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A NEW CRITERION FOR DISK-LIKE CRYSTALLOGRAPHIC REPTILES

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Abstract. Let Γ be a planar crystallographic group and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be an expanding affine mapping satisfying $g\Gamma g^{-1} \subset \Gamma$. Then choose a complete set of right coset representatives $\delta_1, \dots, \delta_n$ of $\Gamma/g\Gamma g^{-1}$. We call the attractor T of the iterated function system

$$g(T) = \delta_1(T) \cup \ldots \cup \delta_n(T)$$

a *crystallographic reptile* if the collection

$$\{\gamma(T) \ : \ \gamma \in \Gamma\}$$

tiles the plane. In the present paper we give a criterion for the set T to be homeomorphic to a closed disk.

Using results from a previous paper by the same authors, we give applications of our criterion to examples of crystallographic tiles. The special case $\Gamma = \mathbb{Z}^2$ of our criterion provides a new proof of Bandt and Wang's result on self-affine lattice tiles that are homeomorphic to a closed disk.

1. Introduction

Topological properties of self-affine tiles have been of interest, especially when T is related to problems from other mathematical

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fields (*cf.* for instance [1, 4, 9, 13, 15, 17, 18]). We refer to Wang's survey [19] for fundamentals of self-affine tiles, and to [1] for results on topological properties of tiles related to number systems.

This paper is devoted to the study of self-affine sets that tile the plane by the action of a *crystallographic group.* (Especially we will emphasize on the planar case where there exist 17 different planar crystallographic groups also called *wallpaper groups*; see Grünbaum and Shephard [8, p. 40ff] for a definition of crystallographic groups.) Such *crystallographic reptiles* have been defined by Gelbrich [7], who studied some fundamental properties of these objects. In Loridant *et al.* [12] the authors of the present paper started the systematic study of topological properties of crystallographic reptiles. The topological structure of a tile in a tiling strongly depends on the relation of this tile to its neighbors. Tools are given in [12] in order to determine the set of neighbors of a crystallographic reptile T and to characterize particular kinds of neighbors of T, like vertex neighbors, which meet T at only one point, or adjacent neighbors, which have an edge in common with T. In the present paper we will prove a criterion for a crystallographic tile to be homeomorphic to a closed disk, or *disk-like* for short. This criterion depends on the structure of the set of neighbors of a tile in the tiling.

We arrange this paper as follows. In Section 2 we will state the main result and recall some known disk-like results obtained earlier (*cf.* for instance [4, 7, 14]). Section 3 provides a complete proof of the main theorem. Section 4 gives a new proof for the Bandt and Wang Theorem (see [4]), which concerned the case of lattice reptiles. After that we add some new examples related to crystallographic reptiles and discuss a few open questions.

2. Basic definitions and statement of the main result

A compact set T in $\mathbb{R}^n$ that equals the closure of its interior $\mathrm{int}(T)$, $T = \overline{\mathrm{int(T)}}$, is said to induce a *tiling* if there exists a countable collection

$$\mathcal{T} := \{\gamma(T) : \gamma \in \Gamma\},$$

where Γ is a set of isometries of $\mathbb{R}^n$, such that $\mathcal{T}$ is a cover of $\mathbb{R}^n$ and every two elements of $\mathcal{T}$ do not have common interior points. We also say that T *tiles* $\mathbb{R}^n$ and that the collection $\mathcal{T}$ is a *tiling* *of* $\mathbb{R}^n$, using the single set T. Without loss of generality we may assume that Γ contains id, the identity map of $\mathbb{R}^n$, and $\mathrm{id}(T) = T$ is called the *central tile* of the tiling.

In a tiling $\mathcal{T}$ using a single tile T, two distinct sets $\gamma_1(T), \gamma_2(T) \in \mathcal{T}$ (or the corresponding isometries, if the tiling is fixed) are called *neighbors* if they intersect each other, and they are called *adjacent* or *edge neighbors* if the interior of $\gamma_1(T) \cup \gamma_2(T)$ contains a point of $\gamma_1(T) \cap \gamma_2(T)$. Moreover, let

$$\mathcal{S} := \{\gamma \in \Gamma \setminus \{\mathrm{id}\} \ : \ T \cap \gamma(T) \neq \emptyset\}$$

denote the set of neighbors of id, and

$$\mathcal{A} := \{\gamma \in \Gamma \setminus \{\mathrm{id}\} \ : \ T \text{ and } \gamma(T) \text{ are adjacent}\}$$

the subset of $\mathcal{S}$ comprising the adjacent neighbors of id. In the tilings considered in this paper, the set of neighbors and the set of adjacent neighbors of a tile $\gamma(T)$ ($\gamma \in \Gamma$) are then $\gamma\mathcal{S}$ and $\gamma\mathcal{A}$ respectively.

If Γ is a crystallographic group, we say that $\mathcal{T}$ is a *crystallographic tiling*. Recall that by a theorem of Bieberbach (*cf.* for instance [5]), a crystallographic group in dimension n always contains a maximal abelian subgroup Λ called *lattice* such that the *point group* Γ/Λ is finite. The main object of our interest is defined as follows.

Definition 2.1. A *crystallographic reptile* (or simply *crystile*) with respect to a crystallographic group Γ is a compact non-empty set $T \subset \mathbb{R}^n$ with the following properties:

- The family $\{\gamma(T) : \gamma \in \Gamma\}$ is a tiling of $\mathbb{R}^n$.
- There is an expanding affine map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$, and there exists a complete set of right coset representatives $\mathcal{D} \subset \Gamma$ of $\Gamma/g\Gamma g^{-1}$, called *digit set*, such that
$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

We refer the reader to [7] and [12] for further information about crystallographic reptiles.

Before we can formulate our main result we recall some notions and definitions related to graphs. A *graph* is a pair $G = (V, E)$ of sets such that $E \subset V \times V$, with V and E finite or infinite; thus, the elements of E are 2-element subsets of V. The elements of V are called *vertices* (or *points*) of the graph G, the elements of E are called *edges* (or *lines*). Two vertices are *incident* if they constitute an edge. The two vertices incident with an edge are its *end-vertices* or *ends*, and an edge *joins* its ends. An edge $\{x, y\}$ is usually written as xy (or yx). Eventually, if $V' \subset V$, the *subgraph of* G *induced by* V' is the graph $G' = (V', E \cap V' \times V')$.

The usual way to draw a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. If we can "draw" a graph G on the plane in such a way that no two edges meet in a point other than a common end, G is *planar* and the drawing is a *plane graph*. Here, we give definitions for *drawing* and *plane graph* which are a bit different from the ones given in [6].

Definition 2.2. Let $G = (V, E)$ be a planar graph with set of vertices V and set of edges E. Then a *drawing* of G is a mapping $\pi : (V, E) \to \mathbb{R}^2$ such that $\pi(V)$ is a discrete set of the plane, $\pi(xy)$ is a simple arc joining $\pi(x)$ and $\pi(y)$, and

$$\pi(xy) \cap \pi(uv) = \{\pi(x), \pi(y)\} \cap \{\pi(u), \pi(v)\} \quad (xy, uv \in E \text{ disjoint}).$$

We also say that $\pi(G)$ is a drawing of G. For every planar graph G with a drawing π, the set $\mathbb{R}^2 \setminus \pi(G)$ is an open set; its components are the *faces* of $\pi(G)$ (or G).

Given a planar graph $G = (V, E)$ and a drawing π of G, let us consider the *derived graph of* G, *i.e.*, the graph $G_1 = (V, E_1)$ emerging from G with the same set of vertices and where two vertices x, y are incident if their images $\pi(x), \pi(y)$ belong to the closure of the same face of G. E_1 contains E, and we extend π to a map π_1 on E_1 by joining the images of vertices corresponding to a new edge by a simple open arc inside one of their common faces. Such an *extension* π_1 is not unique. Also note that it need not be a drawing in the above sense because the derived graph of a planar graph is not necessarily planar.

For a crystallographic tiling $\mathcal{T} := \{\gamma(T) : \gamma \in \Gamma\}$ of the plane which uses a single crystile T, define the *adjacency graph* G_A as the graph with vertex set Γ for which a 2-element set $\{\gamma_1, \gamma_2\} \subset \Gamma$ is an edge whenever the two tiles $\gamma_1(T), \gamma_2(T)$ are adjacent. If we define a 2-element subset $\{\gamma_1, \gamma_2\}$ of Γ to be an edge if $\gamma_1(T), \gamma_2(T)$ are neighbors, we have the *neighbor graph* G_N with vertex set Γ.

We need special drawings of the adjacency graph G_A. In the following definition as well as in the rest of the paper we use

$$B_r(x) := \{y \in \mathbb{R}^2 \ : \ |y - x| \leq r\}$$

to denote a closed disk with radius r centered at x.

Definition 2.3. Let $G_A = (\Gamma, E)$ be the adjacency graph of a crystallographic tiling. Assume that G_A is planar. We say that a drawing π of G_A is *admissible* if there is a $p \in \mathbb{R}^2$ with $\gamma_1(p) \neq \gamma_2(p)$ for all $\gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$ such that:

- $\pi(\gamma) = \gamma(p) \quad (\gamma \in \Gamma)$.
- There is a constant $c \in \mathbb{R}$ such that for all $e \in E$ joining the vertices x and y, we have:

$$\pi(e) \subset B_c(x) \cap B_c(y). \tag{2.1}$$

Moreover, let π_1 an extension of π as defined above. We call π_1 admissible if it satisfies (2.1) for all $e \in E_1$ and the same constant c.

Theorem 2.4. *Assume that $T \subset \mathbb{R}^2$ is a planar crystallographic reptile with respect to a crystallographic group Γ. Then T is disk-like, if and only if the following three conditions all hold:*

(i) *The adjacency graph G_A is a connected planar graph.*
(ii) *The digit set $\mathcal{D}$ induces a connected subgraph in G_A.*
(iii) *G_A has an admissible drawing $\pi : G_A \to \mathbb{R}^2$ such that the derived graph of G_A is exactly the neighbor graph G_N.*

Remark 2.5. Condition (iii) says that two tiles $\gamma_1(T), \gamma_2(T)$ are neighbors if and only if the vertices $\pi(\gamma_1), \pi(\gamma_2)$ lie on the boundary of a single face of the drawing $\pi(G_A)$.

This is not the first result on disk-like fractal sets. Luo *et al.* [14] proved that if an attractor of an iterated function system has connected interior then it is disk-like. The proof of this result is essentially based on Torhorst's Theorem [11, §61, II, Theorem 4]. Bandt and Wang [4] use this criterion in order to show that a lattice reptile is disk-like if it has 6 or 8 neighbors and a digit set that fulfills certain properties (see Section 4 for details). Luo and Zhou [15] show that a class of lattice tiles is disk-like by proving that its boundary is a simple closed curve. In Akiyama and Thuswaldner [2] a class of lattice tiles related to number systems in quadratic number fields is shown to be disk-like. In this paper also the above mentioned result [14] was used. The study of disk-like crystiles was started in [12], where some examples of crystiles presented in Gelbrich's paper [7] were shown to be disk-like.

3. Proof of the main theorem

We recall the following definitions and facts from planar topology which will be used in the proof.

Definition 3.1 ([11, §46, VII])**.** A set C is said to *separate the plane between A and B* if there exist two sets M, N such that

$$\mathbb{R}^2 \setminus C = N \cup M, \quad \left(\overline{M} \cap N\right) \cup \left(\overline{N} \cap M\right) = \emptyset, \quad A \subset M, B \subset N.$$

Definition 3.2 ([11, §54, IV])**.** Let $X \subset Y$ and $f : X \to Y$ be a continuous function. If there exists a continuous function $h : X \times [0,1] \to Y$ such that $h(x,0) = x$ and $h(x,1) = f(x)$, the set $f(X)$ is said to be obtained from X by a *deformation* in Y.

Lemma 3.3 ([11, §59, IV, Theorem 2])**.** *If a compact set $F \subset \mathbb{R}^2$ separates between two points p and q, then every set obtained from F by a deformation in $\mathbb{R}^2 \setminus \{p,q\}$ separates between p and q.*

In a topological space, a *continuum* is a connected compact set, and a set is *locally connected* if for each of its points, every open neighborhood contains a connected neighborhood.

Lemma 3.4 ([11, §61, II, Theorem 5])**.** *Every locally connected continuum which separates the plane between two continua A and B contains a simple closed curve which separates the plane between A and B.*

Hereafter in this section, we assume that $T \subset \mathbb{R}^2$ is a planar crystallographic reptile with respect to a crystallographic group Γ, an expanding affine map g and a digit set $\mathcal{D}$.

Proof of Theorem 2.4. We split the proof in two parts.

Sufficiency. [1] Assume that conditions (i), (ii) and (iii) of Theorem 2.4 hold. In view of the result of [14, Theorem 1.1] mentioned

[1]Some parts of this proof are inspired by the proof of the Bandt and Wang Theorem in [4].

at the end of the preceding section, it suffices to show that the interior of T is connected. We fix an admissible drawing π for G_A, which gives an associated constant c and a point p defined in Definition 2.3 (w.l.o.g., $p \in T$). G_N is by assumption the derived graph of G_A and we call π_1 an admissible extension of the drawing π.

For each $k \in \mathbb{N}$, we define the set $\mathcal{D}^k$ of elements in Γ such that

$$g^k(T) = \bigcup_{\gamma \in \mathcal{D}^k} \gamma(T).$$

Thus we have $\mathcal{D}^1 = \mathcal{D}$, and using condition (ii), it can be shown recursively that the subgraph G_A^k of G_A induced by $\mathcal{D}^k$ is connected for every $k \in \mathbb{N}$.

Our aim is to find a curve in $\pi(G_A^k)$ intersecting a curve in $\pi_1(G_N \setminus G_A^k)$ and to derive a contradiction ($G_N \setminus G_A^k$ is the subgraph of G_N induced by the set of vertices $\Gamma \setminus \mathcal{D}^k$).

We denote by D the diameter of the tiles (*i.e.*, the maximal distance between two points of a tile $\gamma(T)$) and by L the minimal distance between two disjoint tiles. We set $M := \max\{D, L, c\}$.

Suppose that $\mathrm{int}(T)$ is disconnected, and let z_1 and z_2 be two points in different components of $\mathrm{int}(T)$. Let $k \in \mathbb{N}$ be large enough such that $B_{6M}(g^k(z_i)) \subset g^k(\mathrm{int}(T))$ ($i \in \{1, 2\}$). For $i = 1, 2$, we denote by γ^i an element of $\mathcal{D}^k$ such that the tile $\gamma^i(T)$ contains $g^k(z_i)$ and by Ω_i the component of $g^k(\mathrm{int}(T))$ containing $g^k(z_i)$. In the following, A_p ($1 \le p \le 6$) will stand for the unbounded connected region $\mathbb{R}^2 \setminus \big(B_{pM}(g^k(z_1)) \cup B_{pM}(g^k(z_2))\big)$.

Then the boundary $\partial\Omega_1$ of the component Ω_1 is contained in A_6 and separates between $\mathcal{B}_1 := B_{2M}(g^k(z_1))$ and $\mathcal{B}_2 := B_{2M}(g^k(z_2))$. Consider the the finite collection

$$\mathcal{U} := \{\gamma \in \Gamma \setminus \mathcal{D}^k \ : \ \gamma(T) \cap \partial\Omega_1 \neq \emptyset\}.$$

This definition implies that $\bigcup_{\gamma \in \mathcal{U}} \gamma(T)$ is contained in A_5 and separates between $\mathcal{B}_1$ and $\mathcal{B}_2$. By Lemma 3.4, there is a simple closed curve C in $\bigcup_{\gamma \in \mathcal{U}} \gamma(T)$ which separates between $\mathcal{B}_1$ and $\mathcal{B}_2$ too. We suppose that $\mathcal{B}_1$ lies in the bounded component of $\mathbb{R}^2 \setminus C$.

We denote by $(C(t), t \in [0, 1])$ a parametrization of C with $C(0) = C(1)$.

As C is uniformly continuous, we may find a constant $\delta > 0$ such that $d(C(t), C(t')) < L$ as soon as $|t - t'| < \delta$. Let $m \ge 2$ and $(t_j)_{0 \le j \le m}$ be a subdivision of $[0, 1]$ with $0 = t_0 < t_1 < \ldots <$

$t_{m-1} < t_m = 1$ and $t_{j+1} - t_j < \delta$. Then, setting $C_j := C(t_j)$, we have $d(C_j, C_{j+1}) < L$ for all $0 \le j \le m-1$. Choose now for each $j \in \{0, \dots, m\}$ an element $\alpha_j \in \mathcal{U}$ with $\alpha_0 = \alpha_m$ and such that $C_j \in \alpha_j(T)$. Then for all $0 \le j \le m-1$, $d(\alpha_j(T), \alpha_{j+1}(T)) < L$. Thus, by the definition of L, two consecutive tiles are neighbors, *i.e.*, $\alpha_j^{-1}\alpha_{j+1} \in \mathcal{S} \cup \{\mathrm{id}\}$.

We now construct a closed curve that is homotopic to C in A_2 and is made by pieces $\pi(e)$ for some edges $e \in G_N \setminus G_A^k$. By Lemma 3.3, this closed curve will also separate between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Let $j \in \{0, \dots, m-1\}$. Consider the union R_j of the intersecting balls $B_{2M}(\alpha_j(p))$ and $B_{2M}(\alpha_{j+1}(p))$. Then $R_j \subset A_2$. Moreover, the line segments $\overline{\alpha_j(p)C_j}$ and $\overline{C_{j+1}\alpha_{j+1}(p)}$, the arc

$$E_j := \pi_1(\alpha_j(p)\alpha_{j+1}(p))$$

as well as the piece

$$C^j := \{C(t) \ : \ t \in [t_j, t_{j+1}]\}$$

of the curve C are all contained in the simply connected set R_j. Thus the arc E_j can be obtained from the union of $\overline{\alpha_j(p)C_j}$, C^j and $\overline{C_{j+1}\alpha_{j+1}(p)}$ by a deformation in R_j.

Consequently, the union

$$\mathcal{E} := \bigcup_{0 \le j \le m-1} E_j$$

of the arcs is obtained from the union

$$\mathcal{F} := \bigcup_{0 \le j \le m-1} \left(\overline{\alpha_j(p)C_j} \cup C^j \cup \overline{C_{j+1}\alpha_{j+1}(p)} \right)$$

by a deformation in $\bigcup_{0 \le j \le m-1} R_j \subset A_2$.

Note that $\mathcal{F}$ separates between $\mathcal{B}_1$ and $\mathcal{B}_2$. By Lemma 3.3, so does $\mathcal{E}$. Thus every curve from $\gamma^1(p) \in \mathcal{B}_1$ to $\gamma^2(p) \in \mathcal{B}_2$ intersects $\mathcal{E}$. Since the subgraph G_A^k of G_A induced by $\mathcal{D}^k$ is connected and $\gamma^1, \gamma^2 \in \mathcal{D}_k$, there is a connected path $\gamma_1 := \gamma^1, \gamma_2, \dots, \gamma_{q-1}, \gamma_q := \gamma^2$ in G_A with $\gamma_i \in \mathcal{D}^k, i = 1, \dots, q$. The image by π of this path is a curve in $\pi(G_A^k)$ joining $\gamma^1(p)$ and $\gamma^2(p)$. It is intersected by $\mathcal{E}$, which is a closed curve in $\pi_1(G_N \setminus G_A^k)$. Thus, an arc $\pi_1(\alpha_j(p)\alpha_{j+1}(p))$ (maybe degenerated in the sense that $\alpha_i = \alpha_{j+1}$) must intersect an arc $\pi(\gamma_i(p)\gamma_{i+1}(p))$. But by the assumption on the drawing, either

$\pi_1(\alpha_j(p)\alpha_{j+1}(p))$ is in $\pi(G_A)$ or

$$\pi_1(\alpha_j(p)\alpha_{j+1}(p)) \setminus \{\alpha_j(p), \alpha_{j+1}(p)\}$$

is all contained in a face of the drawing. In both cases, these arcs must share a common end point to intersect, contradicting the disjointness of $\{\alpha_0, \ldots, \alpha_m\}$ and $\{\gamma_1, \ldots, \gamma_q\}$.

Necessity. Assume that T is disk-like. We have to check conditions (i), (ii) and (iii) of Theorem 2.4. Condition (ii) can be directly inferred from the disk-likeness of T. Thus we have to show that conditions (i) and (iii) are also satisfied.

As $\gamma(T) \cap \gamma'(T)$ is either empty or a connected set for all distinct elements $\gamma, \gamma' \in \Gamma$ (*cf.* [7, 12]), the boundary ∂T consists of arcs $P_1, \ldots, P_k$, where $3 \le k \le 6$ (*cf.* [7, p. 128]) and $P_j = T \cap \gamma_j(T)$. Here $\gamma_1, \ldots, \gamma_k$ are the edge neighbors of id. The arcs $P_1, \ldots, P_k$ can be arranged to a *circular chain* in the sense that for all distinct $i, j \in \{1, 2, \ldots, k\}$ the intersection $P_i \cap P_j$ is a singleton if $j \equiv i + 1 (\operatorname{mod} k)$, and is an empty set otherwise.

Choose an interior point x_1 of P_1 (in subspace topology), and let $\operatorname{Orb}(x_1) = \{\gamma(x_1) : \gamma \in \Gamma\}$ be the Orbit of x_1 under the transformation group Γ. Clearly, the number of points in $\operatorname{Orb}(x_1) \cap \partial T$ is between 1 and k. If it is not k, we can choose a least integer i_1 such that $\operatorname{Orb}(x_1) \cap P_{i_1} = \emptyset$. Let x_2 be an interior point of P_{i_1}, then $\operatorname{Orb}(x_1) \cap \operatorname{Orb}(x_2) = \emptyset$ and the number of points in $(\operatorname{Orb}(x_1) \cup \operatorname{Orb}(x_2)) \cap \partial T$ is between 2 and k. Going on with this procedure for at most $k-1$ steps, we will find k' points $x_1, \ldots, x_{k'}$ on ∂T, where $k' \le k$, such that the number of points in

$$\left(\bigcup_{i=1}^{k'} \operatorname{Orb}(x_i) \right) \cap \partial T$$

is exactly k. Rename the k points of the above intersection as $y_1, \ldots, y_k$ with $y_l \in P_l$.

By the Schönflies Theorem [16], choose a homeomorphism $h : T \to \{re^{\mathbf{i}t} : 0 \le t < 2\pi, 0 \le r \le 1\}$ such that $h(y_l) = e^{\frac{2\pi l}{k}\mathbf{i}}$. Let $R_l = \{re^{\frac{2\pi l}{k}\mathbf{i}} : 0 \le r \le 1\}$ be the radius joining the origin 0 and the point $h(y_l)$, for $1 \le l \le k$. Then $W = h^{-1}\left(\cup_{l=1}^{k} R_l\right)$ is a union of arcs in T which are disjoint except at their common endpoint $h^{-1}(0) \in \operatorname{int}(T)$.

Now, we can see that $\bigcup_{\gamma\in\Gamma}\gamma(W)$ is an admissible drawing of G_A and that condition (i) is satisfied. Clearly, each triple point of the tiling $\{\gamma(T) : \gamma \in \Gamma\}$ must be enclosed in a face of the above drawing. Since an arc $\gamma \circ h^{-1}(R_l)$ is contained in the boundary of a face containing a triple point x if and only if $x \in \gamma(P_l)$, and since two tiles $\gamma(T), \gamma'(T)$ are neighbors if and only if their intersection $\gamma(T) \cap \gamma'(T)$ contains a triple point x, we see that condition (iii) is satisfied. □

4. Applications, examples and further questions

First we show that Theorem 2.4 contains as a special case a theorem due to Bandt and Wang [4] yielding a criterion for a self-affine lattice reptile to be homeomorphic to a closed disk. Note that the lattice is the easiest of the planar crystallographic groups. It is the one having trivial point group.

Before we recall the formulation of this theorem, we need the following definition. For two sets $\mathcal{D}$ and $\mathcal{F}$ of isometries in $\mathbb{R}^2$, we say that $\mathcal{D}$ is *$\mathcal{F}$-connected* if for every disjoint pair (d, d') of elements in $\mathcal{D}$, there exist an $n \geq 1$ and elements $d =: d_0, d_1, \dots, d_{n-1}, d_n := d'$ of $\mathcal{D}$ such that $d_i^{-1} d_{i+1} \in \mathcal{F}$ for each $i \in \{0, \dots, n-1\}$.

Theorem 4.1 (Bandt-Wang Theorem). *Let T be a self-affine lattice reptile with digit set $\mathcal{D}$.*

- (i) *Suppose that the neighbor set $\mathcal{S}$ of T has not more than six elements. Then T is disk-like if and only if the digit set $\mathcal{D}$ is $\mathcal{S}$-connected.*
- (ii) *Suppose that the neighbor set $\mathcal{S}$ of T has eight elements $\{a^{\pm 1}, b^{\pm 1}, (ab)^{\pm 1}, (ab^{-1})^{\pm 1}\}$. Then T is disk-like if and only if $\{a^{\pm 1}, b^{\pm 1}\}$ are the only adjacent neighbors and the digit set $\mathcal{D}$ is $\{a^{\pm 1}, b^{\pm 1}\}$-connected.*

Proof. We prove this theorem by showing that if a lattice tile T satisfies the conditions of this theorem, it must satisfy the conditions of Theorem 2.4 too.

- (i) By [4, Theorem 3.2] we can assume without loss of generality that the lattice Γ has two elements $a, b \in \Gamma$, such that the tile T has exactly 6 neighbors

$$\{a, a^{-1}, b, b^{-1}, ab, a^{-1}b^{-1}\}.$$

In a similar way as in [12, Proposition 5.4] it can be shown that these neighbors are all adjacent ones. Then the adjacency graph G_A has a drawing as in Figure 1. It is identical with the neighbor graph G_N. In this case, it is clear that T satisfies conditions (i) and (iii) in Theorem 2.4. Since condition (ii) is satisfied by assumption, we are done.

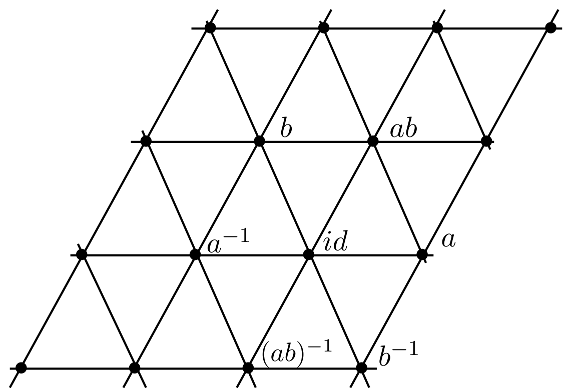

FIGURE 1. The graph G_A for a lattice tile with six edge neighbors. In this case, G_A is equal to its derived graph.

(ii) Suppose that the tile T has exactly 8 neighbors

$$\{a^{\pm 1}, b^{\pm 1}, ab, a^{-1}b^{-1}, a^{-1}b, ab^{-1}\}.$$

It follows that exactly the neighbors $a^{\pm 1}, b^{\pm 1}$ are the adjacent ones. Then, the adjacency graph has a drawing as in Figure 2. In this case, T satisfies all the conditions in Theorem 2.4 if the digit set $\mathcal{D}$ is $\{a^{\pm 1}, b^{\pm 1}\}-$connected. The neighbor graph is exactly the derived graph of the drawing of G_A (see Figure 2).

□

We will now apply the new criterion to $p2$-reptiles.

Example 4.2. Assume that T is a connected reptile with respect to a $p2$-group Γ, with digit set $\mathcal{D} \supset \{\text{id}\}$.

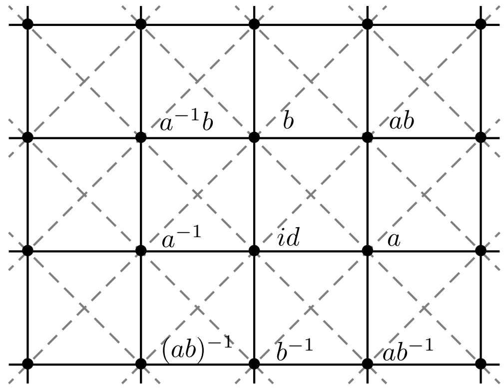

FIGURE 2. The graph G_A for a lattice tile with four adjacent neighbors (full edges) and its derived graph (full and dashed edges).

- Suppose that there exist two translations $a, b \in \Gamma$ and a π-rotation $c \in \Gamma$ with

$$\Gamma = \{a^i b^j c^k : i, j \in \mathbb{Z}, k \in \{0, 1\}\},$$

such that the tile T has exactly 6 neighbors of the following shape.

$$\mathcal{A} := \{b, b^{-1}, c, a^{-1}c, bc, a^{-1}bc\} \quad \text{or} \tag{4.1}$$

$$\mathcal{A} := \{b, b^{-1}, c, a^{-1}c, bc, a^{-1}b^{-1}c\}. \tag{4.2}$$

Then, each neighbor is an adjacent one by [12, Proposition 5.4]. Thus the adjacency graph has a drawing as in Figure 3 for $\mathcal{A}$ as in (4.1). The neighbor graph is equal to the adjacency graph. In this case, it is clear that T satisfies conditions (i) and (iii) in Theorem 2.4. Thus T is disk-like if and only if the digit set is $\mathcal{A}$-connected. The case where $\mathcal{A}$ is as in (4.2) can be treated likewise.
- We give here a concrete example of $p2$-crystile with seven neighbors. It corresponds to a disk-like candidate of Gelbrich (see [7, p. 252, (c)]). Let

$$\begin{aligned} a(x, y) &= (x + 1, y), \\ b(x, y) &= (x, y + 1), \\ c(x, y) &= (-x, -y). \end{aligned}$$

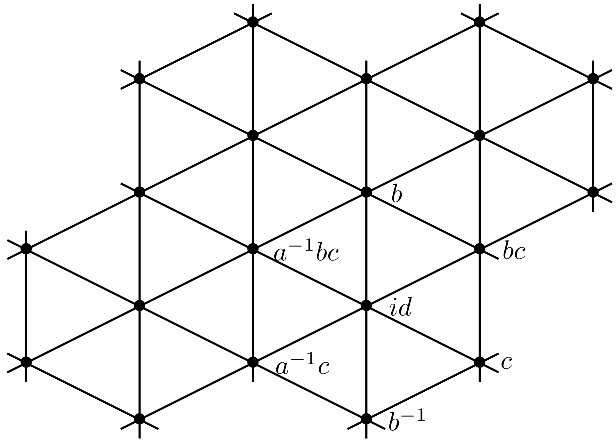

FIGURE 3. The graph G_A for a p2 crystile with six edge neighbors. In this case, G_A is equal to its derived graph.

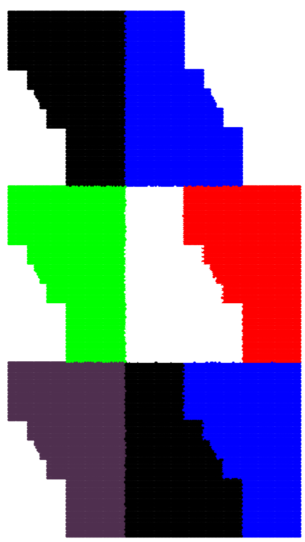

FIGURE 4. $p2$-crystile with seven neighbors, digit set $\{\mathrm{id}, b, c\}$.

Then the $p2$ group Γ can be written as

$$\Gamma = \{a^i b^j c^k : i, j \in \mathbb{Z}, k \in \{0, 1\}\}.$$

The expanding map g is chosen as $g(x, y) = (y, 3x + 1)$, the digit set as $\mathcal{D} = \{\mathrm{id}, b, c\}$, thus the tile T is defined by

$$g(T) = T \cup b(T) \cup c(T).$$

It is depicted in Figure 4.

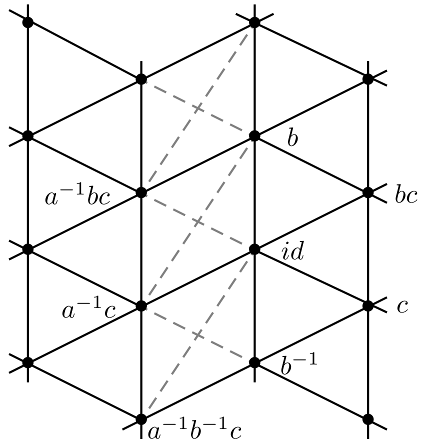

FIGURE 5. The graph G_A for a p2 crystile with five edge neighbors (full edges) and its derived graph (full and dashed edges).

It can be shown with the tools developed in [12] that T has exactly 7 neighbors

$$\mathcal{S} = \{b, b^{-1}, c, b^{-1}c, a^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\},$$

and that its adjacent neighbors are

$$\mathcal{A} = \{b, b^{-1}, c, b^{-1}c, a^{-1}c\}.$$

Thus the adjacency graph has a drawing as in Figure 5. Its neighbor graph G_N is exactly the derived graph of the drawing of G_A (see Figure 5 also). In this case, T satisfies all the conditions in Theorem 2.4 because the digit set $\mathcal{D}$ is $\mathcal{A}$-connected. Hence it is disk-like.

Example 4.3. This example is devoted to a $p3$-crystile with ten neighbors which is called "terdragon". It also occurs in Gelbrich's paper (see [7, p. 255]).

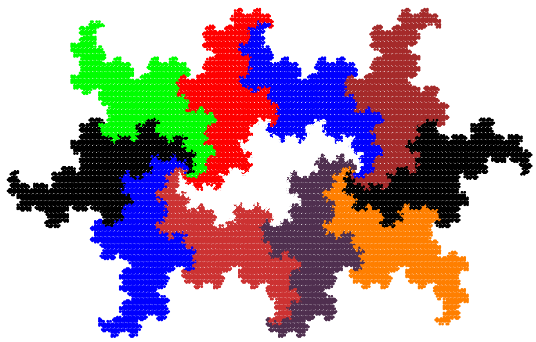

FIGURE 6. The "terdragon", a $p3$-crystile with ten neighbors, digit set $\{\mathrm{id}, ac^2, bc^2\}$.

Let

$$\begin{aligned} a(x,y) &= (x+1,y), \\ b(x,y) &= (x+1/2, y+\sqrt{3}/2), \\ c(x,y) &= (-x-\sqrt{3}y)/2, (\sqrt{3}x-y)/2. \end{aligned}$$

The crystallographic group p3 is then generated by a, b and c, *i.e.*,

$$\Gamma = \{a^i b^j c^k : i,j \in \mathbb{Z}, k \in \{0,1,2\}\}.$$

The expanding map g is chosen as $g(x,y) = \sqrt{3}(y,-x)$, the digit set as $\mathcal{D} = \{\mathrm{id}, ac^2, bc^2\}$, thus the tile T is defined by

$$g(T) = T \cup ac^2(T) \cup bc^2(T).$$

It is depicted in Figure 6. Using the tools developed in [12] it can be shown that T has exactly the 10 neighbors

$$\mathcal{S} = \{a, a^{-1}, c, c^2, ac, ac^2, bc^2, ab^{-1}c, abc^2, a^2b^{-1}c\}$$

four of which are adjacent. These are given by

$$\mathcal{A} = \{ac, ac^2, bc^2, ab^{-1}c\}.$$

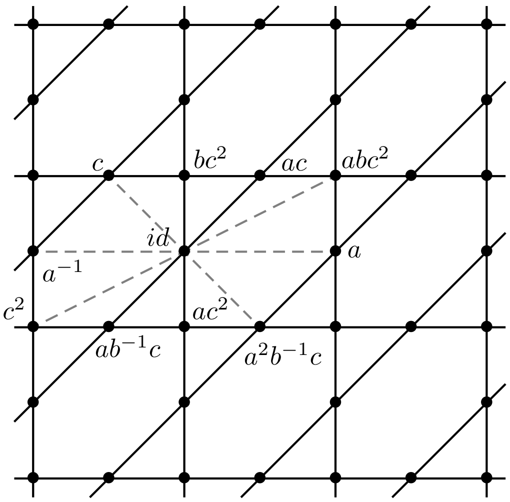

FIGURE 7. The graph G_A for a p3 crystile with four edge neighbors. The additional edges connected to *id* in the derived graph are also represented (dashed).

Thus the adjacency graph has a drawing as in Figure 7. The neighbor graph G_N is exactly the derived graph of the drawing of G_A in Figure 7. In this case, T satisfies all the conditions in Theorem 2.4 because the digit set $\mathcal{D}$ is $\mathcal{A}$-connected. Hence it is disk-like.

We will provide a few open questions which in some sense extend the idea of the Bandt-Wang Theorem. Before stating these questions, we recall some concepts from tiling theory. For further details, we refer to [8].

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^n$. A *symmetry* of $\mathcal{T}$ is an isometry which maps each element of $\mathcal{T}$ onto another element of $\mathcal{T}$. The collection of all the symmetries of a tiling $\mathcal{T}$ is called the *symmetry group* of $\mathcal{T}$, denoted by $S(\mathcal{T})$.

If $\mathcal{T}$ is a tiling of $\mathbb{R}^2$ and T is disk-like, the symmetry group has only 24 possibilities [8], 17 of which are the planar crystallographic groups.

It is known (*cf.* [7]) that a crystallographic tiling using a single disk-like prototile is *normal* in the sense that

(N.1) each tile of the tiling is disk-like,

(N.2) the intersection of two tiles is either empty or a connected set, and

(N.3) there exists two positive numbers R and r such that each tile is covered by a closed disk with radius R and contains a disk with radius r.

Assume that T is a disk-like reptile with respect to a 2-dimensional crystallographic group Γ. It can be shown that the corresponding tiling $\mathcal{T} = \{\gamma(T) : \gamma \in \Gamma\}$ is a normal tiling. From the whole list of *valence-types* (*i.e.*, types of vertex constellations, see [8, p. 176] for an exact definition) given in [8, Theorem 4.3.1], one can easily check all the possible number of neighbors for the tile T. In particular, T has at least 6 neighbors.

Question 4.4. *Let T be a connected crystile with respect to the group Γ. Suppose that each tile of the tiling*

$$\mathcal{T} = \{\gamma(T) : \gamma \in \Gamma\}$$

has exactly 6 neighbors. Does this imply that T is disk-like?

Essentially the Bandt-Wang Theorem says that if T has a "small" number of neighbors, and if the neighbor set is of a particularly "good" form, then T is disk-like. Therefore, for a connected crystile T which has 6 or more neighbors, it is of natural interest to find out the "good" forms of neighbor sets so that T is disk-like. All in all, we pose the following question.

Question 4.5. *Let Γ be an arbitrary planar crystallographic group. Can one give a complete list of all the forms of neighbor sets and adjacency graphs so that the crystile T (w.r.t. Γ) is disk-like?*

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