beta([0; a_1, a_2, \ldots]_\beta) = [0; a_2, \ldots]_\beta$.

Remark 2.4. If $\beta = (X^i)_{i\in\mathbb{Z}}$, then the transformation T_β describes the regular continued fraction over the field of Laurent series and has been introduced by Artin [1].

Definition 2.5. *The expression* $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]_\beta$ *is called the* (β, n)*-th convergent of* f *and the sequence* $(a_i)_{i\in\mathbb{N}} \in \mathcal{H}(\beta)^{\mathbb{N}}$ *is called the sequence of* β*-partial quotients of* f*.*

Lemma 2.6. *For* $n \geq 0$ *the* p_n *and* q_n *are expressed by the recurrence*

$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad p_n = a_n p_{n-1} + \beta_0^2 p_{n-2}, \tag{2.5}$$

$$q_{-2} = \beta_0^{-2}, \qquad q_{-1} = 0, \qquad q_n = a_n q_{n-1} + \beta_0^2 q_{n-2}. \tag{2.6}$$

Moreover, for $n \geq 1$*, we have*

$$|q_n| > |\beta_0||q_{n-1}| \quad \text{and thus} \quad |q_n| = |a_1 \cdots a_n|. \tag{2.7}$$

Proof. The Lemma follows along the same lines as for classical continued fractions. We sketch the proof of the second assertion. Let $n \geq 1$ and $a_n \in \mathcal{H}(\beta)$, then, because $q_0 = 1$ and $|a_i| > |\beta_0|$ we get

$$q_1 = a_1 q_0 \Rightarrow |q_1| > |\beta_0|.$$

Now suppose that

$$|q_n| > |\beta_0||q_{n-1}|,$$

then

$$|a_{n+1}||q_n| > |\beta_0||q_n| > |\beta_0|^2|q_{n-1}|.$$

Since

$$q_{n+1} = a_{n+1}q_n + \beta_0^2 q_{n-1},$$

it follows that

$$|q_{n+1}| = |a_{n+1}q_n| > |\beta_0||q_n|.$$

The fact that $|q_n| = |a_1 \cdots a_n|$ now follows from the recurrence relation for the sequence (q_n) since $|a_n q_{n-1}| > |\beta_0^2 q_{n-2}|$. □

Remark 2.7. We will often use (2.7) in the form

$$\deg q_n = \deg a_1 + \cdots + \deg a_n. \tag{2.8}$$

The proofs of the following Theorems run along the same lines as for classical continued fraction expansions. Thus, we will omit them.

Theorem 2.8. *Let* $f \in D(0, |\beta_0|)$ *such that*

$$f = [0; a_1, \ldots, a_n, a_{n+1}, \ldots]_\beta,$$

then

$$\left| f - \frac{p_n}{q_n} \right| = \frac{|\beta_0|^{2n+1}}{|q_n q_{n+1}|} = \frac{|\beta_0|^{2n+1}}{|a_{n+1}||q_n|^2} < \frac{|\beta_0|^{2n}}{|q_n|^2}. \tag{2.9}$$

Theorem 2.9. *Let $(a_i)_{i\geq 1}$ be a sequence with $a_i \in \mathcal{H}(\beta)$ for each $i \geq 1$. Then $\frac{p_n}{q_n} = [0; a_1, \dots, a_n]_\beta$ converges to an element of $\mathbb{F}_q((X^{-1}))$ if n tends to infinity.*

Theorem 2.10. *Every formal Laurent series can be uniquely expanded into a β-continued fraction.*

3. Metric properties of the β-continued fractions

In this section, we give some metric and ergodic properties of the transformation T_β. The metric properties of classical continued fractions of Laurent series have been studied in [2, 4, 5, 6, 7, 8].

Let $\beta = (\beta_i)_{i\in\mathbb{Z}}$ be a base sequence.

Proposition 3.1. *Let $a_1, \dots, a_n \in \mathcal{H}(\beta)$, $\frac{p_n}{q_n} = [0; a_1, \dots, a_n]_\beta$ and*

$$\Delta(a_1, \dots, a_n) := \{[0; a_1, \dots, a_n + \theta]_\beta, \theta \in D(0, |\beta_0|)\}.$$

Then

$$\Delta(a_1, \dots, a_n) = D\left(\frac{p_n}{q_n}, \frac{|\beta_0|^{2n+1}}{|q_n|^2}\right) = \overline{D}\left(\frac{p_n}{q_n}, \frac{|\beta_0|^{2n+1}}{q|q_n|^2}\right)$$

and thus

$$|\Delta(a_1, \dots, a_n)| = \frac{|\beta_0|^{2n+1}}{q|q_n|^2} = \frac{|\beta_0|^{2n+1}}{q|a_1|^2 \cdots |a_n|^2}.$$

Proof. Observe that

$$[0; a_1, \dots, a_n + \theta]_\beta = \frac{(a_n+\theta)p_{n-1} + \beta_0^2 p_{n-2}}{(a_n+\theta)q_{n-1} + \beta_0^2 q_{n-2}}.$$

Developing this in a power series with respect to θ and observing that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}\beta_0^{2n-2}$$

(which follows by induction on n) the result follows. □

Remark 3.2. The closed disk $\overline{D}(0, |\beta_0|)$ is compact because it is isomorphic to $\mathbb{F}_q^\infty$. A natural measure on $\mathbb{F}((X^{-1}))$ is the normalized Haar measure μ given by

$$\mu(D(a, q^{-n})) = q^{-(n+\deg\beta_0)} = \frac{q^{-n}}{|\beta_0|}.$$

Theorem 3.3. *The transformation T_β conserves the Haar measure μ.*

To prove this theorem, we need the following Lemmas.

Lemma 3.4. *Let $k > \deg\beta_0$ be an integer, then*

$$\sum_{\substack{\deg a = k \\ a\in\mathcal{H}(\beta)}} \frac{1}{|a|^2} = \frac{q-1}{q^k|\beta_0|}.$$

Proof. Let $s \in \mathbb{N}$, such that $\deg \beta_s \leq k < \deg \beta_{s+1}$. Let $a \in \mathcal{H}(\beta)$ such that $\deg a = k$ and $a = d_0\beta_0 + d_1\beta_1 + \cdots + d_s\beta_s$. Then

$$\begin{aligned}
\sum_{\substack{\deg a = k \\ a \in \mathcal{H}(\beta)}} \frac{1}{|a|^2} &= \sum_{\substack{\deg a = k \\ a \in \mathcal{H}(\beta)}} \frac{1}{q^{2k}} \\
&= \frac{q^{\delta_0} \ldots q^{\delta_{s-1}} (q-1) q^{k - \deg \beta_s}}{q^{2k}} \\
&= \frac{q-1}{q^k |\beta_0|}.
\end{aligned}$$

□

Remark 3.5. Let $\beta = (\beta_i)_{i \in \mathbb{Z}}$ be a base sequence, then

$$\sum_{a \in \mathcal{H}(\beta)} |a|^{-2} = \frac{1}{|\beta_0|^2}.$$

Lemma 3.6. *We have the decomposition*

$$T_\beta^{-1} \Delta(a_1, \ldots, a_n) = \bigcup_{b \in \mathcal{H}(\beta)} \Delta(b, a_1, \ldots, a_n)$$

where the union on the right hand side is disjoint.

Proof. Let $\theta = [0; b_1, \ldots, b_n, b_{n+1}, \ldots] \in T_\beta^{-1} \Delta(a_1, \ldots, a_n)$, or equivalently $T_\beta \theta = [0; a_1, \ldots, a_n, \ldots]$. If $T_\beta \theta = [0; b_2, \ldots, b_{n+1}, \ldots]$, then $b_2 = a_1, \ldots, b_{n+1} = a_n$. Therefore $\theta \in T_\beta^{-1} \Delta(a_1, \ldots, a_n)$ if and only if there exists $b' \in \mathcal{H}(\beta)$ such that $\theta \in \Delta(b', a_1, \ldots, a_n)$. Then from Theorem 2.10, we have

$$\Delta(b, a_1, \ldots, a_n) \cap \Delta(b', a_1, \ldots, a_n) \neq \emptyset \Leftrightarrow b' = b.$$

Then the proof of the Lemma follows immediately. □

Now we proceed with the proof of Theorem 3.3.

Proof of Theorem 3.3. Let us prove that T_β conserves the Haar measure μ over any disk of the form $\Delta(a_1, \ldots, a_n)$. We know from Proposition 3.1 that

$$\mu(\Delta(a_1, \ldots, a_n)) = \frac{|\beta_0|^{2n}}{|a_1 \cdots a_n|^2}.$$

Then from Remark 3.5 and Lemma 3.4 3.6 we derive

$$\begin{aligned}
\mu(T_\beta^{-1} \Delta(a_1, \ldots, a_n)) &= \frac{|\beta_0|^{2n+2}}{|a_1 \cdots a_n|^2} \sum_{b \in \mathcal{H}_\beta} \frac{1}{|b|^2} \\
&= \frac{|\beta_0|^{2n}}{|a_1 \cdots a_n|^2} = \mu(\Delta(a_1, \ldots, a_n)).
\end{aligned}$$

Thus T_β conserves the Haar measure μ □

Lemma 3.7. *For any given $\Delta(a_1,\dots,a_n)$ and $E \in D(0,|\beta_0|)$, we have*

$$\mu((T_\beta^{-n}E) \cap \Delta(a_1,\dots,a_n)) = \mu(E)\mu(\Delta(a_1,\dots,a_n)).$$

Proof. It suffices to prove the property for the disks $\Delta(b_1,\dots,b_m)$. Let $\theta = [0; c_1,\dots,c_n,\dots]_\beta \in D(0,|\beta_0|)$, then $\theta \in T_\beta^{-n}\Delta(b_1,\dots,b_m)$ if and only if

$$\begin{cases} c_{n+1} &= b_1 \\ c_{n+2} &= b_2 \\ & \dots \\ c_{n+m} &= b_m \end{cases}$$

which implies that

$$T_\beta^{-n}(\Delta(b_1,\dots,b_m)) = \bigcup_{(c_1,\dots,c_n)\in\mathcal{H}_\beta^n} \Delta(c_1,\dots,c_n,b_1,\dots,b_m).$$

Therefore we get

$$T_\beta^{-n}(\Delta(b_1,\dots,b_m)) \cap \Delta(a_1,\dots,a_n) = \Delta(a_1,\dots,a_n,b_1,\dots,b_m).$$

Taking measures we arrive at

$$\mu(T_\beta^{-n}(\Delta(b_1,\dots,b_m)) \cap \Delta(a_1,\dots,a_n)) = \mu(\Delta(a_1,\dots,a_n))\mu(\Delta(b_1,\dots,b_m))$$

which proves the lemma. □

Theorem 3.8. *The transformation T_β is ergodic with respect to the measure μ.*

Proof. Suppose that the transformation T_β satisfies $T_\beta^{-1}(E) = E$ for some set $E \subset D(0,|\beta_0|)$, then $T_\beta^{-n}(E) = E$ for any positive integer n. From Lemma 3.7 we get

$$\mu(E \cap \Delta(a_1,\dots,a_n)) = \mu(E)\mu(\Delta(a_1,\dots,a_n))$$

for any given $\Delta(a_1,\dots,a_n)$. Thus $\mu(E\cap F) = \mu(E)\mu(F)$ holds for each $F \subset D(0,|\beta_0|)$. In particular, for $F = D(0,|\beta_0|)\setminus E$ we get $\mu(E)\mu(D(0,|\beta_0|)\setminus E) = 0$. Thus either $\mu(E) = 0$ or $\mu(D(0,|\beta_0|)\setminus E) = 0$ and we are done. □

It follows from the ergodic theorem that if f is an integrable function on $D(0,|\beta_0|)$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T_\beta^{n-1}\omega) = \int_{D(0,|\beta_0|)} f\, d\mu \qquad a.e. \tag{3.1}$$

In order to give some applications using the ergodic theorem, we need the following Lemma.

Proposition 3.9. *For μ-almost all $\omega = [0, a_1(\omega),\cdots]_\beta \in D(0,|\beta_0|)$ we have*

(i) $\displaystyle\lim_{N\to+\infty}\frac{1}{N}\,\mathrm{card}\{n \le N,\ \deg a_n(\omega) = k\} = \frac{q-1}{q^k}.$

(ii) *For* $N \to \infty$ *we have* $\displaystyle\sum_{n=1}^{N} \deg a_n(\omega) \sim \frac{(q-1)\deg \beta_0 + q}{(q-1)} N.$

(iii) *For* $n \to \infty$ *we have* $\displaystyle\log_q \left| \omega - \frac{p_n}{q_n} \right| \sim -\frac{2qn}{q-1}.$

Proof. Let $n \in \mathbb{N}$, ω be a formal power series over $D(0, |\beta_0|)$ such that

$$\omega = [0; a_1(\omega), a_2(\omega), \ldots, a_n(\omega), \ldots]_\beta.$$

Then

$$T_\beta^{n-1}(\omega) = [0; a_n(\omega), a_{n+1}(\omega), \ldots]_\beta.$$

(i) Let $k > \deg \beta_0$ and

$$\begin{aligned} f : D(0, |\beta_0|) &\to \{0, 1\} \\ \omega &\mapsto \begin{cases} 1 & \text{if} \quad \deg\, [\beta_0^2/\omega]_\beta = k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that

$$f(T_\beta^{n-1}(\omega)) = \begin{cases} 1 & \text{if} \quad \deg a_n(\omega) = k, \\ 0 & \text{otherwise.} \end{cases}$$

From (4.1) , we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ \deg a_n(\omega) = k}}^{N} 1 = \int_{D(0,|\beta_0|)} f d\mu.$$

Now, partitioning $D(0, |\beta_0|)$ according to the value of a_1 in the first convergent $\omega = [0; a_1, \cdots]_\beta$, we get

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \leq N,\ \deg a_n(\omega) = k\} &= \int_{D(0,|\beta_0|)} f d\mu \\ &= \int_{\{\omega \in D(0,|\beta_0|) : f(\omega) = 1\}} d\mu \\ &= \mu([0, a_1(\omega), \cdots]_\beta : \deg a_1(\omega) = k) \\ &= \mu\left(\bigcup_{\deg a_1(\omega) = k} D\left(\frac{\beta_0^2}{a_1}, \frac{|\beta_0|^2}{|a_1|^2} \right) \right) \\ &= \sum_{\deg a_1(\omega) = k} \mu\left(D\left(\frac{\beta_0^2}{a_1}, \frac{|\beta_0|^2}{|a_1|^2} \right) \right) \\ &= \sum_{\deg a_1(\omega) = k} \frac{|\beta_0|}{|a_1^2|}. \end{aligned}$$

From Lemma 3.4, we get

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{card}\{n \leq N,\ \deg a_n(\omega) = k\} = \frac{q-1}{q^k}.$$

(ii) Let

$$\begin{array}{rcl} f : D(0,|\beta_0|) & \to & \mathbb{N} \\ \omega & \mapsto & \begin{cases} \deg\ [\beta_0^2/\omega]_\beta & \text{if} \quad \omega \neq 0, \\ \ 0 & \text{else.} \end{cases} \end{array}$$

Then we get

$$\sum_{n=1}^{N} f(T_\beta^{n-1}(\omega)) = \sum_{n=1}^{N} \deg a_n(\omega).$$

From (4.1), we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg a_n(\omega) = \int_{D(0,|\beta_0|)} \deg\ \left[\beta_0^2/\omega\right]_\beta d\mu.$$

Partitioning $D(0,|\beta_0|)$ according to the value of a in the first convergent $\omega = [0; a, \cdots]_\beta$ and observing that $a = [\beta_0^2/\omega]_\beta$ yields

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg a_n(\omega) &= \sum_{k>\deg\beta_0} \sum_{\deg a=k} \int_{D\left(\frac{\beta_0^2}{a}, \frac{|\beta_0|^2}{|a|^2}\right)} \deg a d\mu, \\ &= \sum_{k>\deg\beta_0} \sum_{\substack{a\in\mathcal{H}(\beta) \\ \deg a=k}} k\mu\left(D\left(\frac{\beta_0^2}{a}, \frac{|\beta_0|^2}{|a|^2}\right)\right) \\ &= |\beta_0| \sum_{k>\deg\beta_0} k \sum_{\substack{a\in\mathcal{H}(\beta) \\ \deg a=k}} \frac{1}{|a|^2} \end{aligned}$$

Applying Lemma 3.4 finally yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \deg a_n(\omega) = (q-1) \sum_{k>\deg\beta_0} \frac{k}{q^k} = \frac{(q-1)\deg\beta_0 + q}{(q-1)}.$$

(iii) The result follows immediately from (2.8), (2.9) and (ii).

□

Corollary 3.10. *Let $q_n(\omega)$ denote the denominator of the (n,β)-convergent $\frac{p_n(\omega)}{q_n(\omega)}$ of $\omega \in D(0,|\beta_0|)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \deg q_n(\omega) = \frac{(q-1)\deg\beta_0 + q}{(q-1)} \quad a.e.$$

Proof. The proof of the Corollary follows immediately from (2.8). □

4. Hausdorff dimensions of bounded type sets for β-continued fractions

In this section, we consider the Hausdorff dimensions of sets of Laurent series related to β-continued fraction expansions. If U is a countable collection of discs $J_1, J_2, \cdots \subset \overline{D}(0, |\beta_0|)$, we define

$$\Lambda_s(U) = \sum_{i=1}^{\infty} |J_i|^s$$

for any real $s \geq 0$. If S is any set of formal power series, we define

$$L_{s,\delta}(S) = \inf \Lambda_s(U)$$

where the infimum is taken over all collections U of open discs J_i which satisfy $|J_i| < \delta$ and whose union contains S. We call such a set U an (open) cover of S. Finally we define

$$L_s(S) = \lim_{\delta \to 0} L_{s,\delta}(S).$$

The Hausdorff dimension $\dim(S)$ is the supremum of those s such that $L_s(S) = +\infty$ or the infimum of those s such that $L_s(S) = 0$.

Lemma 4.1. [3, Mass distribution principle 4.2] *Let $E \subset \overline{D}(0, |\beta_0|)$ and τ is a measure with $\tau(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$\tau(D) \leq c|D|^s$$

for all discs D with diameter $|D| \leq \delta$. Then

$$\dim E \geq s.$$

Remark 4.2. Since the valuation $|\cdot|$ is non archimedean, it follows that if two discs $\Delta(a_1, \ldots, a_n)$ and $\Delta(b_1, \ldots, b_n)$ intersect, then one contains the other.

Let now $S = \{a_1, \ldots, a_m\}$ be a non-empty finite set of elements of $\mathcal{H}(\beta)$ and

$$E_S = \{\omega \in \overline{D}(0, |\beta_0|) : a_i(\omega) \in S \text{ for } i \geq 1\}.$$

In the following, we adopt the same method as in [12] to give the Hausdorff dimension of E_S.

Theorem 4.3. *Let t be the unique real number determined by*

$$\sum_{k=1}^{m} \left| \frac{\beta_0}{a_k} \right|^{2t} = 1,$$

then

$$\dim E_S = t.$$

Proof. Referring to the definition of E_S, we get

$$E_S = \bigcap_{n=1}^{\infty} \bigcup_{(b_1,\ldots,b_n)\in S^n} \Delta(b_1,\ldots,b_n)$$

where $\Delta(b_1,\ldots,b_n)$ is defined in Proposition 3.1. Then

$$\begin{aligned} \sum_{(b_1,\ldots,b_n)\in S^n} |\Delta(b_1,\ldots,b_n)|^t &= \sum_{(b_1,\ldots,b_n)\in S^n} \frac{|\beta_0|^{(2n+1)t}}{q^t|b_1\cdots b_n|^{2t}} \\ &= \frac{|\beta_0|^t}{q^t}\left(\sum_{b_1\in S}\frac{|\beta_0|^{2t}}{|b_1|^{2t}}\right)\cdots\left(\sum_{b_n\in S}\frac{|\beta_0|^{2t}}{|b_n|^{2t}}\right) \\ &= \frac{|\beta_0|^t}{q^t}, \end{aligned}$$

so

$$H_t(E_S) \le (|\beta_0|/q)^t.$$

Combining this with the definition of H, we get

$$\dim E_S \le t.$$

In order to establish the lower bound, we shall apply Lemma 4.1. Define a probability measure τ on E_S by

$$\tau(\Delta(b_1,\ldots,b_n)) = \prod_{k=1}^{n}\left|\frac{\beta_0}{b_k}\right|^{2t}.$$

It is easily proved that τ is well defined since

$$\sum_{(b_1,\ldots,b_n)\in S^n} \tau(\Delta(b_1,\ldots,b_n)) = 1,$$

and

$$\sum_{b_{n+1}\in S} \tau(\Delta(b_1,\ldots,b_{n+1})) = \tau(\Delta(b_1,\ldots,b_n)).$$

Next we estimate $\tau(\overline{D}(\omega,q^{-k}))$, where $\omega \in \overline{D}(0,|\beta_0|)$ and

$$k > \max_{1\le i\le n}\{\deg a_i\} - 3\deg\beta_0 + 1.$$

Consider now each infinite sequence $(b_1,b_2,\cdots)\in S^\infty$ such that

$$|\Delta(b_1,\ldots,b_n)| \le q^{-k} \quad\text{and}\quad |\Delta(b_1,\ldots,b_{n-1})| > q^{-k}.$$

Let $\mathcal{A}$ denote the finite set of all finite sequences admitting these properties and consider the two following sets defined by

$$\begin{aligned} \mathcal{B} &= \{\Delta(b_1,\ldots,b_{n-1}) : (b_1,\ldots,b_n)\in\mathcal{A}\}, \\ \mathcal{C} &= \{\Delta(b_1,\ldots,b_{n-1})\in\mathcal{B} : \Delta(b_1,\ldots,b_{n-1})\cap\overline{D}(\omega,q^{-k})\neq\emptyset\}. \end{aligned}$$

We select all discs in $\mathcal{C}$ which are maximal, and denote by $\widetilde{\mathcal{C}}$ the set of all maximal $\Delta(b_1, \ldots, b_{n-1}) \in \mathcal{C}$. We claim that

$$\mathrm{card}(\widetilde{\mathcal{C}}) \leq 1.$$

In fact, let us suppose that there exist two discs

$$\Delta(b_1, \ldots, b_{n-1}) \in \widetilde{\mathcal{C}} \quad \text{and} \quad \Delta(b'_1, \ldots, b'_{n-1}) \in \widetilde{\mathcal{C}}.$$

Since the two discs fulfil

$$|\Delta(b_1, \ldots, b_{n-1})| > q^{-k} \quad \text{and} \quad |\Delta(b'_1, \ldots, b'_{n-1})| > q^{-k},$$

we get, by Remark 4.2,

$$\overline{D}(\omega, q^{-k}) \subset \Delta(b_1, \ldots, b_{n-1}) \quad \text{and} \quad \overline{D}(\omega, q^{-k}) \subset \Delta(b'_1, \ldots, b'_{n-1}),$$

thus either

$$\Delta(b_1, \ldots, b_{n-1}) \subset \Delta(b'_1, \ldots, b'_{n-1}) \quad \text{or} \quad \Delta(b'_1, \ldots, b'_{n-1}) \subset \Delta(b_1, \ldots, b_{n-1})$$

which contradicts the maximality of elements of $\widetilde{\mathcal{C}}$. If $\mathrm{card}(\widetilde{\mathcal{C}}) = 0$, then

$$\overline{D}(\omega, q^{-k}) \cap E_S = \emptyset, \quad \text{and} \quad \tau(\overline{D}(\omega, q^{-k})) = 0. \tag{4.1}$$

If $\mathrm{card}(\widetilde{\mathcal{C}}) = 1$, let $\widetilde{\mathcal{C}} = \{\Delta(b_1, \ldots, b_{n-1})\}$. Choose $b_n \in S$ such that $\Delta(b_1, \ldots, b_{n-1}, b_n) \in \mathcal{A}$. Then

$$\begin{aligned}
\tau(\overline{D}(\omega, q^{-k})) &\leq \tau(\Delta(b_1, \ldots, b_{n-1})), \\
&= \prod_{k=1}^{n-1} \left|\frac{\beta_0}{b_k}\right|^{2t} = \left|\frac{b_n}{\beta_0}\right|^{2t} \prod_{k=1}^{n} \left|\frac{\beta_0}{b_k}\right|^{2t} \\
&= q^{(2\deg b_n - 2\deg\beta_0)t} |\Delta(b_1, \ldots, b_{n-1}, b_n)|^t \\
&\leq q^{(2\max(\deg a_1, \ldots, \deg a_n) - 2\deg\beta_0)t} |\overline{D}(\omega, q^{-k})|^t.
\end{aligned}$$

Finally, from (4.1) and Lemma 4.1, we get $\dim(E_S) \geq t$, which completes the proof of the Theorem. $\square$

5. On sets of series having a given rate of convergence

Following [11], it is natural to consider the following sets

$$A(\alpha) = \Big\{\omega \in \overline{D}(0, |\beta_0|) : \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \deg a_k(\omega) = \alpha\Big\},$$

where $\alpha \geq \deg\beta_0 + 1$.

Theorem 5.1. *For any* $\alpha \geq \deg \beta_0 + 1$, *we have*

$$\dim A(\alpha) = \frac{\log\left(\frac{(\alpha - \deg \beta_0)q}{\alpha - \deg \beta_0 - 1}\right)}{2 \log q} + \frac{\log((\alpha - \deg \beta_0 - 1)(q-1))}{2(\alpha - \deg \beta_0) \log q}$$

where $0 \cdot \log 0 := 0$.

In order to get the lower bound of Hausdorff dimension, we need the following result. If λ is a finite measure on $\overline{D}(0, |\beta_0|)$, the lower pointwise dimension of λ at $\omega \in \overline{D}(0, |\beta_0|)$ is given by

$$\underline{\dim}_{\mathrm{loc}} \lambda(\omega) = \liminf_{r \to 0} \frac{\log \lambda(\overline{D}(\omega, r))}{\log r}.$$

Lemma 5.2. *Let* E *be a non-empty Borel set. Then*

$$\dim(E) = \sup \Big\{ s : \text{there exists a measure } \lambda \text{ with } 0 < \lambda(E) < \infty$$
$$\text{and } \underline{\dim}_{\mathrm{loc}} \lambda(\omega) \geq s \text{ for } \lambda - \text{almost all } \omega \in E \Big\}.$$

Proof of Theorem 5.1. We distinguish two cases.

Case 1. $\alpha > 1 + \deg \beta_0$. For any $t < -\log q$ and for any sequence $\{b_1, \dots, b_n\} \subset \mathcal{H}(\beta)$ such that $\deg b_j > \deg \beta_0$ $(1 \leq j \leq n)$, we define a probability measure λ_t on $\overline{D}(0, |\beta_0|)$ by letting

$$\lambda_t(\Delta(b_1, \dots, b_n)) = \exp(t \sum_{j=1}^{n} \deg b_j - nP(t)),$$

where

$$P(t) = \log(q(q-1)) - \log(e^{-t} - q) + t \deg \beta_0.$$

It can be easily seen that this measure is well defined since

$$\sum_{(b_1, \dots, b_n)} \lambda_t(\Delta(b_1, \dots, b_n)) = 1,$$

and

$$\sum_{b_{n+1}} \lambda_t(\Delta(b_1, \dots, b_{n+1})) = \lambda_t(\Delta(b_1, \dots, b_n)).$$

These equalities are assured by the following result

Lemma 5.3.

$$\sum_{b \in \mathcal{H}(\beta)} e^{t \deg b} = \frac{q(q-1)}{e^{-t} - q} e^{t \deg \beta_0} = e^{P(t)}$$

Proof. Let $\delta_i = \deg \beta_{i+1} - \deg \beta_i$. As usual, we divide the proof into two parts. Part 1. If $\deg \beta_1 = \deg \beta_0 + 1$, then $\mathcal{H}_0(\beta) = \emptyset$ and we get

$$\begin{aligned}
\sum_{b\in\mathcal{H}(\beta)} e^{t\deg b} &= \sum_{n=1}^{\infty} \sum_{b\in\mathcal{H}_n(\beta)} e^{t\deg b} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{\delta_n} (q^{\delta_0} \cdots q^{\delta_{n-1}}(q-1)q^k) e^{t(k+\deg \beta_n)} \\
&= (q-1) \sum_{n=1}^{\infty} q^{\deg \beta_n - \deg \beta_0} e^{t\deg \beta_n} \sum_{k=0}^{\delta_n} e^{tk} q^k \\
&= (q-1) \sum_{n=1}^{\infty} q^{\deg \beta_n - \deg \beta_0} e^{t\deg \beta_n} \frac{1-(qe^t)^{\delta_n+1}}{1-qe^t} \\
&= \frac{q(q-1)}{e^{-t}-q} e^{t\deg \beta_0} = e^{P(t)}.
\end{aligned}$$

Part 2. If $\deg \beta_1 > \deg \beta_0 + 1$, then

$$\begin{aligned}
\sum_{b\in\mathcal{H}(\beta)} e^{t\deg b} &= \sum_{n=1}^{\infty} \sum_{b\in\mathcal{H}_n(\beta)} e^{t\deg b} + \sum_{b\in\mathcal{H'}_0(\beta)} e^{t\deg b} \\
&= \frac{q-1}{|\beta_0|(1-qe^t)} |\beta_1| e^{t\deg \beta_1} + (q-1) \sum_{k=1}^{\deg \beta_1 - \deg \beta_0 - 1} (qe^t)^k. \\
&= e^{P(t)}
\end{aligned}$$

Indeed, the latter equality is implied from

$$\sum_{k=n}^{m} q^k = \frac{q^{m+1}-q^n}{q-1}. \tag{5.1}$$

$\square$

Notice that the condition $t < -\log q$ is required.

Remark 5.4. The sequence $\{a_n(\omega)\}_{n\geq 1}$ is a sequence of independent and identically distributed random variables with respect to the probability measure λ_t.

Lemma 5.5.

$$I = \int_{\overline{D}(0,|\beta_0|)} \deg a_1(\omega) d\lambda_t(\omega) = P'(t) = \frac{1}{1-e^t q} + \deg \beta_0. \tag{5.2}$$

Proof. We know that $\overline{D}(0,|\beta_0|)=\bigcup_{B\in\mathcal{H}_\beta}[0,B,\cdots]$, then we get

$$\begin{aligned}
I &= \sum_{B\in\mathcal{H}_\beta}\int_{[0,B,\cdots]}\deg a_1(\omega)d\lambda_t\\
&= \sum_{B\in\mathcal{H}_\beta}\deg B\int_{[0,B,\cdots]}d\lambda_t\\
&= \sum_{s\geq 0}\sum_{B\in\mathcal{H}_s(\beta)}\deg B\ \lambda_t(\omega\in[0,B,\cdots];\ a_1(\omega)=B)\\
&= \sum_{s\geq 0}\sum_{B\in\mathcal{H}_s(\beta)}\deg B\exp(t\deg B-P(t)).
\end{aligned}$$

Now we divide the proof into two parts.
Part 1. If $\deg\beta_1>\deg\beta_0+1$, then

$$\begin{aligned}
I &= \sum_{s\geq 1}\sum_{B\in\mathcal{H}_s(\beta)}\deg Be^{t\deg B-P(t)}+\sum_{B\in\mathcal{H}'_0(\beta)}\deg Be^{t\deg B-P(t)}\\
&= \sum_{s\geq 1}\sum_{\substack{B\in\mathcal{H}_s(\beta)\\ \deg B=\deg\beta_s}}^{\deg\beta_{s+1}-1}\deg Be^{t\deg B-P(t)}+\sum_{B\in\mathcal{H}'_0(\beta)}\deg Be^{t\deg B-P(t)}\\
&= \sum_{s\geq 1}\sum_{\deg\beta_s=k}^{\deg\beta_{s+1}-1}\sum_{\substack{B\in\mathcal{H}_s(\beta)\\ \deg B=k}}ke^{tk-P(t)}+\sum_{k=1}^{\delta_0-1}\frac{(\deg\beta_0+k)(q-1)q^k}{e^{-t(\deg\beta_0+k)+P(t)}}\\
&= \frac{q-1}{|\beta_0|e^{P(t)}}\sum_{s\geq 1}\sum_{\deg\beta_s=k}^{\deg\beta_{s+1}-1}k(qe^t)^k+\frac{(q-1)e^{t\deg\beta_0}}{e^{P(t)}}\sum_{k=1}^{\delta_0-1}(\deg\beta_0+k)(qe^t)^k\\
&= \frac{1+\deg\beta_0(1-qe^t)}{1-qe^t}=P'(t).
\end{aligned}$$

The latter equalities are implied by (5.1) and the fact that

$$\sum_{k=n}^{m}kq^k=\frac{q^n}{(q-1)^2}(mq^{m-n+2}+(1-n)q+n-(m+1)q^{m-n+1}).$$

Part 2. If $\deg\beta_1=\deg\beta_0+1$, then $\mathcal{H}_0(\beta)=\emptyset$ and the proof follows immediately from Part 1.
□

For any $\varepsilon > 0$ and $\omega \in A(\alpha)$, there exists an integer $N(\omega)$ such that $\forall n \geq N(\omega)$,

$$n(\alpha-\varepsilon) \leq \sum_{j=1}^{n} \deg a_j(\omega) \leq n(\alpha+\varepsilon).$$

It follows that

$$A(\alpha) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \in \overline{D}(0, |\beta_0|) : n(\alpha-\varepsilon) \leq \sum_{j=1}^{n} \deg a_j(\omega) \leq n(\alpha+\varepsilon)\}.$$

Let $\mathcal{J}(n,\alpha,\varepsilon)$ be the family of all $\Delta(b_1,\ldots,b_n)$ such that

$$\deg b_j \geq 1 + \deg \beta_0 \quad \text{and} \quad n(\alpha-\varepsilon) \leq \sum_{j=1}^{n} \deg b_j(\omega) \leq n(\alpha+\varepsilon)$$

For $N \geq 1$, we select all discs in $\bigcup_{n=N}^{\infty} \mathcal{J}(n,\alpha,\varepsilon)$ which are maximal. We denote by $\mathcal{M}(n,\alpha,\varepsilon)$ the set of all maximal discs in $\bigcup_{n=N}^{\infty} \mathcal{J}(n,\alpha,\varepsilon)$. It is clear that $\bigcup_{n=N}^{\infty} \mathcal{J}(n,\alpha,\varepsilon)$ is a cover of $A(\alpha)$, it follows that $\mathcal{M}(n,\alpha,\varepsilon)$ is a cover of $A(\alpha)$. For any $t < -\log q$, let

$$d(t) = \frac{P(t) - (\alpha+\varepsilon)}{2(\alpha - \deg \beta_0 - \varepsilon)\log q}$$

For any $\Delta(b_1,\ldots,b_n) \in \mathcal{M}(n,\alpha,\varepsilon)$, we have

$$\begin{aligned}
\lambda_t(\Delta(b_1,\ldots,b_n)) &= e^{t\sum_{j=1}^{n}\deg b_j - nP(t)} \\
&\geq e^{tn(\alpha+\varepsilon)-nP(t)} \\
&\geq q^{-2d(t)\sum_{j=1}^{n}\deg b_j}|\beta_0|^{2nd(t)}
\end{aligned}$$

It follows that

$$\sum_{\Delta(b_1,\ldots,b_n)\in\mathcal{M}(n,\alpha,\varepsilon)} |\Delta(b_1,\ldots,b_n)|^{d(t)}$$

$$\begin{aligned}
&= \sum_{\Delta(b_1,\ldots,b_n)\in\mathcal{M}(n,\alpha,\varepsilon)} \frac{|\beta_0|^{2nd(t)}}{q^{d(t)}|b_1\cdots b_n|^{2d(t)}} \\
&\leq q^{-d(t)} \sum_{\Delta(b_1,\ldots,b_n)\in\mathcal{M}(n,\alpha,\varepsilon)} q^{-2d(t)\sum_{i=1}^{n}\deg b_i}|\beta_0|^{2nd(t)} \\
&\leq q^{-d(t)} \sum_{\Delta(b_1,\ldots,b_n)\in\mathcal{M}(n,\alpha,\varepsilon)} \lambda_t(\Delta(b_1,\ldots,b_n)) \\
&\leq q^{-d(t)}.
\end{aligned}$$

Combine this with the definition of the Hausdorff dimension and the fact that ε is arbitrary small, we get

$$\dim A(\alpha) \leq \frac{P(t) - \alpha t}{2(\alpha - \deg \beta_0)\log q},$$

Notice that the derivative $P' : (-\infty, -\log q) \to (1 + \deg \beta_0, +\infty)$ is one to one and increasing. Let $\beta \in (-\infty, -\log q)$ be the unique solution of $\alpha = P'(\beta)$. Then

$$\beta = \log(\frac{\alpha - \deg \beta_0 - 1}{(\alpha - \deg \beta_0)q}).$$

It follows that

$$\dim A(\alpha) \leq \frac{P(\beta) - \alpha\beta}{2(\alpha - \deg \beta_0) \log q}.$$

To prove the inverse inequality, we use the fact that $\{a_n(\omega)\}_{n\geq 1}$ is a sequence of independent and identically distributed random variables with respect to λ_β. By the law of large numbers and (5.2), we have:
(i) $\lambda_\beta(A(\alpha)) = 1$.
(ii) $n^{-1} \log \lambda_\beta(\Delta(b_1(\omega), \ldots, b_n(\omega))) \to \beta\alpha - P(\beta) \quad \lambda_\beta - a.e$;
(iii) $n^{-1} \log \lambda_\beta |(b_1(\omega), \ldots, b_n(\omega)|) \to (-\alpha + 2\deg \beta_0) \log q \quad \lambda_\beta - a.e.$
Notice that for any $m \geq 1$, and any $\omega \in \overline{D}(0, |\beta_0|)$ having an infinite continued fraction expansion, there exists an integer $n(\omega)$ such that

$$2\sum_{j=1}^{n(\omega)} \deg b_j(\omega) + 1 \leq m < 2 \sum_{j=1}^{n(\omega)+1} \deg b_j(\omega) + 1.$$

So

$$\Delta(b_1(\omega), \ldots, b_{n+1}(\omega)) \subset \overline{D}(\omega, q^{-m}) \subset \Delta(b_1(\omega), \ldots, b_n(\omega)).$$

Combine this with (ii) and (iii), we get

$$\lim_{m\to\infty} \frac{\log \lambda_\beta(\overline{D}(\omega, q^{-m}))}{\log |\overline{D}(\omega, q^{-m})|} = \frac{P(\beta) - \beta\alpha}{2(\alpha - \deg \beta_0) \log q}, \quad \lambda_\beta - a.e.$$

From lemma 5.2 and (i), we get

$$\dim A(\alpha) \geq \frac{P(\beta) - \beta\alpha}{2(\alpha - \deg \beta_0) \log q}.$$

Case 2. $\alpha = 1 + \deg \beta_0$. In this case we will prove that

$$\dim A(1 + \deg \beta_0) = \frac{\log(q(q-1))}{2 \log q} = \lim_{\alpha\to 1+\deg \beta_0} \dim A(\alpha).$$

Using the same argument as in case 1, we show that for any $\varepsilon > 0$,

$$\dim A(1 + \deg \beta_0) \leq \frac{P(t) - t(1 + \deg \beta_0 + \varepsilon)}{2(1 + \deg \beta_0 - \varepsilon) \log q}, \quad \forall t < -\log q.$$

And we get the upper bound by letting $\varepsilon \to 0$ and $t \to -\infty$.
Now to get the inverse inequality, let $\mathcal{H}_\beta^{1+\deg \beta_0} = \{b_j \in \mathcal{H}_\beta,$ such that $\deg b_j =$

$1+\deg\beta_0, \forall 1 \leq j \leq n\}$. We know that $\Delta(b_1,\ldots,b_n)$ is a disc with diameter q^{-2n-1}. In fact,

$$\begin{aligned} |\Delta(b_1,\ldots,b_n)| &= \frac{|\beta_0|^{2n}}{q|b_1\cdots b_n|^2} \\ &= |\beta_0|^{2n}q^{-2\sum_1^n \deg b_j - 1} \\ &= |\beta_0|^{2n}q^{-2(n+n\deg\beta_0)-1} \\ &= q^{-2n-1}. \end{aligned}$$

Let now $(b_1,\ldots,b_n) \neq (b'_1,\ldots,b'_n)$ then

$$\Delta(b_1,\ldots,b_n) \cap \Delta(b'_1,\ldots,b'_n) = \emptyset$$

Let

$$E_n = \bigcup_{b_j \in \mathcal{H}_\beta^{1+\deg\beta_0}} \Delta(b_1,\ldots,b_n),$$

then

$$E_n = \{\omega \in \overline{D}(0,|\beta_0|) : \deg a_1(\omega) = \cdots \deg a_n(\omega) = 1+\deg\beta_0\},$$

and E_n consists of $q^n(q-1)^n$ disjoint discs of diameter q^{-2n-1}. Set

$$E = \bigcap_{n=1}^{\infty} E_n.$$

It is clear that

$$E = \{\omega \in \overline{D}(0,|\beta_0|) : \deg a_k(\omega) = 1+\deg\beta_0, \quad \text{for any } k \geq 1\}$$

It follows that $E \subset A(1+\deg\beta_0)$. Now, in order to give a lower bound of Hausdorff dimension of E, define a masse distribution μ supported on E, by

$$\mu(\Delta(b_1,\ldots,b_n)) = (q-1)^{-n}q^{-n},$$

for all $n \geq 1$ and $b_k \in \mathcal{H}_\beta^{1+\deg\beta_0}$, $1 \leq k \leq n$. For any $\omega \in \overline{D}(0,|\beta_0|)$ and $m \geq 3$, choose $n \geq 2$ such that

$$2n-1 \leq m \leq 2n+1,$$

so $\overline{D}(\omega,q^{-m})$ intersect at most $(n-1)$ discs in E_{n-1}. Therefore

$$\mu(\overline{D}(\omega,q^{-m})) \leq (q-1)^{-n+1}q^{-n+1} \leq ((q-1)q)^{\frac{-m+3}{2}} \leq ((q-1)q)^{\frac{3}{2}}(q^{-m})^{\frac{\log(q(q-1))}{2\log q}}.$$

From Lemma 5.2, we get

$$\dim E \geq \frac{\log(q(q-1))}{2\log q},$$

which achieve the proof.

□

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