INTERIOR COMPONENTS OF A TILE ASSOCIATED TO A QUADRATIC CANONICAL NUMBER SYSTEM

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Abstract. Let $\alpha = -2+\sqrt{-1}$ be a root of the polynomial $p(x) = x^2+4x+5$. It is well known that the pair $(p(x), \{0,1,2,3,4\})$ forms a *canonical number system*, *i.e.*, that each $x \in \mathbb{Z}[\alpha]$ admits a finite representation of the shape $x = a_0 + a_1\alpha + \cdots + a_\ell\alpha^\ell$ with $a_i \in \{0,1,2,3,4\}$. The set $\mathcal{T}$ of points with integer part 0 in this number system

$$\mathcal{T} := \left\{ \sum_{i=1}^{\infty} a_i \alpha^{-i},\ a_i \in \{0,1,2,3,4\} \right\}$$

is called the *fundamental domain* of this canonical number system. It has been studied extensively in the literature. Up to now it is known that it is a plane continuum with nonempty interior which induces a tiling of the $\mathbb{C}$. However, its interior is disconnected. In the present paper we describe some of (the closures of) the components of its interior as attractors of graph-directed self-similar sets. The associated graph can also be used in order to determine the Hausdorff dimension of the boundary of these components. Amazingly, this dimension is strictly smaller than the Hausdorff dimension of the boundary of $\mathcal{T}$.

1. Introduction and basic definitions

We are interested in describing the topology of a plane self-similar set with disconnected interior that is related to a quadratic canonical number system (see Figure 1). More precisely, we want to describe the closure of some components of its interior by a graph directed self-similar set. Moreover, we are able to calculate the Hausdorff dimension of the boundary of these components.

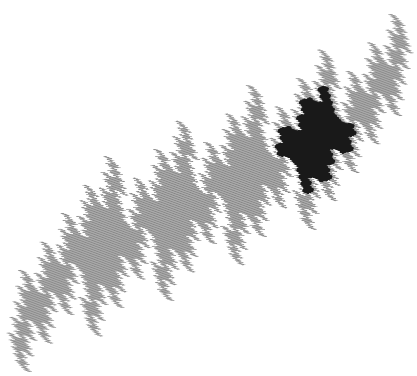

Figure 1. Tile associated to the base $-2+\sqrt{-1}$ with interior component containing 0.

In general, it seems to be difficult to study fine topological properties of self-similar fractals. This

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paper aims to present a paradigm[1] and we hope that the methods we used will be capable of being generalized to admit a more systematic study of components of self-similar sets.

We start with the necessary definitions. It is known (see [9, 14, 15]) that the root $\alpha = -2+\sqrt{-1}$ of the polynomial x^2+4x+5 together with $\mathcal{N} := \{0,1,2,3,4\}$ forms a *canonical number system* (or *CNS*) $(\alpha,\mathcal{N})$, *i.e.*, each element $x \in \mathbb{Z}[\alpha]$ has a unique representation

$$x = \sum_{i=0}^{\ell(x)} a_i \alpha^i$$

for some non-negative integer $\ell(x)$ and $a_i \in \mathcal{N}$ with $a_{\ell(x)} \neq 0$ for $x \neq 0$. We define the natural embedding

$$\begin{array}{rcl} \Phi : \mathbb{C} & \to & \mathbb{R}^2 \\ x & \mapsto & (\Re(x), \Im(x)). \end{array}$$

Then the multiplication by α can be represented by the 2×2 matrix

$$\mathbf{A} := \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix},$$

i.e., for every $x \in \mathbb{C}$,

$$\Phi(\alpha x) = \mathbf{A}\Phi(x).$$

The set $\mathcal{T}$ of points of integer part zero in the base α embedded into the plane is defined by

$$\mathcal{T} := \left\{ \sum_{i=1}^{\infty} \Phi(\alpha^{-i} a_i),\ (a_i)_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} \right\} = \left\{ \sum_{i=1}^{\infty} \mathbf{A}^{-i} \Phi(a_i),\ (a_i)_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} \right\} \tag{1.1}$$

and is depicted in Figure 1. $\mathcal{T}$ is often called the *fundamental domain* of the number system $(\alpha,\mathcal{N})$. Thus each point of this set can be represented by an *infinite string* $w = (a_1, a_2, a_3, \ldots)$ with $a_i \in \mathcal{N}$. The set $\mathcal{T}$ satisfies the equation

$$\mathcal{T} = \bigcup_{i=0}^{4} \psi_i(\mathcal{T}), \tag{1.2}$$

where $\psi_i, (i = 0, \ldots, 4)$ are contractions defined via the matrix $\mathbf{A}$ and the embedding Φ by

$$\psi_i(x) = \mathbf{A}^{-1}(x + \Phi(i)),\ x \in \mathbb{R}^2 \qquad (0 \le i \le 4). \tag{1.3}$$

$\mathcal{T}$ is a self-similar connected compact set (or *continuum*) with nonempty interior (see [13]). It induces a *tiling* of the plane by its translates. We recall that a *tiling* (*cf.* [11, 25]) of the plane is a decomposition of $\mathbb{R}^2$ into sets whose interiors are pairwise disjoint (so-called *non-overlapping sets*), each set being the closure of its interior and having a boundary of Lebesgue measure zero. Properties of tiles and tilings can be found for instance in [3, 7, 16, 23, 26]. It was shown in [13] that the family of sets

$$\{\mathcal{T} + \Phi(\omega),\ \omega \in \mathbb{Z}[\alpha]\} \tag{1.4}$$

is a tiling of the plane. We call $\mathcal{T}$ the *central tile* of this tiling.

Remark 1.1. This tile is an example of the large class of tiles associated to a root β of a quadratic polynomial $p(x) = x^2 + Ax + B$. Indeed, if $p(x)$ satisfies the conditions

$$-1 \le A \le B, \quad B \ge 2,$$

then $(\beta, \{0, 1, \ldots, B-1\})$ is a CNS (see *e.g.* [5, 6, 14, 15]). One can associate a tile to these CNS in the same way as we did for the special case $A = 4$, $B = 5$. It is shown in [1] that for $2A - B \ge 3$, which is also the case for our tile $\mathcal{T}$, these tiles have disconnected interior.

[1] see also Bailey *et al.* [2] where the components of the interior of the Lévy dragon are studied; interestingly, their structure is totally different from the ones studied in the present paper.

Some research on the structure of the components of the interior of self-similar and self-affine tiles has already been done. In Bailey *et al.* [2] investigate the interior of the Lévy dragon, which is a self-affine continuum with disconnected interior providing a tiling of the plane. They stated many conjectures concerning the geometrical shape of the connected components of the interior. Ngai and Nguyen [19] study the components of the Heighway dragon. Moreover, Ngai and Tang [20, 21] gave general results on components of the interior of self-affine tiles. As an example they consider our tile $\mathcal{T}$ in [20]. They prove that the closure of each component of its interior is homeomorphic to a closed disk.

The aim of our paper is to describe the closure C_0 of the connected component containing $\Phi(0)$ (or 0, for short) of the interior of $\mathcal{T}$. To make this precise we need some notations: a *finite digit string* is a finite sequence $w = (a_1, \dots, a_n)$ with $n \in \mathbb{N}$ and $a_i \in \mathcal{N}$. The integer n is then called the *length* of the string w (we write $|w| = n$). For a finite string $w = (a_1, \dots, a_n)$ we define the map ψ_w by

$$\psi_w(x) := \psi_{a_1} \circ \ldots \circ \psi_{a_n}(x) = \mathbf{A}^{-n}x + \sum_{i=1}^{n} \mathbf{A}^{-i}\Phi(a_i), \qquad x \in \mathbb{R}^2. \tag{1.5}$$

The set $\psi_w(\mathcal{T})$ is called an *n-th level subpiece of* $\mathcal{T}$. So by definition, it contains all the points represented by an infinite string of the shape $(a_1, \dots, a_n, d_1, d_2, \dots)$ with $d_i \in \mathcal{N}$.

Note that iterating (1.2) we have for every $n \geq 1$ the *subdivision principle*

$$\mathcal{T} = \bigcup_{w, |w|=n} \psi_w(\mathcal{T}). \tag{1.6}$$

Our description of C_0 will be in terms of n-th level subpieces with $n \geq 0$. Indeed, it will be shown that C_0 can be obtained as the closure of the union of such subpieces; the strings w involved in this union will be read off from a graph $\mathcal{G}$ presented in the next section. The set C_0 can be viewed as the attractor of a graph-directed construction (see Definition 2.1).

A similar description will be obtained in a forthcoming paper for a large class of tiles associated to quadratic number systems; nevertheless, this class will not contain the present example.

2. Statement of the main results

As indicated at the end of the previous section, the description of C_0 will be given via a graph $\mathcal{G}$. We will present this graph in the present section, explain how the set C_0 can be derived from this graph, and state our main theorems together with a sketches of the proofs.

2.1. Graph $\mathcal{G}$.

This graph is depicted in Figure 2. For the so-called *accepting state* $\circ$, there is by convention an edge $\circ \xrightarrow{a} \circ$ for every $a \in \mathcal{N}$.

We note here that we found this graph with help of computer calculations. More precisely, we approximated C_0 from inside by subpieces of $\mathcal{T}$. Listing the addresses of these subpieces we found some repetitions that led us to the construction of the graph $\mathcal{G}$.

We state some definitions and make some remarks about this graph. The graph $\mathcal{G}$ is *right resolving*, *i.e.*, each walk of $\mathcal{G}$ is uniquely defined by its starting state together with its labeling. Thus we will write $w = (A; a_1, \dots, a_n)$ for a walk w starting in A with labeling $(a_1, \dots, a_n)$. For subsets of the walks in $\mathcal{G}$ we adopt the following notations:

p	set of all walks in $\mathcal{G}$,
p_n	set of all walks in $\mathcal{G}$ having length n,
$p(A_1)$	set of walks in p starting at node A_1,
$p_n(A_1)$	set of walks in p_n starting at node A_1,
$p(A_1, A_2)$	set of walks in $p(A_1)$ ending at node A_2,
$p_n(A_1, A_2)$	set of walks in $p_n(A_1)$ ending at node A_2.

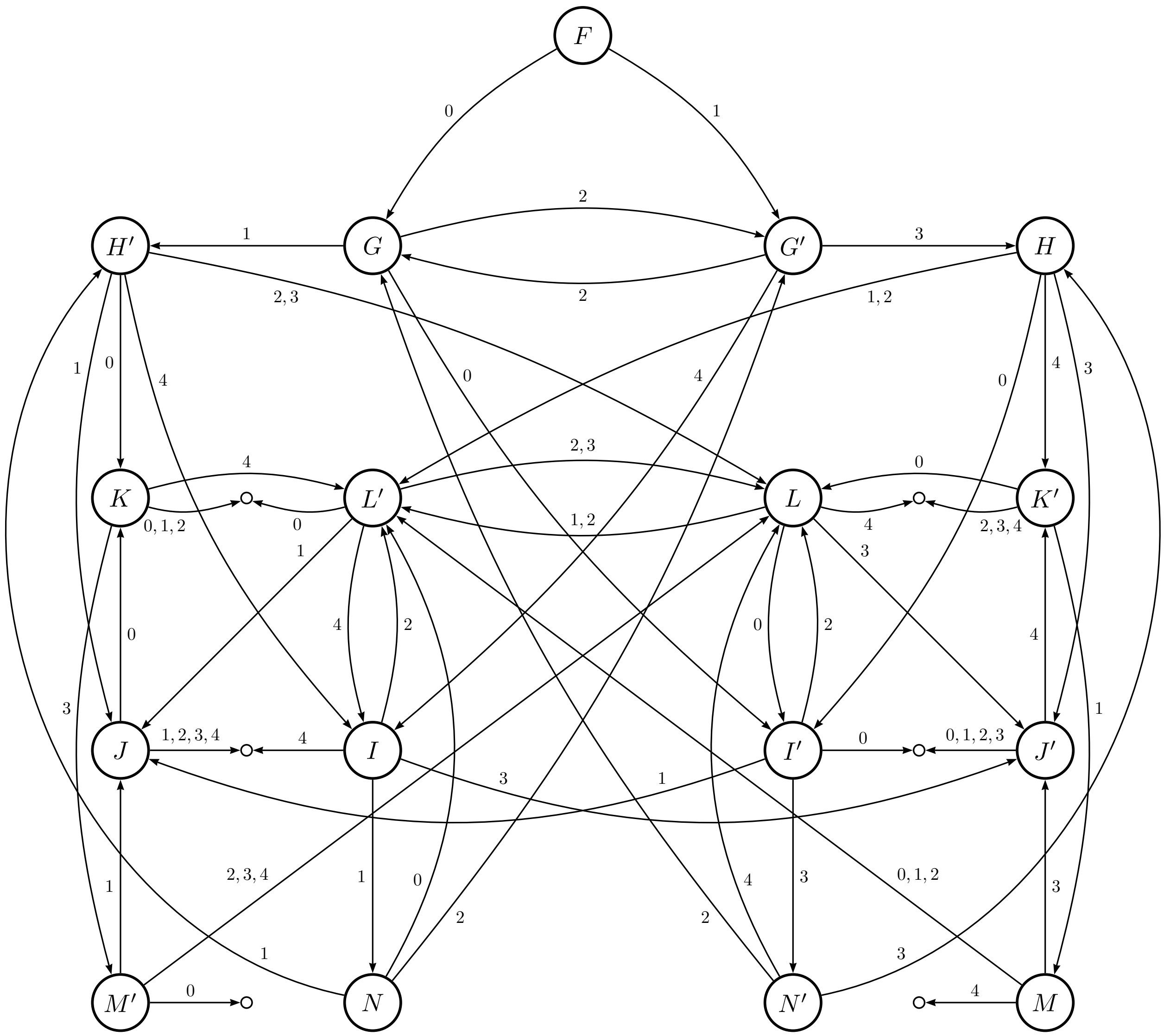

FIGURE 2. Graph $\mathcal{G}$ of the closure of the interior component of $\mathcal{T}$ containing 0.

If w is a walk in $\mathcal{G}$ with labeling $(a_1,\ldots,a_n)$, then we denote the walk which corresponds to w in the transposed graph $\mathcal{G}^T$ by w^T (backwards walk). Its labeling is obviously $(a_n,\ldots,a_1)$. The terminal state of a walk w in $\mathcal{G}$ shall be denoted by $t(w)$. If w_1 and w_2 are two walks in $\mathcal{G}$ and w_2 starts at the terminal state of w_1 then we write $w_1\&w_2$ for the concatenation of these two walks. If we emphasize on the labeling $(a_1,\ldots,a_n)$ of a walk w we will write $w=(a_1,\ldots,a_n)$. For instance, if we concatenate $w_1=(A_1;a_1,\ldots,a_n)$ and $w_2=(A_2;b_1,\ldots,b_m)$ we will often write $(A_1,a_1,\ldots a_n)\&(b_1,\ldots,b_m)$ because the starting state of w_2 is defined via w_1. For a walk w of length n and $k\leq n$ we denote by $w|_k$ the walk consisting of the first k edges of w, *i.e.*, $(a_1,\ldots,a_n)|_k=(a_1,\ldots,a_k)$. If $v=w|_k$ we write $v\prec w$.
If A is a state of $\mathcal{G}$, we call A' its *dual*. By convention we set $F'=F$, $\circ=\circ'$ and $A''=A$ for all the other states of $\mathcal{G}$. Note that in $\mathcal{G}$, every edge $A_1\xrightarrow{a}A_2$ has a *dual edge* $A_1'\xrightarrow{4-a}A_2'$.

2.2. **Graph directed sets.** The graph $\mathcal{G}$ can be used to describe the set C_0. We recall to this matter the notion of graph directed self-affine (self-similar) sets.

Definition 2.1. A *geometric graph directed construction* of $\mathbb{R}^d$ consists of

(1) finitely many compact subsets $J_1, \dots, J_q$ of $\mathbb{R}^d$ such that each J_i has nonempty interior;
(2) a directed graph $G(V, E)$ with set of vertices $V = \{1, \dots, q\}$ and to each edge $e \in E$ a uniform contraction T_e having the following properties:
  a) Each vertex has outgoing edges.
  b) Let E_{ij} be the set of edges leading from i to j. Then $\bigcup_j \{T_e(J_j) \mid e \in E_{ij}\}$ is a non-overlapping family and

$$J_i \supset \bigcup_j \{T_e(J_j) \mid e \in E_{ij}\} \quad (i \in \{1, \dots, q\}). \tag{2.1}$$

A very similar definition can be found in Mauldin and Williams [17] (*cf.* also [4] and [8]). Despite the definition of geometric graph directed construction in [17] is more restrictive in some regards the following result is still valid with the same proof.

Proposition 2.2 ([17, Theorem 1])**.** *There exists a unique vector $(K_1, \dots, K_q)$ of compact subsets of $\mathbb{R}^d$ such that for each $i \in \{1, \dots, q\}$*

$$K_i = \bigcup_j \{T_e(K_j) \mid e \in E_{ij}\} \tag{2.2}$$

holds. $(K_1, \dots K_q)$ is called a system of graph directed sets. If the K_i are affinities the system is called self-affine, if they are similarities, it is called self-similar.

By attaching the contraction ψ_a defined in (1.3) to each edge labelled by a in $\mathcal{G}$, the graph $\mathcal{G}$ leads to a system of graph directed sets. For each state A of $\mathcal{G}$ let

$$\mathbf{M}(A) := \left\{ x = \sum_{i \geq 1} \mathbf{A}^{-i} \Phi(a_i),\ w = (a_1, a_2, \dots) \text{ infinite walk of } p(A) \right\}. \tag{2.3}$$

Then we have the following result.

Proposition 2.3. *The vector $\{\mathbf{M}(A), A \in \mathcal{G}\}$ together with the graph $\mathcal{G}$ defines a system of graph directed sets. It is even a system of self-similar graph directed sets.*

Proof. We have to verify the conditions in Definition 2.1 and Proposition 2.2. $\mathbf{M}(A)$ is obviously bounded. The fact that it is closed follows by a Cantor diagonal argument very similar to the one used in Kátai [12].

The family $\bigcup_B \{\psi_e(\mathbf{M}(B)) \mid e \in E_{AB}\}$ is non-overlapping because $\mathbf{M}(B) \subset \mathcal{T}$ and $\mathcal{G}$ is right resolving (note that $(\alpha, \mathcal{N})$ admits unique representations). Furthermore, it is easy to see that $\{\mathbf{M}(A) \mid A \in \mathcal{G}\}$ fulfills (2.2). □

In particular,

$$\mathbf{M} := \mathbf{M}(F)$$

is a compact set and $\mathbf{M} \subset \mathcal{T}$. This paper aims at showing that $\mathbf{M} = C_0$, as stated in our main result.

2.3. Main results.

Theorem 2.4. *Let $(\alpha = -2 + \sqrt{-1},\ \mathcal{N} = \{0, 1, 2, 3, 4\})$ be the quadratic canonical number system related to the polynomial $x^2 + 4x + 5$. Let $\mathcal{T}$ be the fundamental domain associated to $(\alpha, \mathcal{N})$. Then $\mathrm{Int}(\mathbf{M})$ is the component of $\mathrm{Int}(\mathcal{T})$ containing 0. Moreover, $\mathbf{M}$ is the closure of its interior, hence $\mathbf{M} = C_0$, the closure of the component of $\mathrm{Int}(\mathcal{T})$ containing 0.*

Remark 2.5. Note that the above mentioned result of Ngai and Tang [20] implies that C_0 is homeomorphic to a closed disk.

For the proof of this theorem we will consider approximations of the set $\mathbf{M}$ in terms of finite walks of the graph $\mathcal{G}$.

Definition 2.6. For some $n \in \mathbb{N}$ and some state A of $\mathcal{G}$ let $W \subset p_n(A)$ be a set of walks. Then we set

$$\mathcal{M}(W) := \bigcup_{w \in W} \psi_w(\mathcal{T}).$$

Here, according to (1.5), $\psi_w(\mathcal{T})$ is the subpiece associated to the labeling of w. The approximating sets are obtained by taking for W the sets

$$G_n := p_n(F, \circ) \qquad (n \geq 3) \tag{2.4}$$

It is easy to see that for every $n > 3$, $\mathcal{M}(G_{n-1}) \subset \mathcal{M}(G_n) \subset \mathbf{M}$. Note that there exists no walk in $p_n(F, \circ)$ if $n < 3$. This means that there are no subpieces $\psi_w(\mathcal{T})$ with $|w| < 3$ entirely contained in $\mathbf{M}$.

We will show that $\mathbf{M} = C_0$ in the following way. First we show that the interior of $\mathbf{M}$ is connected and contained in the interior of $\mathcal{T}$, hence $\mathrm{Int}(\mathbf{M}) \subset \mathrm{Int}(C_0)$. In a second step we show that its boundary lies on the boundary of $\mathcal{T}$, which implies that $C_0 \subset \mathbf{M}$. By proving that $\mathbf{M}$ is the closure of its interior, *i.e.*, $\overline{\mathrm{Int}(\mathbf{M})} = \mathbf{M}$, this will finally yield $\mathbf{M} = C_0$.

The connectivity of $\mathrm{Int}(\mathbf{M})$ will be obtained by considering the approximations $\mathcal{M}(G_n)$ for $n \geq 3$, since we will show that

$$\mathrm{Int}(\mathbf{M}) \subset \bigcup_{n \geq 3} \mathcal{M}(G_n).$$

Using this fact, we will be able to connect a point from $\mathrm{Int}(\mathbf{M})$ to 0 by a path going from subpiece to subpiece with increasing size (*i.e.*, over subpieces $\psi_w(\mathcal{T})$ with decreasing $|w|$) within $\mathrm{Int}(\mathcal{T})$.

The second main result concerns the Hausdorff dimension of the boundary of the interior component containing zero. It reads as follows.

Theorem 2.7. *Let* $(\alpha = -2+\sqrt{-1},\ \mathcal{N} = \{0,1,2,3,4\})$ *be the quadratic canonical number system related to the polynomial* $x^2 + 4x + 5$. *Let* $\mathcal{T}$ *be the fundamental domain associated to* $(\alpha, \mathcal{N})$ *and denote by* C_0 *the component of* $\mathrm{Int}(\mathcal{T})$ *containing* 0. *Then*

$$\dim_H \partial C_0 = \frac{2\log 3}{\log 5} = 1.36521\ldots.$$

Remark 2.8. Since it is well-known (*cf. e. g.* Gilbert [10]) that the Hausdorff dimension of $\partial\mathcal{T}$ is given by

$$\dim_H \partial T = \frac{2\log\beta}{\log 5} = 1.60858\ldots$$

where β is the dominant root of the polynomial $x^3 - 3x^2 - x + 5$ we have that

$$\dim_H \partial C_0 < \dim_H \partial T.$$

The proof of Theorem 2.7 is essentially done by standard techniques from fractal geometry.

The remaining part of this paper is organized as follows. In Section 3 we present an automaton $\mathcal{B}_0$ which is helpful to determine whenever subpieces intersect each other. This leads to the definition of an action of $\mathcal{B}_0$ on $\mathcal{G}$ in a way that is stated in this section too. Sections 4 and 5 will be helpful for the proof of the connectivity of $\mathrm{Int}(\mathbf{M})$. Section 6 shows that the boundary of $\mathbf{M}$ is contained in the boundary of $\mathcal{T}$. Section 7 is devoted to the construction of connected paths within the interior of $\mathcal{T}$ together with some of its neighbors, that will be also used to show the connectivity of $\mathrm{Int}(\mathbf{M})$ inside $\mathrm{Int}(\mathcal{T})$. Section 8 contains the proof of Theorem 2.4 and in Section 9 we prove Theorem 2.7.

3. Counting automaton $\mathcal{B}_0$ and its action on the graph $\mathcal{G}$

We define a counting automaton and an action of this automaton on the preceding graph $\mathcal{G}$.

3.1. **Counting automaton $\mathcal{B}_0$.** The counting automaton $\mathcal{B}_0$ is given in [22, 24] for bases of quadratic canonical number systems in general and reproduced in Figure 3 for $\alpha = -2 + \sqrt{-1}$.

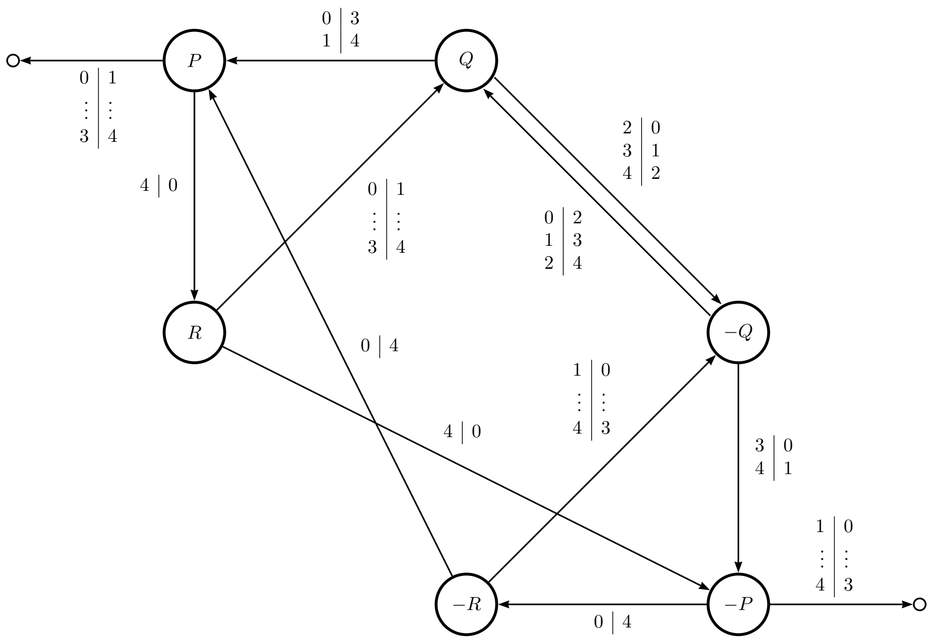

FIGURE 3. Counting automaton $\mathcal{B}_0$.

Its states are defined by

$$\pm P := \pm\Phi(1), \qquad \pm Q := \pm\Phi(3+\alpha), \qquad \pm R := \pm\Phi(-4-\alpha),$$

and $\circ$ denotes the *accepting state* 0.

The edges of $\mathcal{B}_0$ are defined as follows. There exists an edge from a state S to a state S' in $\mathcal{B}_0$ labelled by $a|a'$ with $a, a' \in \mathcal{N}$ if and only if

$$S + \Phi(a) = \mathbf{A}S' + \Phi(a').$$

In particular, since $\circ$ denotes 0, there is an edge $\circ \xrightarrow{a|a} \circ$ for each $a \in \mathcal{N}$ (these edges are not represented in Figure 3).

The numbers a and a' in a label $a|a'$ are called *input* and *output digits* respectively.

Remark 3.1. Note that $\mathcal{B}_0$ is right resolving: to any state S and any input digit $a \in \mathcal{N}$, there is exactly one state S' and one output digit a' such that the addition in the graph can be performed, *i.e.*, such that $S \xrightarrow{a|a'} S' \in \mathcal{B}_0$.

Thus the automaton $\mathcal{B}_0$ can also perform the addition of $S + \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(a_{n-i})$ for $S \in \mathcal{B}_0$ and $a_i \in \mathcal{N}$, simply by feeding $\mathcal{B}_0$ with the input digit string $(a_n, \dots, a_1)$ from left to right starting from S and collecting the output digit string $(a'_n, \dots, a'_1)$ and the landing state S': to

$$S \xrightarrow{a_n|a'_n} S_1 \xrightarrow{a_{n-1}|a'_{n-1}} \cdots \xrightarrow{a_1|a'_1} S' \tag{3.1}$$

corresponds the addition

$$S + \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(a_{n-i}) = \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(a'_{n-i}) + \mathbf{A}^n S'. \tag{3.2}$$

Note that for $S'=0$, *i.e.*, for a walk leading from S to 0 in $\mathcal{B}_0$, the term $\mathbf{A}^n S'$ vanishes. In this case the automaton produces from the string $(a_n,\dots,a_1)$ corresponding to the "**A**-adic" expansion of $z=\sum_{i=0}^{n-1}\mathbf{A}^i\Phi(a_{n-i})$ the string $(a'_n,\dots,a'_1)$ which is the string of the "**A**-adic" expansion of $z+S$.

Remark 3.2. By Remark 3.1, the outputs S' and $(a'_n,\dots,a'_1)$ are uniquely defined by the inputs S and $(a_n,\dots,a_1)$.

The automaton emerging from $\mathcal{B}_0$ by leaving away the accepting state is called $\mathcal{B}$. It is helpful in order to characterize the boundary of $\mathcal{T}$, as the following results show.

Proposition 3.3 (Scheicher and Thuswaldner [23])**.** *The following equation holds for the boundary $\partial\mathcal{T}$ of $\mathcal{T}$:*

$$\partial\mathcal{T}=\bigcup_{S\in\mathcal{B}}(\mathcal{T}\cap(\mathcal{T}+S)). \tag{3.3}$$

Thus, even if $\mathcal{T}$ has more neighbors than the six presented here (see [1]), these neighbors are sufficient to describe the whole boundary.

Proposition 3.4 (Müller *et al.* [18])**.** *For $S\in\{\pm P,\pm Q,\pm R\}$ let $B_S:=\mathcal{T}\cap(\mathcal{T}+S)$. Then $B_S\neq\emptyset$. Furthermore, if there exists an infinite walk*

$$S\xleftarrow{a_1|a'_1}S_1\xleftarrow{a_2|a'_2}\dots$$

in $\mathcal{B}$ such that $x=\sum_{i\geq 1}\mathbf{A}^{-i}\Phi(a_i)$ then $x\in B_S$.

As a consequence of these propositions and of the definition of $\mathcal{B}_0$, we have the following way to characterize that two n-th level subpieces of $\mathcal{T}$ have common points.

Characterization 3.5. *Let $n\in\mathbb{N}$ and $w=(a_1,\dots,a_n)$, $w'=(a'_1,\dots,a'_n)$ be two strings of length n. If there is a walk*

$$S_n\xrightarrow{a_n|a'_n}S_{n-1}\dots\xrightarrow{a_1|a'_1}\circ$$

in $\mathcal{B}_0$, then

$$\psi_w(\mathcal{T})\cap\psi_{w'}(\mathcal{T})\neq\emptyset.$$

3.2. **Graph action of $\mathcal{B}_0$ on $\mathcal{G}$.** The structure of **M** will be understood with the help of the following graph action of $\mathcal{B}_0$ on the graph $\mathcal{G}$.

Definition 3.6. Let S be a state in $\mathcal{B}_0$, A a state of $\mathcal{G}$ and let $w=(A;a_1,\dots,a_n)\in p_n(A)$. Take $(a_n,\dots,a_1)$ as the input string for the automaton $\mathcal{B}_0$ with starting state S and denote the output string by $(a'_n,\dots,a'_1)$. Then we define $\Psi_S(w):=(A;a'_1,\dots a'_n)$. Ψ_S is called the *addition of* S. If the automaton $\mathcal{B}_0$ rests in $\circ$ after reading $(a_n,\dots,a_1)$ and if $\Psi_S(w)\in p_n(A)$ then we say that the addition of S is *admissible* for w. Note that for a walk $w=(A)$ of length zero only $\Psi_\circ(w)=w$ is admissible.

Remark 3.7. By Characterization 3.5, the admissible addition of S to a string w produces a string $w':=\Psi_S(w)$ such that $\psi_w(\mathcal{T})\cap\psi_{w'}(\mathcal{T})\neq\emptyset$.

Definition 3.8. Fix $n\in\mathbb{N}$, a state $A\in\mathcal{G}$ and let $w_1,w_2\in W\subset p_n(A)$. Let $\psi_{w_1}(\mathcal{T})$ and $\psi_{w_2}(\mathcal{T})$ be the corresponding subsets of $\mathcal{T}$. We say that w_1 and w_2 are W-equivalent to each other, if there exist finitely many states $S_1,\dots,S_m$ of $\mathcal{B}$ such that the following conditions hold with admissible additions Ψ_{S_j}.

$$\begin{aligned}\Psi_{S_m}\circ\dots\circ\Psi_{S_1}(w_1) &= w_2 \quad\text{and}\\ \Psi_{S_j}\circ\dots\circ\Psi_{S_1}(w_1) &\in W \quad (1\leq j\leq m).\end{aligned}$$

We denote this by $w_1\sim w_2\,(W)$ or simply by $w_1\sim w_2$ if the underlying set W is clear from the context. In this case we also call the corresponding sets $\psi_{w_1}(\mathcal{T})$ and $\psi_{w_2}(\mathcal{T})$ W-equivalent and use the same notation $\psi_{w_1}(\mathcal{T})\sim\psi_{w_2}(\mathcal{T})$.

If $w_1=\Psi_S(w_2)$ we also write in a slight abuse of notation $w_1\sim_S w_2$ or $w_2\;{}_S\!\sim w_1$.

It is easy to check that $\sim$ is an equivalence relation.

Remark 3.9. We want to give some comments on these definitions.

1) Let $w = (A; a_1, \dots, a_n) \in p_n(A)$ be a walk and assume that Ψ_{S_n} with $S_n \in \mathcal{B}_0$ is an admissible addition for w. Then from Definition 3.6 it follows that there exists a walk
$$S_n \xrightarrow{a_n|a'_n} S_{n-1} \xrightarrow{a_{n-1}|a'_{n-1}} \dots \xrightarrow{a_2|a'_2} S_1 \xrightarrow{a_1|a'_1} \circ$$
in $\mathcal{B}_0$ such that $w' = (A; a'_1, \dots, a'_n) \in p_n(A)$. Furthermore, we can perform this addition "digit wise", *i.e.*,
$$\Psi_{S_n}(A; a_1, \dots, a_n) = \Psi_{S_j}(A; a_1, \dots, a_j)\&(a'_{j+1}, \dots, a'_n) = (A; a'_1, \dots, a'_n).$$
Note that from this we easily see that
$$\psi_{\Psi_{S_n}(w)}(\mathcal{T}) = \psi_w(\mathcal{T} + S_n).$$
We even have, if $S_n^{(1)}, \dots, S_n^{(m)}$ are m states of $\mathcal{B}_0$ such that the additions $\Psi_{S_n^{(j)}} \circ \dots \circ \Psi_{S_n^{(1)}}(w)$ are admissible for $1 \le j \le m$, that:
$$\psi_{\Psi_{S_n^{(m)}} \circ \dots \circ \Psi_{S_n^{(1)}}(w)}(\mathcal{T}) = \psi_w(\mathcal{T} + S_n^{(m)} + \dots + S_n^{(1)}).$$
2) Let w_1, w_2 be W-equivalent for some $W \subset p_n(A)$. Then there exist $v_1, \dots, v_m$ with $v_1 := w_1$ and $v_m := w_2$ such that
$$\psi_{v_j}(\mathcal{T}) \cap \psi_{v_{j+1}}(\mathcal{T}) \neq \emptyset \quad (1 \le j \le m-1).$$
This follows immediately from Remark 3.7 together with Definition 3.8.
3) Let $W \subset p_n(A)$ and let $k < n$ be integers. Let $w_1, w_2 \in W$ such that $w_1|_k = w_2|_k =: \sigma$. Then there exist
$$\tau_1, \tau_2 \in W_\sigma := \{\tau \,|\, \sigma\&\tau \in W\}$$
such that $w_i = \sigma\&\tau_i$ $(i = 1, 2)$. If τ_1 and τ_2 are equivalent in W_σ then w_1 and w_2 are equivalent in W. This follows from the following fact together with Definition 3.8. Let $\tau, \tau' \in W_\sigma$ and $S \in \mathcal{B}_0$. Then
$$\Psi_S(\tau) = \tau' \quad \Longrightarrow \quad \Psi_S(\sigma\&\tau) = \Psi_\circ(\sigma)\&\tau' = \sigma\&\tau'.$$
This implies that
$$\tau \sim \tau'(W_\sigma) \quad \Longrightarrow \quad \sigma\&\tau \sim \sigma\&\tau'(W),$$
and this means that in order to examine equivalences of walks it often suffices to examine equivalences of their tails.

Definition 3.10. Fix $n \in \mathbb{N}$ and let $W \subset p_n(A)$ be a set of strings. Then W and the set
$$\mathcal{M}(W) := \{\psi_w(\mathcal{T}) \mid w \in W\}$$
are called *transitive* if we have $w_1 \sim w_2(W)$ for each two $w_1, w_2 \in W$.

Remark 3.11. Since the subpiece $\psi_w(\mathcal{T})$ is arcwise connected for every string w (remember that $\mathcal{T}$ is arcwise connected), by definition of the equivalence relation in W, a transitive set $W \subset p_n(F, \circ)$ yields an arcwise connected subset $\mathcal{M}(W)$ of $\mathbf{M}$.

We end this section with a last definition.

Definition 3.12. Let A be a node of $\mathcal{G}$ and S a state of $\mathcal{B}_0$. If Ψ_S is admissible for all walks in $p(F, A)$ then we call Ψ_S an *admissible graph action for A on $\mathcal{G}$*, or an *A-action*, for short.

If Ψ_S is an A-action then we call
$$F(\Psi_S, A) := \{t(w') \,|\, w' = \Psi_S(w) \text{ for a walk } w \in p(F, A)\}$$
the *ending set of* (Ψ_S, A).

Remark 3.13. Let the assertion : Ψ_S *is an A-action with ending set* $F(\Psi_S, A)$, then we define the dual assertion : Ψ_{-S} *is an A'-action with ending set* $\{Z', Z \in F(\Psi_S, A)\}$.

4. Admissibility of all the additions for a class of walks in $\mathcal{G}$

This section is devoted to the proof of the following result.

Proposition 4.1. *Let w be a finite walk in $p(F,\circ)$. Then for each state S of $\mathcal{B}$ the addition $\Psi_S(w)$ is admissible for w.*

Remark 4.2. By Remarks 3.7 and 3.9.1 , this implies that for $w \in p_n(F,\circ)$, all the sets $\psi_w(\mathcal{T}+S)$ with $S \in \mathcal{B}_0$ are subpieces of $\mathcal{T}$ that have non-empty intersection with $\psi_w(\mathcal{T})$.

Suppose that w is a finite walk in $p(F,\circ)$. Then w is of the shape

$$F \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} A_k \xrightarrow{a_{k+1}} \circ \xrightarrow{a_{k+2}} \cdots \xrightarrow{a_n} \circ \qquad (A_k \neq \circ) \tag{4.1}$$

for some $2 \leq k < n$. Let $S_n := S$. Note that S_n together with the labels $(a_1,\ldots,a_n)$ defines uniquely the walk

$$S_n \xrightarrow{a_n|a'_n} S_{n-1} \xrightarrow{a_{n-1}|a'_{n-1}} \cdots \xrightarrow{a_{k+2}|a'_{k+2}} S_{k+1} \xrightarrow{a_{k+1}|a'_{k+1}} S_k \xrightarrow{a_k|a'_k} \cdots \xrightarrow{a_1|a'_1} S_0 \tag{4.2}$$

in $\mathcal{B}_0$ (recall that $\mathcal{B}_0$ is right resolving by Remark 3.1). By the definition of Ψ_S this walk yields the identities

$$\Psi_S(w) = \Psi_{S_j}(a_1,\ldots,a_j)\&(a'_{j+1},\ldots,a'_n) = \Psi_{S_k}(a_1,\ldots,a_k)\&(a'_{k+1},\ldots,a'_n). \tag{4.3}$$

We want to show that $\Psi_S(w)$ is a walk in $p(F)$ and that $S_0 = \circ$ for all states S of $\mathcal{B}$. We first need the following lemma.

Lemma 4.3. *Ψ_S is an A-action in the following cases :*

(L,-Q): *Ψ_{-Q} is an L-action with $F(\Psi_{-Q},L) \subset \{\circ, L', K'\}$.*
(L,R): *Ψ_R is an L-action with $F(\Psi_R,L) \subset \{\circ, I', J', L', M'\}$.*
(L,P): *Ψ_P is an L-action with $F(\Psi_P,L) \subset \{L, M, I, N'\}$.*
(L,-P): *Ψ_{-P} is an L-action with $F(\Psi_{-P},L) \subset \{L, H, J\}$.*
(M,-Q): *Ψ_{-Q} is an M-action with $F(\Psi_{-Q},M) \subset \{\circ, J'\}$.*
(M,R): *Ψ_R is an M-action with $F(\Psi_R,M) \subset \{\circ, L'\}$.*
(M,P): *Ψ_P is an M-action with $F(\Psi_P,M) \subset \{\circ\}$.*
(M,-P): *Ψ_{-P} is an M-action with $F(\Psi_{-P},M) \subset \{L\}$.*
(H,-Q): *Ψ_{-Q} is an H-action with $F(\Psi_{-Q},H) \subset \{I', K\}$.*
(H,P): *Ψ_P is an H-action with $F(\Psi_P,H) \subset \{I, L\}$.*
(H,-P): *Ψ_{-P} is an H-action with $F(\Psi_{-P},H) \subset \{G\}$.*
(I,-Q): *Ψ_{-Q} is an I-action with $F(\Psi_{-Q},I) \subset \{H', J\}$.*
(I,R): *Ψ_R is an I-action with $F(\Psi_R,I) \subset \{\circ, L', K, I'\}$.*
(I,-P): *Ψ_{-P} is an I-action with $F(\Psi_{-P},I) \subset \{H, L\}$.*
(J,Q): *Ψ_Q is an J-action with $F(\Psi_Q,J) \subset \{I, K'\}$.*
(J,-Q): *Ψ_{-Q} is an J-action with $F(\Psi_{-Q},J) \subset \{\circ, J', M'\}$.*
(J,R): *Ψ_R is an J-action with $F(\Psi_R,J) \subset \{\circ, L'\}$.*
(J,P): *Ψ_P is an J-action with $F(\Psi_P,J) \subset \{L\}$.*
(J,-P): *Ψ_{-P} is an J-action with $F(\Psi_{-P},J) \subset \{\circ, K\}$.*
(K,Q): *Ψ_Q is an K-action with $F(\Psi_Q,J) \subset \{H, J'\}$.*
(K,-Q): *Ψ_{-Q} is an K-action with $F(\Psi_{-Q},K) \subset \{\circ, L'\}$.*
(K,-R): *Ψ_{-R} is an K-action with $F(\Psi_{-R},K) \subset \{\circ, I\}$.*
(K,P): *Ψ_P is an K-action with $F(\Psi_P,K) \subset \{\circ, J\}$.*

Moreover, the duals of these assertions are also true, that is to say: if for some pair $S \in \mathcal{B}$ and $A \in \mathcal{G}$

(A,S): Ψ_S is an A-action with $F(\Psi_S,A) \subset \{A_1,\ldots,A_k\}$

holds then also the dual statement

(A',-S): Ψ_{-S} is an A'-action with $F(\Psi_{-S},A') \subset \{A'_1,\ldots,A'_k\}$

holds.

Remark 4.4. Note that $\Psi_\circ$ is an A-action (with set of ending states $\{A\}$) for every $A \in \mathcal{G}$.

Proof. The statement will be proved by induction on the length of the walks $w \in p(F)$. The assertion $(A,S)_n$ stands for: *(A,S) holds for all walks up to length n.* If there is no walk in $p_k(F,A)$ for $k \leq n$, then $(A,S)_n$ is true.
For $n \leq 1$ the statements $(A,S)_n$ for the pairs (A,S) in the proposition are all true.
Suppose now that $(A,S)_{n-1}$ is true for all pairs (A,S) of the proposition and their duals. We show that all $(A,S)_n$'s and their duals are also true. We will show how to proceed for the case of $(L,-Q)_n$ and sum up the results for all the cases in a table.
We have to show that Ψ_{-Q} is an L-action for all walks w of length at most n with ending set $F(\Psi_{-Q}, L) \subset \{\circ, L', K'\}$. Let $w = (a_1, \dots, a_n)$ such that there is a walk

$$A \xrightarrow{a_1} A_1 \xrightarrow{a_2} \dots \xrightarrow{a_{n-2}} A_{n-2} \xrightarrow{a_{n-1}} A_{n-1} \xrightarrow{a_n} L$$

in $\mathcal{G}$. Then by Remark 3.1 the input digits $(a_n, \dots, a_1)$ define a unique walk in $\mathcal{B}_0$ starting from $-Q$:

$$-Q \xrightarrow{a_n|a'_n} S_{n-1} \xrightarrow{a_{n-1}|a'_{n-1}} S_{n-2} \xrightarrow{a_{n-2}|a'_{n-2}} \dots \xrightarrow{a_1|a'_1} S_0.$$

First suppose that $a_n \in \{0,1,2\}$, *i.e.*, by $\mathcal{B}_0$, $S_{n-1} = Q$. Then

$$\Psi_{-Q}(w) = \Psi_Q(w|_{n-1}) \& (a'_n).$$

According to $\mathcal{G}$, since w ends up in L and $a_n \in \{0,1,2\}$, $w|_{n-1}$ can end up in H', K', I', L' or M', *i.e.*, $A_{n-1} \in \{H', K', I', L', M'\}$. If $A_{n-1} = H'$, then $a_n = 2$, because the edge leading from H' to L in $\mathcal{G}$ has only the labels $\{2,3\}$, and we assumed $a_n \in \{0,1,2\}$. Thus $a'_n = 4$, as indicated by the edge $-Q \xrightarrow{2|4} Q$ of $\mathcal{B}_0$. Moreover, by $(H', Q)_{n-1}$, which is the dual of $(H, -Q)_{n-1}$, we have $Q \xrightarrow{w|_{n-1}^T} \circ$ in $\mathcal{B}_0$ and $\Psi_Q(w|_{n-1})$ ends up in $\{I, K'\}$. Thus $-Q \xrightarrow{w^T} \circ$, *i.e.*, $S_0 = \circ$, and $\Psi_{-Q}(w)$ ends up in $\{\circ\}$, because of the edges $I \xrightarrow{4} \circ$ and $K' \xrightarrow{4} \circ$ in $\mathcal{G}$. We can argue along the same lines if $A_{n-1} \in \{K', I', L', M'\}$. All these cases lead to walks $\Psi_{-Q}(w)$ ending in $\circ$, K' or L'.

Secondly, suppose that $a_n \in \{3,4\}$, *i.e.*, by $\mathcal{B}_0$, $S_{n-1} = -P$. Then

$$\Psi_{-Q}(w) = \Psi_{-P}(w|_{n-1}) \& (a'_n).$$

According to $\mathcal{G}$, since w ends up in L and $a_n \in \{3,4\}$, $w|_{n-1}$ can only end up in H', L', M' or N', *i.e.*, $A_{n-1} \in \{H', L', M', N'\}$. The first three cases can be treated as above and lead to $\Psi_{-Q}(w)$ ending in $\circ$ or L', so let us assume that $A_{n-1} = N'$. Then, the only edge in $\mathcal{G}$ leading from N' to L being $N' \xrightarrow{4} L$, we have $a_n = 4$, and since there is only one edge landing in N' ($I' \xrightarrow{3} N'$), we even have $A_{n-2} = I'$ and $a_{n-1} = 3$. Thus, by the edges $-Q \xrightarrow{4|1} -P \xrightarrow{3|2} \circ$ of $\mathcal{B}_0$, we read $a'_n = 1$, $a'_{n-1} = 2$ and $S_{n-2} = \circ$. Consequently, we have $\circ \xrightarrow{w|_{n-2}^T} \circ$, and $\Psi_\circ(w|_{n-2}) = w|_{n-2}$ ends up in $\{I'\}$. Thus again $S_0 = \circ$ and

$$\Psi_{-Q}(w) = \Psi_\circ(w|_{n-2}) \& (a'_{n-1}, a'_n) = w|_{n-2} \& (2,1)$$

ends up in $\{L'\}$, as it can be checked on $\mathcal{G}$ by considering the edges $I' \xrightarrow{2} L \xrightarrow{1} L'$.

Thus in all cases $\Psi_{-Q}(w)$ is a walk in $p(F)$ ending up in $\{\circ, L', K'\}$. Thus $(L,-Q)_n$ is true and we are done.

All the other assertions can be treated likewise. The occurring cases are summed up in Tables 1 and 2 from which the complete proof can be read off easily. In these tables, A_{n-2}, a'_{n-1} and S_{n-2} are given if they are needed, and in this case we use $i = 2$ in the 6-th column, otherwise $i = 1$. □

Proposition 4.1 will be proved inductively: let w be a finite walk in $p(F, \circ)$ and $j \geq k+1$, where k is defined by (4.1). We will show that for every $j \geq k+1$, the addition $\Psi_S(w|_j)$ is admissible

$(A,S)_n$	a_n	A_{n-1} (A_{n-2})	a'_n (a'_{n-1})	S_{n-1} (S_{n-2})	end of $\Psi_{S_{n-i}}(w\|_{n-i})$	end of $\Psi_S(w)$
$(L,-Q)_n$	$0,1,2$	H'	4	Q	I,K'	$\circ$
		K'	2		$\circ,L$	$\circ,L'$
		I'	4		H,J'	K'
		L'	4		$\circ,L,K$	$\circ,L'$
		M'	4		$\circ,J$	$\circ$
	$3,4$	H'	0	$-P$	I',L'	$\circ$
		L'	0		I',L',M',N	$\circ,L'$
		M'	$0,1$		$\circ$	$\circ$
		N' (I')	1 (2)	$-P$ $(\circ)$	I'	L'
$(L,R)_n$	$0,1,2,3$	H'	$3,4$	Q	I,K'	$\circ,J'$
		K'	1		$\circ,L$	$\circ,L'$
		L'	$3,4$		$\circ,L,K$	$\circ,J',M',L'$
		I'	3		H,J'	$\circ,J'$
		M'	$3,4$		$\circ,J$	$\circ$
	4	M'	0	$-P$	$\circ$	$\circ$
		N'	0 (2)	$-P$ $(\circ)$	I'	I'
$(L,P)_n$	$0,1,2,3$	H'	$3,4$	$\circ$	H'	I,L
		K'	1		K'	M
		L'	$3,4$		L'	L,I
		I'	3		I'	N'
		M'	$3,4$		M'	L
	4	M' (K)	0 (4)	R (Q)	H,J'	L
		N' (I')	0 (4)		J',H	L
$(L,-P)_n$	$1,2,3,4$	H'	$1,2$	$\circ$	H'	J,L
		L'	$1,2$		L'	J,L
		I'	1		I'	J
		M'	$1,2,3$		M'	J,L
		N'	3		N'	H
	0	K' (J')	4 (3)	$-R$ $(-Q)$	I',K	L
$(M,-Q)_n$	1	K'	3	Q	$\circ,L$	$\circ,J'$
$(M,R)_n$	1	K'	2	Q	$\circ,L$	$\circ,L'$
$(M,P)_n$	1	K'	2	$\circ$	K'	$\circ$
$(M,-P)_n$	1	K'	0	$\circ$	K'	L
$(H,-Q)_n$	3	N' (I')	0 (2)	$-P$ $(\circ)$	I'	I'
		G' $(F;G,N)$	0 (0;1)	$-P$ $(\circ)$	F,G,N	I',K
$(H,P)_n$	3	N',G'	4	$\circ$	N',G'	L,I
$(H,-P)_n$	3	N',G'	2	$\circ$	N',G'	G
$(I,-Q)_n$	4	H'	1	$-P$	I',L'	J
		L'			I',M',L',N	J,H'
		G' $(F;G,N)$	1 (0;1)	$-P$ $(\circ)$	F,G,N	H',J
$(I,R)_n$	4	H'	0	$-P$	I',L'	$\circ,L'$
		L'			I',M',L',N	$\circ,L'$
		G' $(F;G,N)$	0 (0;1)	$-P$ $(\circ)$	F,G,N	I',K
$(I,-P)_n$	4	G',L',H'	3	$\circ$	G',L',H'	H,L
$(J,Q)_n$	1	H'	4	P	G'	I
		M'			L'	I
		I'			H',L'	I
		L'			H',L',J'	I,K'
$(J,-Q)_n$	1	H'	3	Q	K',I	$\circ,J'$
		M'			$\circ,J$	$\circ$
		I'			H,J'	$\circ,J'$
		L'			$\circ,L,K$	$\circ,J',M'$

TABLE 1. Proof of Lemma 4.3.

$(A,S)_n$	a_n	A_{n-1} (A_{n-2})	a'_n (a'_{n-1})	S_{n-1} (S_{n-2})	end of $\Psi_{S_{n-i}}(w\|_{n-i})$	end of $\Psi_S(w)$
$(J,R)_n$	1	H'	2	Q	K', I	$\circ, L'$
		M'			$\circ, J$	$\circ$
		I'			H, J'	$\circ, L'$
		L'			$\circ, L, K$	$\circ, L'$
$(J,P)_n$	1	H', M', I', L'	2	$\circ$	H', M', I', L'	L
$(J,-P)_n$	1	H', M', I', L'	0	$\circ$	H', M', I', L'	$\circ, K$
$(K,Q)_n$	0	H'	3	P	G'	H
		J			L	J'
$(K,-Q)_n$	0	H', J	2	Q	K', I	$\circ, L'$
$(K,-R)_n$	0	H'	4	P	G'	I
		J			L	$\circ$
$(K,P)_n$	0	H', J	1	$\circ$	H', J	$\circ, J$

TABLE 2. Proof of Lemma 4.3: end of the preceding table.

for each state S of $\mathcal{B}$: taking for j the length n of w will yield the result. Lemma 4.5 will contain the induction start, Lemma 4.6 the induction step.

Lemma 4.5. *Suppose that w is a walk of the shape (4.1). Then the following assertions hold.*

(i) $\Psi_P(w|_{k+1})$ *ends in* $\{\circ, M', K', J\}$.
(ii) $\Psi_Q(w|_{k+1})$ *ends in* $\{\circ, J, K, L\}$.
(iii) $\Psi_R(w|_{k+1})$ *ends in* $\{\circ, L', J', I', K\}$.
(iv) $w|_{k+1}$ *ends in* $\{\circ\}$.

Their associated duals also hold ("$\Psi_S(w|_{k+1})$ ends in the set of states $\mathcal{A}$" has the dual "$\Psi_{-S}(w|_{k+1})$ ends in $\mathcal{A}'$ ").

In particular, $\Psi_S(w|_{k+1})$ is a walk in $p(F)$ for all $S \in \mathcal{B}_0$. Moreover, $S \xrightarrow{w|_{k+1}^T} \circ$ for all $S \in \mathcal{B}_0$. That is to say, $\Psi_S(w|_{k+1})$ is admissible for all $S \in \mathcal{B}_0$.

Proof. Let $S \in \mathcal{B}_0$. Note that the following edges exist:

$$\begin{aligned} A_k &\xrightarrow{a_{k+1}} \circ \quad &&\text{in } \mathcal{G} \text{ by definition of } k \text{ and} \\ S &\xrightarrow{a_{k+1}|a'_{k+1}} S' \quad &&\text{in } \mathcal{B} \text{ for some } S' \in \mathcal{B}_0 \end{aligned} \tag{4.4}$$

(the second edge is uniquely defined by S and a_{k+1}). We recall the identity:

$$\Psi_S(w|_{k+1}) = \Psi_{S'}(w|_k)\&(a'_{k+1}). \tag{4.5}$$

To (i): $S = P$. Depending on a_{k+1}, the edge $P \xrightarrow{a_{k+1}|a'_{k+1}} S'$ of (4.4) in $\mathcal{B}_0$ implies that $S' = \circ$ or R, which fix the range of a'_{k+1} ($a'_{k+1} = 0$ if $S' = R$ and $a'_{k+1} \in \{1, \dots, 4\}$ if $S' = \circ$). The possible states A_k are also determined by a_{k+1} via the existence of the edge $A_k \xrightarrow{a_{k+1}} \circ$ in $\mathcal{G}$ (see (4.4)). Using the corresponding assertion (A_k, S') of Lemma 4.3 it is then easy to get the possible endings of $\Psi_{S'}(w|_k)$. Now if $Y \in \mathcal{G}$ is such an ending, then, by (4.5), with the range of a'_{k+1} one obtains the possible endings Z of $\Psi_S(w|_{k+1})$ by looking for all edges $Y \xrightarrow{a'_{k+1}} Z$ in $\mathcal{G}$. Let us consider an example: if $a_{k+1} = 4$, we are considering the edge $P \xrightarrow{4|0} R$ in B_0, thus $S' = R$ and $a'_{k+1} = 0$. Moreover, $A_k \in \{K', L, J, I, M\}$ because these states are the only starting states of edges in $\mathcal{G}$ labelled by 4 and leading to $\circ$. For $A_k = K'$, using (K', R) of Lemma 4.3 we get that $\Psi_R(w|_k)$ ends up in $\circ$ or I'. Consequently, since $a'_{k+1} = 0$, $\Psi_P(w|_{k+1})$ ends up in $\circ$: indeed, we have $\circ \xrightarrow{0} \circ$ and $I' \xrightarrow{0} \circ$ in $\mathcal{G}$. Note that (K', R) also implies $R \xrightarrow{w|_k^T} \circ$ in $\mathcal{B}_0$, thus $P \xrightarrow{w|_{k+1}^T} \circ$.
The results for the other values of a_{k+1} are given in Table 3.

S	a_{k+1}	S'	A_k	end of $\Psi_{S'}(w\vert_k)$	a'_{k+1}	end of $\Psi_S(w\vert_{k+1})$
P	$0,1,2,3$	$\circ$	L'	L'	1	J
			J	J	$2,3,4$	$\circ$
			J'	J'	$1,2,3,4$	$\circ, K'$
			K	K	$1,2,3$	$\circ, M'$
			K'	K'	$3,4$	$\circ$
			M'	M'	$3,4$	$\circ$
	4	R	J	$\circ, L'$		
			I	$\circ, I', K, L'$		
			L	$\circ, J', L', M', I'$	0	$\circ$
			K'	$\circ, I'$		
			M	$\circ, L'$		
Q	$0,1$	P	K	$\circ, J$	$3,4$	$\circ$
			L'	H', J', L'	3	$\circ, L$
			J	L	4	$\circ$
			M'	L'	3	L
			I'	$\circ, K'$	$3,4$	L
			J'	$\circ, K'$	$3,4$	$\circ$
	$2,3,4$	$-Q$	K'	H', J	$0,1,2$	$\circ, K, J, L$
			L	$\circ, L', K'$	2	$\circ, L$
			K	$\circ, L'$	0	$\circ$
			J'	K, I'	$0,1$	$\circ$
			I	H', J	2	$\circ, L$
			J	$\circ, M', J'$	$0,1,2$	$\circ, J, L$
			M	$\circ, J'$	2	$\circ$
R	$0,1,2,3$	Q	K	H, J'	$1,2,3$	$\circ, L'$
			K'	$\circ, L$	$3,4$	$\circ, J'$
			L'	$\circ, L, K$	1	$\circ, L'$
			J	I, K'	$2,3,4$	$\circ, L', J'$
			J'	$\circ, M, J$	$1,2,3,4$	$\circ, J', L'$
			I'	H, J'	1	$\circ, L'$
			M'	$\circ, J$	1	$\circ$
	4	$-P$	K'	$\circ, J'$		$\circ, L'$
			L	H, J, L		I', K
			J	$\circ, K$	0	$\circ$
			I	H, L		I'
			M	L		I'

TABLE 3. Proof of Lemma 4.5.

The proof is the same for the other cases ($S = Q, S = R$) and their duals, it is summed up in Table 3 for $S = Q$ and $S = R$. Item (iv) is clear. □

Again, let w be a finite walk in $p(F, \circ)$ and $j \geq k+1$, where k is defined by (4.1). We call (B_j) the assertion: *for all $S \in \mathcal{B}_0, \Psi_S(w|_j)$ is a walk in $p(F)$ with possible ending states given in Table 4. Moreover, $S \xrightarrow{w|_j^T} S_0 = \circ$ in $\mathcal{B}_0$.* In particular, (B_j) states that $\Psi_S(w|_j)$ is admissible for all $S \in \mathcal{B}_0$.
By Lemma 4.5, (B_{k+1}) already holds. The induction step is contained in the following lemma.

walk	possible ending states
$\Psi_\circ(w\|_j)$	$\{\circ\}$
$\Psi_P(w\|_j)$	$\{\circ, M', K', J\}$
$\Psi_{-P}(w\|_j)$	$\{\circ, M, K, J'\}$
$\Psi_Q(w\|_j)$	$\{\circ, J, K, L, M\}$
$\Psi_{-Q}(w\|_j)$	$\{\circ, J', K', L', M'\}$
$\Psi_R(w\|_j)$	$\{\circ, L', J', I', M', K\}$
$\Psi_{-R}(w\|_j)$	$\{\circ, L, J, I, M, K'\}$

TABLE 4. Table of statement (B_j).

S_{j+1}	S_j	a'_{j+1}	end of $\Psi_{S_j}(w\|_j)$	end of $\Psi_{S_{j+1}}(w\|_{j+1})$
P	$\circ$	$1,2,3,4$	$\circ$	$\circ$
	R	0	$\circ, J', L', I', M', K$	$\circ$
Q	P	$3,4$	$\circ, J, K', M'$	$\circ, L$
	$-Q$	$0,1,2$	$\circ, L', K', J', M'$	$\circ, J, L, M$
R	Q	$1,2,3,4$	$\circ, L, K, J, M$	$\circ, L', J', M'$
	$-P$	0	$\circ, J', K, M$	$\circ, L'$

TABLE 5. Proof of Lemma 4.6.

Lemma 4.6. *If (B_j) holds for some $j \geq k+1$, then (B_{j+1}) holds too.*

Proof. First we deal with the case of $S = P$. Then there is an edge $P \xrightarrow{a_{j+1}|a'_{j+1}} S'$ in $\mathcal{B}_0$, thus $S' \in \{\circ, R\}$. Remember that $\Psi_P(w|_{j+1}) = \Psi_{S'}(w|_j)\&(a'_{j+1})$.

By assumption (B_j), $S' \xrightarrow{w|_j^T} \circ$, thus $P \xrightarrow{a_{j+1}|a'_{j+1}} S' \xrightarrow{w|j^T} \circ$, *i.e.*, $P \xrightarrow{w|j+1^T} \circ$ in $\mathcal{B}_0$.

Moreover, if $S' = \circ$, then $\Psi_P(w|_{j+1}) = w|_j\&(a'_{j+1})$ is a walk in $p(F)$ that ends at $\circ$: indeed, $w|_j$ is in $p(F)$ and $j \geq k+1$, so by (4.1) $w|_j$ already ends at $\circ$; its concatenation with (a'_{j+1}) remains in $p(F)$, because $\circ \xrightarrow{a'_{j+1}} \circ$.

If $S' = R$, then the edge $P \xrightarrow{a_{j+1}|a'_{j+1}} R$ in $\mathcal{B}_0$ indicates that $a'_{j+1} = 0$. Hence $\Psi_P(w_{j+1}) = \Psi_R(w|_j)\&(a'_{j+1})$, with $\Psi_R(w|_j)$ walk in $p(F)$ ending at $Y \in \{\circ, L', J', I', M', K\}$ by assumption (B_j) (see Table 4 for the endings). Now we can check on $\mathcal{G}$ that there is an edge starting from each $Y \in \{\circ, L', J', I', M', K\}$ and labelled by 0. They all lead to $\circ$.

The other cases $(S = Q, S = R)$ as well as their duals are treated in a similar way (see Table 5 for $S = Q, S = R$). □

Proof of Proposition 4.1. Let w be a walk in $p_n(F, \circ)$ of the shape (4.1) and the resulting walk in $\mathcal{B}_0$ given by (4.2). If $n = k+1$, then $w|_{k+1} = w$ and Lemma 4.5 gives the result immediately. Otherwise, starting from the same lemma and going on with Lemma 4.6 from $j = k+1$ to $j = n$, we also obtain the statement of Proposition 4.1. □

5. Equivalences of paths in $p(F, \circ)$

The main result of this section, Proposition 5.2, will be used in Section 8 to construct arcs inside Int$(\mathbf{M})$ from arbitrary points contained in a subpiece $\psi_w(\mathcal{T})$, where $w \in p(F, \circ)$, to the point 0 (contained in $\psi_{(0,0,0)}(\mathcal{T})$).
In the following, the equivalences of walks from the set G_k defined as in (2.4) for some $k \in \mathbb{N}$ will take place in G_k. First we note the following fact about the walks of length 3.

Remark 5.1. We have $G_3 = \{(F; 0,0,0), (F; 1,4,4)\}$ and these walks are equivalent in G_3. Namely, $(F; 1,4,4) = \Psi_{-P}((F; 0,0,0))$. In other words, G_3 is transitive.

Proposition 5.2. *Let $n \geq 4$ and $w \in G_n$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0,\ldots,4\}$.*

In view of Remark 3.9.2 this proposition says the following. Let w be a walk in G_n and let $\psi_w(\mathcal{T})$ be the associated subset of $\mathcal{T}$. Then $\psi_w(\mathcal{T})$ contains a subpiece $\psi_{w\&d}(\mathcal{T})$ with the following property. There exist walks $w\&d = v_1, v_2, \ldots, v_k = w'$ in G_{n+1} such that the associated subpieces satisfy

$$\psi_{v_j}(\mathcal{T}) \cap \psi_{v_{j+1}}(\mathcal{T}) \neq \emptyset \qquad (1 \leq j \leq k-1).$$

Since $w'' := w'|_{n-1} \in G_{n-1}$, this means that one can draw an arc from each piece $\psi_w(\mathcal{T})$ of $\mathcal{M}(G_n)$ to a piece $\psi_{w''}(\mathcal{T})$ of $\mathcal{M}(G_{n-1})$. By induction on n this will lead to a proof of the connectivity of $\mathbf{M}$ because it allows to draw arcs from each point of $\mathbf{M}$ to the connected set $\psi_{(F;0,0,0)}(\mathcal{T}) \subset \mathbf{M}$.

Remark 5.3.
1) If $w\&d \sim w'$ for $w \in G_n$ and $d \in \{0,\ldots,4\}$, then by using $\Psi_{\pm P}$ we even have $w\&d \sim w'$ for every $d \in \{0,\ldots,4\}$.
2) If two walks w and w' of G_n are equivalent, then there exist $d, d' \in \{0,\ldots,4\}$ with $w\&d \sim w'\&d'$. This means that two intersecting pieces $\psi_w(\mathcal{T})$ and $\psi_{w'}(\mathcal{T})$ (*i.e.*, such that $\psi_w(\mathcal{T}) \cap \psi_{w'}(\mathcal{T}) \neq \emptyset$) contain intersecting subpieces $\psi_{w\&d}(\mathcal{T})$ and $\psi_{w'\&d'}(\mathcal{T})$. In particular it is sufficient to find a walk $w' \in G_n$ with $w'|_{n-1} \in G_{n-1}$ and $w \sim w'$. In this case w will automatically fulfil Proposition 5.2.
3) For $p \in \mathbb{N}$, we introduce the notation $w\ _{S^p}\!\sim w'$: this means that w' is obtained after applying Ψ_S to w for p times.

Proposition 5.2 will be shown via the following lemmata. By Remark 3.9.3, we just have to concentrate on the tails of the walks. Moreover, the lemmata correspond to the following classes of walks:

$$E_n(A) := \{w \in G_n,\ w \text{ contains the edge } A \to \circ\ \} \qquad (a \in \mathcal{G} \setminus \{\circ\}).$$

Sloppily spoken the walks contained in $E_n(A)$ are those walks of G_n which reach the accepting state via the state A. Note that

$$G_n = \bigcup_{A \in \mathcal{G}} E_n(A). \tag{5.1}$$

Lemma 5.4. *Let $n \geq 4$ and $w \in E_n(K) \cup E_n(K')$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0,\ldots,4\}$.*

Proof. Let us consider $w \in E_n(K)$. If $w|_{n-1} \in E_{n-1}(K)$, we are ready. We suppose it is not the case. We have the following cases for the tails τ of $w = \sigma\&\tau$ (σ is fixed by w and τ).

(1) $\tau = (J;0,a)$ with $a \in \{0,1,2\}$. Then

$$(J;0,2)\ _{-P}\!\sim (J;0,1)\ _{-P}\!\sim (J;0,0)\ _{Q}\!\sim (J;1,3) \prec (J;1),$$

which ends at $\circ$. Thus $w \sim \sigma\&(J;1,3)$ with $\sigma\&(J;1) \in G_{n-1}$, and we are ready by Remark 5.3.2.

(2) $\tau = (A;d,1,0,a)$ with $(A,d) \in \mathcal{C} := \{(F,0),(G',2),(N',2),(I,1)\}$ and $a \in \{0,1,2\}$. Then for all constellations $(A,d) \in \mathcal{C}$, we have

$$\begin{aligned}(A;d,1,0,0)\ _{P}\!\sim\ & (A;d,1,0,1)\ _{P}\!\sim (A;d,1,0,2)\ _{Q}\!\sim (A;d+1,4,2,0)\ _{Q}\!\sim (A;d+1,4,3,3)\\ _{(-P)^3}\!\sim\ & (A;d+1,4,3,0)\ _{Q}\!\sim (A;d+1,4,4,3) \prec (A;d+1,4,4),\end{aligned}$$

which ends at $\circ$. Thus $w \sim \sigma\&(A;d+1,4,4,3)$ with $\sigma\&(A;d+1,4,4) \in G_{n-1}$, and we are ready by Remark 5.3.2.

One can proceed identically for $w \in E_n(K')$ by considering the dual walks of the previous ones. Thus Lemma 5.4 is proved. □

Lemma 5.5. *Let $n \geq 4$ and $w \in E_n(J) \cup E_n(J')$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0,\ldots,4\}$.*

Proof. Let us consider $w \in E_n(J)$. If $w|_{n-1} \in E_{n-1}(J)$, we are ready. We suppose it is not the case. Then the tail τ of $w = \sigma\&\tau$ has the form $\tau = (A;1,a)$ with $A \in \mathcal{A} := \{L', M', I', H'\}$ and $a \in \{1,2,3,4\}$. We have the following equivalences for $A \in \mathcal{A}$:

$$(A;1,1) \ _P\!\sim (A;1,2) \ _P\!\sim (A;1,3) \ _P\!\sim (A;1,4) \ _{-Q}\!\sim (A;0,1).$$

For $A \in \{L', M', I'\}$, $\sim (A;0,1) \prec (A;0)$, which ends at $\circ$. Thus $w \sim \sigma\&(A;0,1)$ with $\sigma\&(A;0) \in G_{n-1}$, and we are ready by Remark 5.3.2.

For $A = H'$, we have $w = \sigma\&(H';0,1)$ which is now a walk belonging to $E_n(K)$, thus we obtain the required result by using Lemma 5.4.

One can proceed identically for $w \in E_n(J')$, thus Lemma 5.5 is proved. □

Lemma 5.6. *Let $n \geq 4$ and $w \in E_n(L) \cup E_n(L')$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0,\dots,4\}$.*

Proof. Let us consider $w \in E_n(L')$. If $w|_{n-1} \in E_{n-1}(L')$, we are ready. We suppose it is not the case. Then w belongs to one of the following classes of tails τ of $w = \sigma\&\tau$ (σ is fixed by w and τ).

(1.*i*) $\tau = (L;2,0)$. Then $(L;2,0) \ _Q\!\sim (L;3,3)$ which leads to a walk $\sigma\&(L;3,3)$ equivalent to w which belongs to $E_n(J')$, treated in Lemma 5.5.

(1.*ii*) $\tau = (L;1,0)$. We subdivide this class into the following smaller classes:
A. $(A;d,1,0)$ with $(A,d) \in \mathcal{C}_A := \{(K',0),(H',a),(L',a), a \in \{2,3\}\}$.
B. $(K;3,d,1,0)$ with $d \in \{2,3,4\}$.
C. $(I';3,4,1,0)$.
D. $(A;d,0,(3,4,0)^p,2,1,0)$ for some $p \in \mathbb{N}$ and

$$(A,d) \in \mathcal{C}_D := \begin{array}{l}\{(F,0),(K',0),(I',2),(M',4),\\ (G',a),(N',a),(H'a),(L',a),(M',a), a \in \{2,3\}\}.\end{array}$$

Here $(3,4,0)^p$ inside the walk means that the sequence of digits $(3,4,0)$ has to be read p times before going on to the digit 2. This corresponds to the cycle $I' \to N' \to L \to I'$ in the graph of Figure 2.

A. We have for $(A,d) \in \mathcal{C}_A$: $(A;d,1,0) \ _{-Q}\!\sim (A;d+1,4,2) \prec (A;d+1,4)$, which ends at $\circ$.

B. We have for $d = 3,4$ that $(K;3,d,1,0) \ _{-P}\!\sim (K;2,d-3,0,4) \prec (K;2)$, which ends at $\circ$, and $(K;3,2,1,0) \ _{-Q}\!\sim (K;3,3,4,2) \prec (K;3,3,4)$, which ends at $\circ$.

C. We have $(I';3,4,1,0) \ _{-P}\!\sim (I';2,1,0,4) \prec (I';2,1,0)$, which ends at $\circ$ too.

D. The following chain holds for every $(A,d) \in \mathcal{C}_D \setminus \{(M',4)\}$ and $p \geq 0$:

$$\begin{array}{rl}(A;d,0,(3,4,0)^p,2,1,0) \ _{-P}\!\sim & (A;d+1,(3,4,0)^p,3,4,0,4) \succ (A;d+1,(3,4,0)^p,3,4,0,4,4)\\ _Q\!\sim & (A;d+1,(3,4,0)^p,3,4,0,3,1) \ _{P^2}\!\sim (A;d+1,(3,4,0)^p,3,4,0,3,3)\\ _{-Q}\!\sim & (A;d+1,(3,4,0)^p,3,4,0,2,0)_{(-Q)^3}\!\sim (A;d,0,(3,4,0)^p,2,2,0,2)\\ \prec & (A;d,0,(3,4,0)^p,2,2,0),\end{array}$$

which is of type $(L;2,0)$ treated in Item (1.*i*).

If (A,d) happens to be $(M',4)$, we go into smaller classes by considering the tails $(A';1,0,3,4,0,(3,4,0)^p,2,1,0)$ with $A' \in \mathcal{A} := \{G,N,H',M',L',I'\}$ and we have the similar chain for $A' \in \mathcal{A}$:

$$\begin{aligned}
(A';1,0,3,4,0,(3,4,0)^p,2,1,0) \;{}_{-P}\sim\; & (A';2,3,4,0,(3,4,0)^p,3,4,0,4)\\
\succ\; & (A';2,3,4,0,(3,4,0)^p,3,4,0,4,4)\\
{}_{Q}\sim\; & (A';2,3,4,0,(3,4,0)^p,3,4,0,3,1)\\
{}_{P^2}\sim\; & (A';2,3,4,0,(3,4,0)^p,3,4,0,3,3)\\
{}_{-Q}\sim\; & (A';2,3,4,0,(3,4,0)^p,3,4,0,2,0)\\
{}_{(-Q}\sim\; & (A';1,0,3,4,0,(3,4,0)^p,2,2,0,2)\\
\prec\; & (A';1,0,3,4,0,(3,4,0)^p,2,2,0),
\end{aligned}$$

which is again of type $(L;2,0)$ treated in Item $(1.i)$.

(2) $\tau \in \{(I,2),(M,0),(M,a),(H,a), a \in \{1,2\}\}$. Then

$$(A;2,0) \;{}_{Q}\sim (S;3,3) \in E_n(J')$$

for $A \in \{I,M,H\}$, and for the other cases one can consider the smaller classes:

- for $A \in \{G',N'\}$,
$$(A;3,1,0) \;{}_{Q}\sim (A;4,4,2) \prec (A;4,4),$$
which ends at $\circ$.
- for $d \in \{0,1\}$,
$$(K';1,d,0) \;{}_{-Q}\sim (K';2,d+3,2) \prec (K';2),$$
which ends at $\circ$.

(3) $\tau = (N;0,0)$. Then w must end in the form $(I;1,0,0)$, and the following chain holds:

$$\begin{aligned}
(I;1,0,0) \quad & \succ (I;1,0,0,0) \;{}_{Q}\sim (I;1,0,1,3) \;{}_{P}\sim (I;1,0,1,4)\\
{}_{Q}\sim \quad & (I;2,3,3,2)\\
{}_{(-P)^2}\sim \quad & (I;2,3,3,0) \;{}_{Q}(I;2,3,4,3) \prec (I;2,3,4),
\end{aligned}$$

which is of type $(L';3,4)$: this is the dual of the tail $(L;1,0)$, thus it can be treated as in Item $(1.ii)$.

(4) $\tau = (K;4,0)$. Then the walk w ends in the following way: $(A;d,1,0,(4,1,0)^p,4,0)$ for some $p \in \mathbb{N}$ and

$$\begin{aligned}
(A,d) \in \mathcal{C} := \quad & \{(F,0),(G',2),(N',2),(I,1),\\
& (G,1),(N,1),(K,3),(G,0),(L,0),(H,0),\\
& (I,2),(N,0),(M,0),(M,a),(L,a),(H,a), a \in \{1,2\}\}.
\end{aligned}$$

For $(A,d) \in \mathcal{C}$, the following chain holds:

$$\begin{aligned}
(A;d,1,0,(4,1,0)^p,4,0) \succ \quad & (A;d,1,0,(4,1,0)^p,4,0,0) \;{}_{Q}\sim (A;d,1,0,(4,1,0)^p,4,1,3)\\
{}_{P}\sim \quad & (A;d,1,0,(4,1,0)^p,4,1,4) \;{}_{Q}\sim (A;d+1,(4,1,0)^p,4,2,2,3,2)\\
{}_{(-P)^2}\sim \quad & (A;d+1,(4,1,0)^p,4,2,2,3,0) \;{}_{Q}\sim (A;d+1,(4,1,0)^p,4,2,2,4,3).
\end{aligned}$$

If now $p=0$ and $(A,d) \in \{(G',2),(N',2),(I,2),(M,2),(L,2),(H,2)\}$, then we have

$$(A;d+1,(4,1,0)^p,4,2,2,4,3) \prec (A;d+1,4,2),$$

which ends at $\circ$; otherwise

$$(A;d+1,(4,1,0)^p,4,2,2,4,3) \prec (A;d+1,(4,1,0)^p,4,2,2,4),$$

which is of type $(L';3,4)$, *i.e.*, of the dual of the tail $(L;1,0)$, that can be treated similarly as in Item $(1.ii)$.

Proceeding identically for $w \in E_n(L)$, we obtain Lemma 5.6. □

Lemma 5.7. *Let $n \geq 4$ and $w \in E_n(M) \cup E_n(M')$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0,\dots,4\}$.*

Proof. Let us consider $w \in E_n(M')$. If $w|_{n-1} \in E_{n-1}(M')$, we are ready. We suppose it is not the case. Then w belongs to the following classes of tails τ of $w = \sigma\&\tau$: $(A; d, 1, 0, (4, 1, 0)^p), 3, 0)$ for some $p \in \mathbb{N}$ and

$$\begin{aligned}(A, d) \in \mathcal{C} := \quad & \{(F, 0), (G', 2), (N', 2), (I, 1),\\ & (G, 1), (N, 1), (K, 3), (G, 0), (L, 0), (H, 0),\\ & (I, 2), (N, 0), (M, 0), (M, a), (L, a), (H, a), a \in \{1, 2\}\}.\end{aligned}$$

For $(A, d) \in \mathcal{C}$, we have the equivalence

$$(A; d, 1, 0, (4, 1, 0)^p), 3, 0) \;_{-P}\!\sim (A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4).$$

If now $p = 0$ and $(A, d) \in \{(G', 2), (N', 2), (I, 2), (M, 2), (L, 2), (H, 2)\}$, then we have

$$(A; d + 1, (4, 1, 0)^p, 4, 2, 2, 4) \prec (A; d + 1, 4, 2)$$

which ends at $\circ$, otherwise the tail of the equivalent walk is of type $(L; 4)$ which was treated in Lemma 5.6.

We can proceed similarly for the dual case, and Lemma 5.7 is proved. □

Lemma 5.8. *Let $n \geq 4$ and $w \in E_n(I) \cup E_n(I')$. Then there is a walk $w' \in G_{n+1}$ such that $w'|_{n-1} \in G_{n-1}$ and $w\&d \sim w'$ for some $d \in \{0, \dots, 4\}$.*

Proof. Let us consider $w \in E_n(I)$. If $w|_{n-1} \in E_{n-1}(I)$, we are ready. We suppose it is not the case. Then w belongs to the following classes of tails τ of $w = \sigma\&\tau$: $(A; d, 4, (1, 0, 4)^p), 4)$ for some $p \in \mathbb{N}$ and

$$\begin{aligned}(A, d) \in \mathcal{C} := \quad & \{(G, 1), (N, 1), (F, 1), (G, 2), (N, 2)\\ & (I, 2), (K, 4), (M, 0), (M, a), (L, a), (H, a), a \in \{1, 2\}\}.\end{aligned}$$

For $(A, d) \in \mathcal{C} \setminus \{(M, 0)\}$, we have the equivalence

$$(A; d, 4, (1, 0, 4)^p), 4) \;_P\!\sim (A; d - 1, (1, 0, 4)^p, 0, 0).$$

We consider the following cases:

- for $p = 0$ and $(A, d) = (F, 1)$, then $w = (F; 1, 4, 4) \in G_3$ has length $n = 3$;
- for $p = 0$ and $(A, d) \in \{(G, 2), (N, 2)\}$, $(A; d - 1, 0, 0)$ is the tail of a walk belonging to $E_n(K)$, treated in Lemma 5.4;
- for $p = 0$ and $(A, d) = (I, 2)$, $(I; 1, 0, 0)$ is the tail of a walk in $E_n(L')$, treated in Lemma 5.6;
- otherwise $(A; d - 1, (1, 0, 4)^p, 0, 0) \prec (A; d - 1, (1, 0, 4)^p, 0)$, which ends at $\circ$.

For $(A, d) = (M, 0)$, we go into the smaller classes $(A'; 3, 4, 1, 0, 4, (1, 0, 4)^p, 4)$ with $p \geq 0$ and $A' \in \{G', X', I, M, L, H\}$. In these cases,

$$(A'; 3, 4, 1, 0, 4, (1, 0, 4)^p, 4) \;_P\!\sim (A'; 2, 1, 0, 4, (1, 0, 4)^p, 0, 0) \prec (A'; 2, 1, 0, 4, (1, 0, 4)^p, 0),$$

which ends at $\circ$.

Dealing with the walks of $E_n(I')$ in the same way, we obtain Lemma 5.8. □

Proposition 5.2 now follows from Lemmata 5.4 to 5.8 together with the equation (5.1).

6. Boundary of $\mathbf{M}$

As will be seen later, the last two sections assure the connectivity of the subset of $\mathbf{M}$ consisting of the union of the subpieces $\psi_w(\mathcal{T})$ where w is a walk of $\mathcal{G}$ starting at F and ending at the accepting state $\circ$. By definition, this subset is dense in $\mathbf{M}$. The present section now uses the walks of $p(F)$ that do not end at $\circ$ to prove that the boundary $\partial\mathbf{M}$ of $\mathbf{M}$ lies in the boundary of $\mathcal{T}$.

Proposition 6.1. *Let $w_0 := (a_1, \dots, a_n)$ be a walk in $p(F)$ which does not end at $\circ$. Then*

$$\psi_{w_0}(\mathcal{T}) \cap \partial\mathcal{T} \neq \emptyset.$$

For w_0 as in the proposition, we will show that $w := (0,4)\&w_0$ satisfies $\psi_w(\mathcal{T}) \cap B_Q \neq \emptyset$. Note that then $\psi_w(\mathcal{T}) = \psi_0 \circ \psi_4 \circ \psi_{w_0}(\mathcal{T})$; we will see that this piece stays in contact with $\partial\mathcal{T}$ after application of the inverse of $\psi_0 \circ \psi_4$. We need the following lemma.

For A a state of $\mathcal{G}$, $\mathcal{S}$ a subset of $\mathcal{B}$ and $n \geq 3$, let (B_n) be the following assertion:

If w_0 is a walk of length $n-2$ in $p(F)$ ending at $A \neq \circ$, then $w^T := ((0,4)\&w_0)^T$ is the labelling of a walk in $\mathcal{B}$ starting at S and ending at Q for every $S \in \mathcal{S}$, where $(A, \mathcal{S})$ are given in the Table 6 (the duals have to be added, they associate A' to $-\mathcal{S}$).

Lemma 6.2. *The assertion (B_n) holds for every $n \geq 3$.*

Proof. For $n = 3$ we have $w = (0,4,0)$ and $A = G$, or $w = (0,4,1)$ and $A = G'$. It is easily seen on $\mathcal{B}_0$ that $S \xrightarrow{w^T} Q$ for all S in the corresponding $\mathcal{S}$.

Let us suppose (B_n) to be true for an $n \geq 3$. We show that (B_{n+1}) also holds. Let $w = (0, 4, a_3, \dots, a_{n+1})$ with $(F; a_3, \dots, a_{n+1}) =: w_0 \in p(F)$.

If w_0 ends up in $A = G$, then $u := (a_3, \dots, a_n)$ ends up in G' or N' and $a_{n+1} = 2$, because $G' \xrightarrow{2} G$ and $N' \xrightarrow{2} G$ are the only edges of $\mathcal{G}$ leading to G (the case $F \xrightarrow{a_{n+1}} G$ is not possible, since there would be no edge labelled by a_n leading to F). If u ends up in G', then by assumption we have $S \xrightarrow{w|_n^T} Q$ for all $S \in \{Q, -Q, R, -R\}$. Since $R \xrightarrow{2} S = Q$, $-Q \xrightarrow{2} S = Q$ and $-R \xrightarrow{2} S = -Q$, $Q \xrightarrow{2} S = -Q$ all exist in $\mathcal{B}$, we obtain for every $S' \in \{\pm Q, \pm R\}$: $S' \xrightarrow{2} S \xrightarrow{w|_n^T} Q$, *i.e.*, $S' \xrightarrow{w^T} Q$. If u ends up in N', then by assumption we have $S \xrightarrow{w|_n^T} Q$ for all $S \in \{Q, -Q, \}$. Thus one can use the preceding walks in $\mathcal{B}$: $R \xrightarrow{2} S = Q$, $-Q \xrightarrow{2} S = Q$ and $-R \xrightarrow{2} S = -Q$, $Q \xrightarrow{2} S = -Q$ all exist in $\mathcal{B}$, hence for every $S' \in \{\pm Q, \pm R\}$, $S' \xrightarrow{2} S \xrightarrow{w|_n^T} Q$, *i.e.*, $S' \xrightarrow{w^T} Q$.

The results for the other possible endings A of w_0 are summed up in Table 7 (the duals can be treated likewise). □

Proof of Proposition 6.1. Let $w := (0,4)\&w_0$ with w_0 a walk in $p(F)$ that does not end at $\circ$. Then

$$\psi_w(\mathcal{T}) \cap B_Q \neq \emptyset.$$

Indeed, by Proposition 3.4, it suffices to show that there exists a walk $Q \xleftarrow{w} S$ in $\mathcal{B}$. This is what Lemma 6.2 does. Now recall that $\psi_w(\mathcal{T}) = \psi_0 \circ \psi_4 \circ \psi_{w_0}(\mathcal{T})$. Thus again by Proposition 3.4 there exists an infinite walk $Q \xleftarrow{0} S \xleftarrow{4} S' \xleftarrow{w_0} \dots$ in $\mathcal{B}$. This implies that $S' \in \{P, Q, -R\}$, as can be checked on $\mathcal{B}_0$. Thus there is an infinite walk $S' \xleftarrow{w_0} \dots$ in B with $S' \in \mathcal{B}$, and therefore $\psi_{w_0}(\mathcal{T}) \cap \partial\mathcal{T} \neq \emptyset$, as assured by Proposition 3.4. □

A	$\mathcal{S}$
G	$\{Q, -Q, R, -R\}$
H	$\{Q, R, -R\}$
I	$\{P, Q\}$
J	$\{-R\}$
K	$\{-P\}$
L	$\{Q, -R\}$
M	$\{Q\}$
N	$\{Q, -Q\}$

TABLE 6. Table for assertion (B_n).

A	end of $w\|_n$	a_{n+1}	$S' \xrightarrow{a_{n+1}} S$
G	G', N'	2	$R, -Q \to Q$
			$-R, Q \to -Q$
H	G', N'	3	$R \to Q$
			$-R, Q \to -Q$
I	H', G', L'	4	$P \to R$
			$Q \to -Q$
J	H', M', I'	1	$-R \to -Q$
K	H', J	0	$-P \to -R$
L	H', L'	2, 3	$Q \to -Q$
			$-R \to -Q$
	M'	2, 3, 4	$Q \to -Q$
			$-R \to -Q$
	N'	4	$Q \to -Q$
			$-R \to -Q$
	K'	0	$Q \to P$
			$-R \to P$
M	K'	1	$Q \to P$
N	I	1	$Q \to P$
			$-Q \to Q$

TABLE 7. Proof of Lemma 6.2.

In what follows we will use the following notations. We fix a metric $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{R}^2$, and denote the diameter of a compact set C, defined as the maximal distance between two points of C, by $\mathrm{diam}(C)$.

Proposition 6.3. *The boundary of* $\mathbf{M}$ *is contained in the boundary of* $\mathcal{T}$.

Proof. Let $x \in \partial\mathbf{M}$. We will show that for every $\varepsilon > 0$, we have $\mathrm{dist}(x, \partial\mathcal{T}) < \varepsilon$. This will imply that $x \in \partial\mathcal{T}$, since $\partial\mathcal{T}$ is a closed set. We consider two cases.

Case 1. For every $n \geq 3$, $x \notin \mathcal{M}(G_n)$ (see Definition 2.6). The element x belonging to $\mathbf{M}$, we can write $x = \sum_{i=1}^{\infty} \mathbf{A}^{-i}\Phi(a_i)$ with $w_n := (a_1, \ldots, a_n) \in p_n(F)$. In our assumption, for every $n \geq 3$, we have $w_n \notin p_n(F, \circ)$. Let $\varepsilon > 0$ and n_0 such that for $n \geq n_0$, $\mathrm{diam}(\psi_w(\mathcal{T})) < \varepsilon$ for every w of length $|w| = n$. Then

$$x \in \psi_{w_{n_0}}(\mathcal{T}) \text{ with } \begin{cases} w_{n_0} \in p_{n_0}(F) & \text{(by definition)} \\ w_{n_0} \notin p_{n_0}(F, \circ) & \text{(by assumption)} \end{cases}.$$

By Proposition 6.1, $\psi_{w_{n_0}}(\mathcal{T}) \cap \partial T \neq \emptyset$, thus $\mathrm{dist}(x, \partial T) < \varepsilon$ since $\mathrm{diam}(\psi_{w_{n_0}}(\mathcal{T})) < \varepsilon$.

Case 2. There is an $n_0 \geq 3$ with $x \in \mathcal{M}(G_{n_0})$. Because of Equation (1.2), we even have $x \in \mathcal{M}(G_n)$ for all $n \geq n_0$.
We denote by $B_r(0)$ the open ball $\{y \in \mathbb{R}^2,\ \mathrm{dist}(0, y) < r\}$. By compactness of $\mathcal{T}$, it is possible to find $r_1 > 0$ such that

$$\mathcal{T} \subset B_{r_1}(0). \tag{6.1}$$

Since $\{P, Q\}$ is a basis of the lattice $\Phi(\mathbb{Z}[\alpha])$ by (1.4), there exist positive integers m_1, m_2 such that

$$B_{r_1}(0) \subset \bigcup_{\substack{n_1 \in \{-m_1, \ldots, m_1\} \\ n_2 \in \{-m_2, \ldots, m_2\}}} (\mathcal{T} + n_1 P + n_2 Q) \subset B_{r_2}(0). \tag{6.2}$$

The second inclusion follows again from the compactness of $\mathcal{T}$, $r_2 > 0$ is simply chosen large enough.

Let now $\varepsilon > 0$, $n \geq n_0$ such that $\operatorname{diam}(\psi_w(B_{r_2}(0))) < \varepsilon$ for every w of length $|w| = n$, and let $w \in p_n(F,\circ) = G_n$ such that $x \in \psi_w(\mathcal{T})$. Then using (6.1) and (6.2), the following inclusions hold:

$$x \in \psi_w(\mathcal{T}) \subset \psi_w(B_{r_1}(0)) \subset \underbrace{\psi_w\left(\bigcup_{\substack{n_1 \in \{-m_1,\dots,m_1\}\\ n_2 \in \{-m_2,\dots,m_2\}}} (\mathcal{T} + n_1P + n_2Q)\right)}_{= \bigcup_{\substack{n_1 \in \{-m_1,\dots,m_1\}\\ n_2 \in \{-m_2,\dots,m_2\}}} \psi_w(\mathcal{T} + n_1P + n_2Q)} \subset \psi_w(B_{r_2}(0)).$$

Our aim is to find a $y \in \partial\mathcal{T}$ in the union above. Since x and y will then both belong to $\psi_w(B_{r_2}(0))$, which has diameter less than ε, we will be done. Note that $\psi_w(B_{r_1}(0))$ is a neighborhood of x, a point of $\partial\mathbf{M}$, hence this neighborhood has nonempty intersection with the complement of $\mathbf{M}$ in $\mathbb{R}^2$.

Remember that w is a walk of $p_n(F,\circ)$. Now we make the following assumption:

- Each of the following additions is admissible,
- Each of the following additions yields a walk that is contained in $p(F,\circ)$.

$$\begin{aligned}
W_1 &:= \{\Phi_{n_1P} \circ \Phi_{n_2Q}(w) \,|\, 0 \leq n_1 \leq m_1;\, 0 \leq n_2 \leq m_2\},\\
W_2 &:= \{\Phi_{n_1(-P)} \circ \Phi_{n_2Q}(w) \,|\, 0 \leq n_1 \leq m_1;\, 0 \leq n_2 \leq m_2\},\\
W_3 &:= \{\Phi_{n_1P} \circ \Phi_{n_2(-Q)}(w) \,|\, 0 \leq n_1 \leq m_1;\, 0 \leq n_2 \leq m_2\},\\
W_4 &:= \{\Phi_{n_1(-P)} \circ \Phi_{n_2(-Q)}(w) \,|\, 0 \leq n_1 \leq m_1;\, 0 \leq n_2 \leq m_2\}.
\end{aligned}$$

Set $W := W_1 \cup W_2 \cup W_3 \cup W_4$. With a slight abuse of notation we may write

$$W := \{\Phi_{n_1P} \circ \Phi_{n_2Q}(w) \,|\, -m_1 \leq n_1 \leq m_1;\, -m_2 \leq n_2 \leq m_2\}.$$

By assumption all walks of W are contained in $G_n = p_n(F,\circ)$. Thus

$$\begin{aligned}
\psi_w(B_{r_1}(0)) &\subset \bigcup_{n_1,n_2} \psi_w(\mathcal{T} + n_1P + n_2Q)\\
&= \bigcup_{n_1,n_2} \psi_{\Phi_{n_1P}\circ\Phi_{n_2Q}(w)}(\mathcal{T}) \qquad \text{(by Remark 3.9.1)}\\
&\subset \mathcal{M}(G_n)\\
&\subset \mathbf{M},
\end{aligned}$$

which contradicts the fact that $\psi_w(B_{r_1}(0))$ contains points of the complement $\mathbb{R}^2 \setminus \mathbf{M}$. So our assumption is wrong.

Therefore one of the following alternatives must hold:

- at least one of the additions in W is not admissible or
- at least one element $w_0 \in W$ does not belong to G_n.

In view of Proposition 4.1 we conclude that at least one $w_0 \in W$ does not belong to $G_n = p_n(F,\circ)$. Proposition 4.1 also shows that all additions are admissible for each element of G_n, *i.e.*, starting from a word in G_n, each addition $\Psi_{\pm P}, \Psi_{\pm Q}$ leads to a word in $p_n(F)$. Thus, starting at w, by a sequence of admissible additions we can reach a word $w_0 \in W$ which belongs to $p_n(F) \setminus p_n(F,\circ)$. Let us write

$$w_0 = \Phi_{n_1P} \circ \Phi_{n_2Q}(w) \in p_n(F) \setminus p_n(F,\circ)$$

for some $n_1 \in \{-m_1, \dots, m_1\}$, $n_2 \in \{-m_2, \dots, m_2\}$. (Note that by Remark 3.9.1

$$\psi_w(\mathcal{T} + n_1 P + n_2 Q) = \psi_{w_0}(\mathcal{T})$$

because we have a sequence of admissible additions from w to w_0.) For this choice of w_0 we have

- $\psi_{w_0}(\mathcal{T}) \subset \psi_w(B_{r_2}(0))$ by assumption,
- $\psi_{w_0}(\mathcal{T}) \cap \partial\mathcal{T} \neq \emptyset$ by Proposition 6.1.

This implies that $\psi_w(B_{r_2}(0))$ contains x as well as some point of $\partial\mathcal{T}$, hence $\mathrm{dist}(x, \partial T) < \varepsilon$.

Consequently, in both cases for every $\varepsilon > 0$, $\mathrm{dist}(x, \partial T) < \varepsilon$, thus $x \in \partial\mathcal{T}$, and this holds for every $x \in \partial\mathbf{M}$, hence, $\partial\mathbf{M} \subset \partial\mathcal{T}$. □

7. Generalized fundamental inequality and consequences

This section is devoted to a generalization of the fundamental inequality found in [1]. This will lead to the construction of an arcwise connected skeleton inside the interior of the tile $\mathcal{T}$ together with some of its neighbors. We denote by $\mathcal{S}$ the set of elements of $\mathbb{Q}[\alpha]$ with integer part zero with respect to the basis α and the digits $\mathcal{N} = \{0, 1, 2, 3, 4\}$:

$$\mathcal{S} := \left\{ \sum_{i=1}^{l} \Phi(\alpha^{-i} a_i),\ l \in \mathbb{N}, (a_i)_{1 \le i \le l} \in \mathcal{N}^l \right\}.$$

Consequently we have $\overline{\Phi(\mathcal{S})} = \mathcal{T}$.

Remark 7.1. We recall that the tile $\mathcal{T}$ is symmetric with respect to the point $\Phi(c) := \Phi\left(\frac{4}{2(\alpha-1)}\right)$ (see [1, Lemma 3.2]).

Proposition 7.2 (Generalized fundamental inequality)**.** *There is an $\varepsilon > 0$ such that for any $x \in \mathcal{S} + 2\alpha$ we have*

$$\Im(x) > \Im(c) + \varepsilon.$$

Proof. This follows from the minoration

$$\Im(\sum_{i=1}^{l} a_i \alpha^{-i}) \ge -\left|\sum_{i=1}^{l} a_i \alpha^{-i}\right| \ge -\frac{4}{|\alpha - 1|},$$

so that for $x \in \mathcal{S} + 2\alpha$ we have $\Im(x) > 0 > I(c) = -\frac{1}{5}$. □

Corollary 7.3. *Let $\gamma \in \mathbb{Z}[\alpha]$ and put $\gamma = u + v\alpha$ with u, v in $\mathbb{Z}$. Then there exists a constant $\varepsilon > 0$ such that for any $x \in \mathcal{S}$,*

$$\begin{cases} \Im(x) + \Im(\gamma) > \Im(c) + \varepsilon & \text{if } v \ge 2 \\ \Im(x) + \Im(\gamma) < \Im(c) - \varepsilon & \text{if } v \le -2. \end{cases}$$

Proof. This is proved in the same way as in [1, Lemma 4.3]. □

We will now construct a generalized version of the skeleton constructed in [1] where the tiles were disk-like. To this matter set the backbone

$$\mathcal{L} := \{\, \Phi(c) + w\Phi(1) \mid w \in [0, 4] \} .$$

Furthermore, let

$$V_0 := \bigcup_{S \in \mathcal{B}_0} (\mathcal{T} + S).$$

Then the following lemma holds.

Lemma 7.4. *We have $\mathcal{L} \subset \mathrm{Int}(\mathbf{A}V_0)$.*

Proof. Note that

$$\begin{aligned}
\mathrm{Int}(\mathbf{A}(V_0)) &= \mathbb{R}^2 \setminus \bigcup_{x\in\mathbb{Z}^2\setminus\mathcal{B}_0} (\mathbf{A}(\mathcal{T}+x)) \\
&= \mathbb{R}^2 \setminus \Bigg(\bigcup_{u\in\mathbb{Z},\, |v|\geq 2} (\mathcal{T}+\Phi(u+v\alpha)) \cup \\
&\qquad \bigcup_{v=1,\, u\leq -1 \text{ or } u\geq 10} (\mathcal{T}+\Phi(u+v\alpha)) \cup \\
&\qquad \bigcup_{v=0,\, u\leq -6 \text{ or } u\geq 10} (\mathcal{T}+\Phi(u+v\alpha)) \cup \\
&\qquad \bigcup_{v=-1,\, u\leq -6 \text{ or } u\geq 5} (\mathcal{T}+\Phi(u+v\alpha)) \Bigg).
\end{aligned}$$

Thus we have to show that $(\mathcal{T}+\Phi(u+v\alpha))\cap\mathcal{L}=\emptyset$ for all constellations (u,v) occurring in the above unions.
Let first $\gamma = u+v\alpha$ with $u\in\mathbb{Z}$ and $|v|\geq 2$. Then Corollary 7.3 and the fact that $\overline{\Phi(\mathcal{S})}=\mathcal{T}$ imply that $(\mathcal{T}+\Phi(\gamma))\cap\mathcal{L}=\emptyset$.
For the pairs of the shape $(u,0)$, $u\leq -6$ or $u\geq 10$ we see that $(\mathcal{T}+\Phi(\gamma))\cap\mathcal{L}=\emptyset$ in exactly the same way as in [1, Lemma 4.4].
If now $v=-1$, suppose first that $u\leq -6$. If $x\in\ \alpha^{-1}(\mathcal{S}-\alpha^2)+u$, then $\alpha x\in\ \mathcal{S}+5+(u+4)\alpha$. From $u+4\leq -2$ and by Corollary 7.3, we can write $\Im(\alpha x)<\Im(c)-\varepsilon$, and consequently:

$$\forall j\in\{0,\dots,4\},\ \Im(\alpha x)=\Im(\alpha x+j)=\Im(\alpha(x+\alpha^{-1}j))<\Im(c)-\varepsilon,$$

thus

$$\forall j\in\{0,\dots,4\},\ \forall x\in\ \alpha^{-1}(\mathcal{S}-\alpha^2+j)+u,\ \ \Im(\alpha x)<\Im(c)-\varepsilon.$$

From $\overline{\Phi(\mathcal{S})}=\mathcal{T}$, we conclude that this inequality also holds if we replace $\mathcal{S}$ by $\mathcal{T}$, and from the set equation of $\mathcal{T}$ we obtain, taking the union over $j\in\{0,\dots,4\}$,

$$\forall\Phi(x)\in\mathcal{T}+\Phi(u-\alpha),\ \ \Im(\alpha x)\leq\Im(c)-\varepsilon.$$

On the other hand, for an element of $\mathcal{L}$, *i.e.*, $\Phi(x)=\Phi(c)+w\Phi(1), w\in[0,4]$, we have

$$\Im(\alpha x)=w+\Im(\alpha c)\geq\Im(\alpha c)=-1/5=\Im(c),$$

which implies that $(\mathcal{T}+\Phi(u-\alpha))\cap\mathcal{L}=\emptyset$ for all $u\leq -6$.
Suppose secondly that $u\geq 5$. If again $x\in\ \alpha^{-1}(\mathcal{S}-\alpha^2)+u$, then $\alpha x\in\ \mathcal{S}+5+(u+4)\alpha$. From $u\geq 5$ and by Corollary 7.3, we have $\Im(\alpha x)>\Im(c)+\varepsilon+4\Im(\alpha)$, and consequently, by a similar reasoning as above, we obtain

$$\forall\Phi(x)\in\mathcal{T}+\Phi(u-\alpha),\ \ \Im(\alpha x)\geq\Im(c)+\varepsilon+4.$$

On the other hand, for $\Phi(x)\in\mathcal{L}$, we have $\Im(\alpha x)\leq\Im(c)+4$, thus again $(\mathcal{T}+\Phi(u-\alpha))\cap\mathcal{L}=\emptyset$ for all $u\geq 5$.

The remaining case $v=1$ is treated likewise, thus the proof is complete. □

Composing small pieces of backbones, let us define the *n-skeleton* by

$$\mathcal{K}_n=\bigcup_{m=1}^{n}\left(\bigcup_{(a_1,a_2,\dots a_m)}\sum_{i=1}^{m}A^{-i}\Phi(a_i)+\mathbf{A}^{-m-1}(\mathcal{L})\right).$$

Lemma 7.5. *$\mathcal{K}_n$ is arcwise connected and $\mathcal{K}_n\subset\mathrm{Int}(V_0)$.*

Proof. The arcwise connectivity can be shown as in [1, Lemma 4.6].
For the second part, note that for every $(a_1, \dots, a_m)$, by Lemma 7.4 we have for the small pieces

$$\sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a_i) + \mathbf{A}^{-1-m}\mathcal{L} \subset \sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a_i) + \operatorname{Int}\left(\mathbf{A}^{-m}V_0\right).$$

Now remember that using the automaton $\mathcal{B}_0$ one can compute for $S \in \mathcal{B}_0$ that

$$\begin{aligned}
\textstyle\sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a_i) + \mathbf{A}^{-m}S &= \mathbf{A}^{-m}\left(S + \textstyle\sum_{i=0}^{m-1} \mathbf{A}^{-i}\Phi(a_{m-i})\right)\\
&= \mathbf{A}^{-m}\left(\mathbf{A}^{m}S' + \textstyle\sum_{i=0}^{m-1} \mathbf{A}^{-i}\Phi(a'_{m-i})\right) \quad \text{(by (3.2))}\\
&= \textstyle\sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a'_i) + S'
\end{aligned}$$

with $S' \in \mathcal{B}_0$ and $a'_i \in \mathcal{N}$. From this we can conclude that

$$\sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a_i) + \mathbf{A}^{-m}V_0 \subset V_0,$$

hence the union of the backbones remains in $\operatorname{Int}(V_0)$. □

Remark 7.6. Note that the middle points $\sum_{i=1}^{m} \mathbf{A}^{-i}\Phi(a_i)$ of the union of all backbones are dense in $\mathcal{T}$.

8. The component of $\operatorname{Int}(\mathcal{T})$ containing 0

We are almost ready to prove Theorem 2.4 concerning the description of C_0. We will first construct an arc from an arbitrary point in $\operatorname{Int}(\mathbf{M})$ to zero entirely contained in $\operatorname{Int}(\mathbf{M})$.

Lemma 8.1. *If $x \in \operatorname{Int}(\mathbf{M})$, then there is an $n \geq 3$ such that $x \in \mathcal{M}(G_n)$.*

Proof. Let $x = \sum_{j\geq 1} \Phi(\alpha^{-j}aj) \in \operatorname{Int}(\mathbf{M})$. In particular, $w = (a_j)_{j\geq 1}$ is an infinite walk in $p(F)$. Suppose $x \notin \mathcal{M}(G_n)$ for any $n \geq 3$, *i.e.*, $w_n := (F; a_1, a_2, \dots, a_n) \in p_n(F)$ does not end at $\circ$ for any $n \geq 3$. We show that $x \in \partial\mathcal{T}$, which is a contradiction, since $\mathbf{M} \subset \mathcal{T}$, hence $\operatorname{Int}(\mathbf{M}) \subset \operatorname{Int}(\mathcal{T})$. We have by definition that $x \in \psi_{w_n}(\mathcal{T})$ for every $n \geq 3$. Fix $\varepsilon > 0$, then for n large enough we also have that

$$\begin{cases} \operatorname{diam}(\psi_{w_n}(\mathcal{T})) < \varepsilon, \\ \psi_{w_n}(\mathcal{T}) \cap \partial\mathcal{T} \neq \emptyset \text{ (by Lemma 6.1).} \end{cases}$$

Thus for every $\varepsilon > 0$, $\operatorname{dist}(x, \partial\mathcal{T}) < \varepsilon$, hence $x \in \partial\mathcal{T}$, since $\partial\mathcal{T}$ is a closed set. □

Lemma 8.2. *Let $n \geq 3$, $S \in \mathcal{B}$ and $v_1, v_2 \in G_n$ such that $v_2 = \Psi_S(v_1)$. Then $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T})$ contains points of* $\operatorname{Int}(\mathcal{T})$.

Proof. Since both subpieces $\psi_{v_1}(\mathcal{T})$ and $\psi_{v_2}(\mathcal{T})$ are subsets of $\mathcal{T}$, points of their intersection that are not in $\operatorname{Int}(\mathcal{T})$ must be in $\partial\mathcal{T}$. We show that there are at most countably many such points, whereas $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T})$ is uncountable.

The uncountability of $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T})$ follows from the fact that $v_2 = \Psi_S(v_1)$, hence $\psi_{v_2}(\mathcal{T}) = \psi_{v_1}(\mathcal{T} + S)$, with $S \in \mathcal{B}$ (see Remark 3.9.1). Since $\mathcal{T} \cap (\mathcal{T} + S)$ has uncountably many points for $S \in \mathcal{B}$ (see [1, Section 9]), this remains true after applying the homeomorphism ψ_{v_1}.

On the other side, a point x of $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T})$ which lies on $\partial\mathcal{T}$ also belongs to a translate $\mathcal{T} + S'$ of $\mathcal{T}$ with $S' \in \mathcal{B}$, by the boundary equation (3.3). This translate is the union of the subpieces $\psi_w(\mathcal{T}) + S'$ with $|w| = |v_1| =: n$. Let us write $v_1 =: (a_1, \dots, a_n)$ and $w =: (b_1, \dots, b_n)$. Then the point $\mathbf{A}^n x$ belongs to the triple intersection

$$\left(\mathcal{T} + \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(a_{n-i})\right) \cap \left(\mathcal{T} + S + \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(a_{n-i})\right) \cap \left(\mathcal{T} + \mathbf{A}^n S' + \sum_{i=0}^{n-1} \mathbf{A}^i\Phi(b_{n-i})\right),$$

or, equivalently, the point $\mathbf{A}^n x - \sum_{i=0}^{n-1} \mathbf{A}^i a_{n-i}$ belongs to the triple intersection

$$\mathcal{T} \cap (S + \mathcal{T}) \cap (S'' + \mathcal{T}) =: V(S, S'')$$

with $S'' := \mathbf{A}^n S' + \sum_{i=0}^{n-1} \mathbf{A}^i \Phi(b_{n-i}) - \sum_{i=0}^{n-1} \mathbf{A}^i \Phi(a_{n-i})$. Note that $S'' \notin \{\circ, S\}$. Indeed, using (3.1), (3.2) and Remark 3.2, $S'' = \circ$ as well as $S'' = S$ would imply $S' = \circ$.
Thus to each point x of $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T}) \cap \partial\mathcal{T}$ corresponds exactly one point of V_3, the set of all triple points of $\mathcal{T}$ (*i.e.*, where $\mathcal{T}$ meets with two other translates). Since V_3 is at most countable (see [1, Theorem 10.1]), there are at most countably many points in $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T}) \cap \partial\mathcal{T}$. Together with the first part of this proof this means that $\psi_{v_1}(\mathcal{T}) \cap \psi_{v_2}(\mathcal{T}) \cap \mathrm{Int}(\mathcal{T})$ is not empty. □

Proposition 8.3. *Let $x \in \mathrm{Int}(\mathbf{M})$ and an $n \geq 3$ given by Lemma 8.1,* i.e.*, such that $x \in \mathcal{M}(G_n)$. Then there is an arc p from x to an element y of $\mathrm{Int}(\mathcal{M}(G_{n-1}))$ with $p \subset \mathrm{Int}(\mathcal{T})$.*

Proof. In this proof we will often use the fact that $\mathcal{T}$ is the closure of its interior. This has been shown in a more general context in Wang [26]. Let $w \in G_n$ such that $x \in \psi_w(\mathcal{T})$. By Proposition 5.2 there exists a finite chain of walks $v_1, \ldots, v_m \in G_{n+1}$ with the following properties:

$$\begin{aligned} v_1 &= w\&d \text{ for some } d \in \{0, \ldots, 4\}, \\ \Psi_{S_i}(v_i) &= v_{i+1} \text{ for some } S_j \in \mathcal{B} \quad (1 \leq i \leq m-1), \\ v_m \mid_{n-1} &\in G_{n-1}. \end{aligned}$$

Now choose $x_i \in \mathrm{Int}(\psi_{v_i}(\mathcal{T}))$ arbitrary and set $y := x_m$. Note that $\psi_{v_m}(\mathcal{T}) \subset \psi_{v_m|_{n-1}}(\mathcal{T}) \subset \mathcal{M}(G_{n-1})$, thus y has the required property. First we shall construct an arc $p_1 \subset \mathrm{Int}(\mathcal{T})$ from x to x_1. Without loss of generality, one can suppose that $x \in \psi_{v_1}(\mathcal{T})$ (see Remark 5.3.1).

Since $x \in \mathrm{Int}(\mathbf{M})$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathrm{Int}(\mathbf{M})$. Thus there is a $z_1 \in B_\varepsilon(x) \cap \mathrm{Int}(\psi_{v_1}(\mathcal{T}))$. Connect x with z_1 by a straight line segment ℓ_1. Obviously, $\ell_1 \subset \mathrm{Int}(\mathbf{M})$.

Now $x_1, z_1 \in \mathrm{Int}(\psi_{v_1}(\mathcal{T}))$. Thus there exists $\varepsilon_2 > 0$ such that

$$\begin{aligned} B_{\varepsilon_2}(z_1) &\subset \mathrm{Int}(\psi_{v_1}(\mathcal{T})), \\ B_{\varepsilon_2}(x_1) &\subset \mathrm{Int}(\psi_{v_1}(\mathcal{T})). \end{aligned}$$

By Remark 7.6 at the end of the previous section there exists a $j \in \mathbb{N}$ such that $\psi_{v_1}(\mathcal{K}_j)$ contains points z_2, z_3 with

$$\begin{aligned} z_2 &\in B_{\varepsilon_2}(z_1), \\ z_3 &\in B_{\varepsilon_2}(x_1). \end{aligned}$$

Now connect z_1 with z_2 by the line segment ℓ_2 and connect z_3 with x_1 by the line segment ℓ_3. Both of these line segments are obviously contained in $\mathrm{Int}(\mathcal{T})$. Since $\mathcal{K}_j$ is arcwise connected by Lemma 7.5 there exists an arc $q_1 \in \psi_{v_1}(\mathcal{K}_j)$ connecting z_2 with z_3. We have to show that $q_1 \subset \mathrm{Int}(\mathcal{T})$.

What we know from Lemma 7.5 is that

$$q_1 \subset \mathrm{Int}(\psi_{v_1}(V_0)) = \mathrm{Int}\left(\bigcup_{S \in \mathcal{B}_0} \psi_{v_1}(\mathcal{T} + S)\right) = \mathrm{Int}\left(\bigcup_{S \in \mathcal{B}_0} \psi_{\Psi_S(v_1)}(\mathcal{T})\right).$$

We used here Remark 3.9.1). Indeed, since $v_1 \in G_{n+1}$, all the additions $\Psi_S(v_1)$ with $S \in \mathcal{B}$ are admissible by Proposition 4.1. So for all $S \in \mathcal{B}_0$, $\psi_{v_1}(\mathcal{T} + S)$ is contained in $\mathcal{T}$, because it is a subpiece of level $n+1$ of $\mathcal{T}$. This implies that $\mathrm{Int}(\psi_{v_1}(V_0)) \subset \mathrm{Int}(\mathcal{T})$. Thus $q_1 \subset \mathrm{Int}(\mathcal{T})$. Summing up we have constructed an arc $p_1 := \ell_1 \ell_2 q_1 \ell_3$ from x to x_1 which is contained in $\mathrm{Int}(\mathcal{T})$.

In the next step we construct an arc p_{i+1} from x_i to x_{i+1}, still inside $\mathrm{Int}(\mathcal{T})$. Because $\Psi_{S_i}(v_i) = v_{i+1}$ for some $S_i \in \mathcal{B}$, Lemma 8.2 implies the existence of a $z_1 \in \psi_{v_i}(\mathcal{T}) \cap \psi_{v_{i+1}}(\mathcal{T})$ which is contained in $\mathrm{Int}(\mathcal{T})$. Thus there exists an $\varepsilon_1 > 0$ with $B_{\varepsilon_1}(z_1) \subset \mathrm{Int}(\mathcal{T})$. Furthermore,

$$\begin{aligned} z_1 \in \psi_{v_i}(\mathcal{T}) &\implies \exists z_2 \in B_{\varepsilon_1}(z_1) \cap \mathrm{Int}(\psi_{v_i}(\mathcal{T})), \\ z_1 \in \psi_{v_{i+1}}(\mathcal{T}) &\implies \exists z_3 \in B_{\varepsilon_1}(z_1) \cap \mathrm{Int}(\psi_{v_{i+1}}(\mathcal{T})). \end{aligned}$$

Now connect z_2 with z_1 by the line segment ℓ_1 and connect z_1 with z_3 by the line segment ℓ_2. Both of these line segments are obviously contained in $\mathrm{Int}(\mathcal{T})$.

As above we can now construct using the n-skeletons an arc $q_1 \subset \mathrm{Int}(\mathcal{T})$ connecting x_i with z_2 and an arc $q_2 \subset \mathrm{Int}(\mathcal{T})$ connecting z_3 with x_{i+1}. The arc $p_{i+1} := q_1\ell_1\ell_2 q_2 \subset \mathrm{Int}(\mathcal{T})$ now connects x_i with x_{i+1}.

Setting $p := p_1 \dots p_m$ we have a path connecting x with y lying entirely in the interior of $\mathcal{T}$. □

Proposition 8.4. *Let x be a point of* $\mathrm{Int}(\mathbf{M})$. *Then there is an arc connecting x to* 0.

Proof. By Lemma 8.1 there is an $n \geq 3$ such that $x \in \mathcal{M}(G_n)$. By applying Proposition 8.3 $n-3$ times we can construct an arc inside $\mathrm{Int}(\mathcal{T})$ from x to some $y \in \mathrm{Int}(\mathcal{M}(G_3))$. Since $\mathcal{M}(G_3) = \psi_{(0,0,0)}(\mathcal{T}) = \psi_{(1,4,4)}(\mathcal{T})$ with $(F;1,4,4) = \Psi_{-P}((F;0,0,0))$ (see Remark 5.1), an arc q'' from y to $0 \in \psi_{(0,0,0)}(\mathcal{T})$ inside $\mathrm{Int}(\mathcal{T})$ can be constructed in the same way as the arcs p_i in the proof of Proposition 8.3, using the n-skeletons.

Now $q = q'q''$ does the job. □

We obtain directly from the above proposition the following result.

Corollary 8.5. *The set* $\mathrm{Int}(\mathbf{M})$ *is a subset of the interior component of $\mathcal{T}$ containing* 0.

The reverse inclusion is the purpose of the next proposition.

Proposition 8.6. *The component of* $\mathrm{Int}(\mathcal{T})$ *containing* 0 *is a subset of* $\mathrm{Int}(\mathbf{M})$.

Proof. Let y be a point in $\mathrm{Int}(\mathcal{T})$ such that there is an arc $p : [0,1] \to \mathrm{Int}(\mathcal{T})$ connecting 0 and y. Suppose that y does not belong to $\mathrm{Int}(\mathbf{M})$. Since $\partial\mathbf{M} \subset \partial\mathcal{T}$ (see Proposition 6.3), y does not belong to $\mathbf{M}$. Let

$$t_0 := \inf\{t \in [0,1] \,|\, p(t) \notin \mathbf{M}\}.$$

Then $t_0 \in (0,1)$ and $p(t_0) \in \partial\mathbf{M}$, because every neighborhood of this point encounters $\mathbf{M}$ as well as its complement. Since again $\partial\mathbf{M} \subset \partial\mathcal{T}$, we obtain that $p(t_0) \in \partial\mathcal{T}$, a contradiction to the definition of p. □

Lemma 8.7. *The set* $\mathbf{M}$ *is the closure of its interior,* i.e., $\overline{\mathrm{Int}(\mathbf{M})} = \mathbf{M}$.

Proof. Let $x \in \mathbf{M}$ and $\varepsilon > 0$ be arbitrary. Let $n \geq 3$ be large enough such that $\mathrm{diam}(\psi_w(\mathcal{T})) < \varepsilon$ for each $w \in p_n(F)$. There exists a walk $v \in p_n(F)$ such that $x \in \psi_v(\mathcal{T})$. It can easily be read off from the graph $\mathcal{G}$ that each $v \in p_n(F)$ can be extended to a walk $v' = v\&(b_1, b_2) \in G_{n+2} = p_{n+2}(F, \circ)$. Thus

$$\psi_{v'}(\mathcal{T}) \subset \mathbf{M} \Longrightarrow \mathrm{Int}(\psi_{v'}(\mathcal{T})) \subset \mathrm{Int}(\mathbf{M})$$

and

$$\psi_{v'}(\mathcal{T}) \subset \psi_v(\mathcal{T}).$$

Select $y \in \mathrm{Int}(\psi_{v'}(\mathcal{T}))$. Then the above inclusions imply that $y \in \mathrm{Int}(\mathbf{M})$ and $\mathrm{dist}(x,y) < \varepsilon$.

Since ε can be arbitrarily small we have that $x \in \overline{\mathrm{Int}(\mathbf{M})}$. □

Proof of Theorem 2.4. The first part of this theorem follows from Corollary 8.5 and Proposition 8.6. The second part is given by Lemma 8.7. □

9. Dimension calculations

The present section is devoted to the proof of Theorem 2.7. Let $\mathcal{G}'$ be the graph that emerges from $\mathcal{G}$ by removing the states $\circ$ and all edges leading to them. Then $\mathcal{G}'$ defines system of graph directed sets $(\delta\mathbf{M}(A))_A$ where A runs through the states of $\mathcal{G}'$. Let $\delta\mathbf{M} := \delta\mathbf{M}(F)$ be the set corresponding to the state F. The following lemma shows that $\delta\mathbf{M}$ is very close to ∂C_0.

Lemma 9.1. *The symmetric difference*

$$\delta\mathbf{M}\triangle\partial C_0$$

is countable.

Proof. First note that Lemma 8.1 implies that $\delta\mathbf{M}\subset\partial C_0$.

Suppose now that $x\in\partial C_0\setminus\delta\mathbf{M}$. Then the address of x corresponds to the labelling of a walk which is contained in $\mathcal{G}$ but not in $\mathcal{G}'$, *i.e.*, there exists a walk $w\in G_n$ for some n such that $x\in\psi_w(\mathcal{T})$ and, *a fortiori*, $x\in\partial\psi_w(\mathcal{T})$. Since $\partial C_0\subset\partial\mathcal{T}$ holds by Proposition 6.3, $x\in\partial\mathcal{T}$. Thus, x has to lie in another tile of the tiling induced by $\mathcal{T}$. However, in view of Proposition 4.1 and the remark after it,

$$x\in\psi_w(\mathcal{T}+S)$$

where S is a neighbor of $\mathcal{T}$ not contained in $\mathcal{B}$. It is well known (see for instance [1, Chapters 9 to 11]) that there exist only countably many points in $\partial\psi_w(\mathcal{T})$ with this property. Since, moreover, there are only countably many paths contained in

$$\bigcup_{n\geq 3} G_n$$

we conclude that there exist only countably many points x in the set $\partial C_0\setminus\delta\mathbf{M}$. □

Now from basic fractal geometry we get the following corollary.

Corollary 9.2.

$$\dim_H\partial C_0=\dim_H\delta\mathbf{M}.$$

Thus calculating the Hausdorff dimension of ∂C_0 is reduced to calculating the Hausdorff dimension of the GIFS attractor $\delta\mathbf{M}$. However, calculating the Hausdorff dimension of a self similar GIFS satisfying the open set condition can be performed by standard methods from fractal geometry (*cf.* for instance [8, 17]). With that Theorem 2.7 is proved.

10. Concluding remarks

We have shown in this paper how to obtain the closure $\mathbf{M}$ of the interior component of $\mathcal{T}$ that contains the point $\Phi(0)$. This component is depicted in Figure It is of natural interest to wonder how the closure of the interior component of $\mathcal{T}$ containing a given point x could be computed. For the other "big" components (see Figure 4), *i.e.*, containing $\Phi(1),\Phi(2)$ and $\Phi(3)$, the description is similar: it suffices to replace the edges at the top of $\mathcal{G}$ in a way that can be seen in Figure 5.

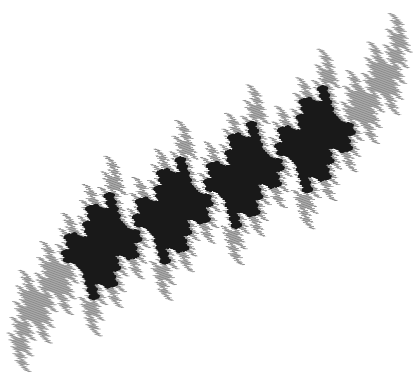

Figure 4. Tile associated to the base $-2+\sqrt{-1}$ with "big" interior components.

Thus the closure of these components are simply images of $\mathbf{M}$ by translations.

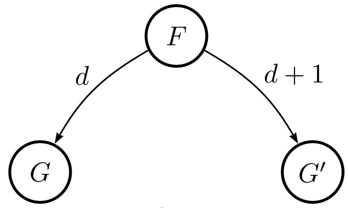

FIGURE 5. Top edges in $\mathcal{G}$ for the closure of the component containing $\Phi(d),\ d = 0, 1, 2, 3$.

For other "smaller components", we conjecture that there closure is also similar to **M**, the similitude may be given by a pre-graph that would be connected to $\mathcal{G}$ via the state F.

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REFERENCES

[1] S. AKIYAMA AND J. M. THUSWALDNER, *The topological structure of fractal tilings generated by quadratic number systems*, Computer and mathematics with applications, 49 (2005), pp. 1439–1485.
[2] S. BAILEY, T. KIM, AND R. STRICHARTZ, *Inside the lévy dragon*, Amer. Math. Monthly, 109 (2002), pp. 689–703.
[3] C. BANDT, *Self-similar sets 5*, Proc. Amer. Math. Soc., 112 (1991), pp. 549–562.
[4] M. F. BARNSLEY, J. H. ELTON, AND D. P. HARDIN, *Recurrent iterated function systems*, Constr. Approx., 5 (1989), pp. 3–31.
[5] H. BRUNOTTE, *On trinomial bases of radix representations of algebraic integers*, Acta Sci. Math. (Szeged), 67 (2001), pp. 521–527.
[6] H. BRUNOTTE, *Characterization of CNS trinomials*, Acta Sci. Math. (Szeged), 68 (2002), pp. 673–679.
[7] P. DUVALL, J. KEESLING, AND A. VINCE, *The Hausdorff dimension of the boundary of a self-similar tile*, J. London Math. Soc. (2), 61 (2000), pp. 748–760.
[8] K. J. FALCONER, *Techniques in Fractal Geometry*, John Wiley and Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1997.
[9] W. J. GILBERT, *Radix representations of quadratic fields*, J. Math. Anal. Appl., 83 (1981), pp. 264–274.
[10] W. J. GILBERT, *Complex bases and fractal similarity*, Ann. Sci. Math. Québec, 11 (1987), pp. 65–77.
[11] B. GRÜNBAUM AND G. C. SHEPHARD, *Tilings and Patterns*, W. H. Freeman and Company, New York, 1987.
[12] I. KÁTAI, *Number systems and fractal geometry*, University of Pécs, 1995.
[13] I. KÁTAI AND I. KŐRNYEI, *On number systems in algebraic number fields*, Publ. Math. Debrecen, 41 (1992), pp. 289–294.
[14] I. KÁTAI AND B. KOVÁCS, *Kanonische Zahlensysteme in der Theorie der Quadratischen Zahlen*, Acta Sci. Math. (Szeged), 42 (1980), pp. 99–107.
[15] ———, *Canonical number systems in imaginary quadratic fields*, Acta Math. Hungar., 37 (1981), pp. 159–164.
[16] J. LAGARIAS AND Y. WANG, *Self-affine tiles in* $\mathbb{R}^n$, Adv. Math., 121 (1996), pp. 21–49.
[17] R. D. MAULDIN AND S. C. WILLIAMS, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc., 309 (1988), pp. 811–829.
[18] W. MÜLLER, J. M. THUSWALDNER, AND R. F. TICHY, *Fractal properties of number systems*, Periodica Math. Hungar., 42 (2001), pp. 51–68.
[19] S.-M. NGAI AND N. NGUYEN, *The Heighway dragon revisited*, Discrete Comput. Geom., 29 (2003), pp. 603–623.
[20] S.-M. NGAI AND T.-M. TANG, *A technique in the topology of connected self-similar tiles*, Fractals, 12 (2004), pp. 389–403.
[21] ———, *Topology of connected self-similar tiles in the plane with disconnected interiors*, Topology Appl., 150 (2005), pp. 139–155.
[22] K. SCHEICHER, *Zifferndarstellungen, lineare Rekursionen und Automaten*, phd. thesis, Technische Universität Graz, Graz, 1997.
[23] K. SCHEICHER AND J. M. THUSWALDNER, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc., 133 (2002), pp. 163–182.
[24] J. M. THUSWALDNER, *Elementary properties of the sum of digits function in quadratic number fields*, in Applications of Fibonacci Numbers, G. E. B. et. al., ed., vol. 7, Kluwer Academic Publisher, 1998, pp. 405–414.
[25] A. VINCE, *Digit tiling of euclidean space*, in Directions in Mathematical Quasicrystals, Providence, RI, 2000, Amer. Math. Soc., pp. 329–370.
[26] Y. WANG, *Self-affine tiles*, in Advances in Wavelet, K. S. Lau, ed., Springer, 1998, pp. 261–285.

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