NORMALITY OF NUMBERS GENERATED BY THE VALUES OF ENTIRE FUNCTIONS

MANFRED G. MADRITSCH, JÖRG M. THUSWALDNER, AND ROBERT F. TICHY

Abstract. We show that the number generated by the q-ary integer part of an entire function of logarithmic order, where the function is evaluated over the natural numbers and the primes, respectively, is normal in base q. This is an extension of related results for polynomials over the real numbers established by Nakai and Shiokawa.

1. Introduction

Let $q \geq 2$ be a fixed integer and $\theta = 0.a_1a_2\dots$ be the q-ary expansion of a real number θ with $0 < \theta < 1$. We write $d_1 \dots d_l \in \{0, 1, \dots, q-1\}^l$ for a block of l digits in the q-ary expansion. By $\mathcal{N}(\theta; d_1 \dots d_l; N)$ we denote the number of occurrences of the block $d_1 \dots d_l$ in the first N digits of the q-ary expansion of θ. We call θ *normal to the base* q if for every fixed $l \geq 1$

$$\mathcal{R}_N(\theta) = \mathcal{R}_{N,l}(\theta) = \sup_{d_1\dots d_l} \left| \frac{1}{N}\mathcal{N}(\theta; d_1 \dots d_l; N) - \frac{1}{q^l} \right| = o(1)$$

as $N \to \infty$, where the supremum is taken over all blocks $d_1 \dots d_l \in \{0, 1, \dots, q-1\}^l$.
We want to look at numbers whose digits are generated by the integer part of entire functions. Let f be any function and $[f(n)]_q$ denote the base q expansion of the integer part of $f(n)$, then define

$$\begin{aligned} \theta_q &= \theta_q(f) = 0.[f(1)]_q[f(2)]_q[f(3)]_q[f(4)]_q[f(5)]_q[f(6)]_q \dots, \\ \tau_q &= \tau_q(f) = 0.[f(2)]_q[f(3)]_q[f(5)]_q[f(7)]_q[f(11)]_q[f(13)]_q \dots, \end{aligned} \tag{1.1}$$

where the sequences of the arguments run through the positive integers and the primes, respectively.

In this paper we consider the construction of normal numbers in base q as concatenation of q-ary integer parts of certain functions. The first result on that topic was achieved by Champernowne [2], who was able to show that

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\dots$$

is normal in base 10. This construction can be easily generalised to any integer base q. Copeland and Erdös [4] were able to show that

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\dots$$

is normal in base 10. These examples correspond to the choice $f(x) = x$ in (1.1). Davenport and Erdös [5] considered the case where $f(x)$ is a polynomial whose values at $x = 1, 2, \dots$ are always integers and showed that in this case the numbers $\theta_q(f)$ and $\tau_q(f)$ are normal. For $f(x)$ a polynomial with *rational* coefficients Schiffer [10] was able to show that $\mathcal{R}_N(\theta_q(f)) = \mathcal{O}(1/\log N)$. Nakai and Shiokawa [8] extended his results and showed that $\mathcal{R}_N(\tau_q(f)) = \mathcal{O}(1/\log N)$. In the case of *real* coefficients Nakai and Shiokawa [7] proved the same estimate for $\mathcal{R}_N(\theta_q(f))$. In this paper we want to discuss the case where $f(x)$ is a *transcendental entire function* (i.e., an entire

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function that is not a polynomial) of small *logarithmic order*. Recall that we say an increasing function $S(r)$ has *logarithmic order* λ if

$$\limsup_{r\to\infty}\frac{\log S(r)}{\log\log r}=\lambda. \tag{1.2}$$

We define the *maximum modulus* of an entire function f to be

$$M(r,f):=\max_{|x|\leq r}|f(x)|. \tag{1.3}$$

If f is an entire function and $\log M(r,f)$ has logarithmic order λ, then we call f an *entire function of logarithmic order* λ.

To achieve our results we combine the following ingredients.

- The first part of the proofs concerns the estimation for the number of solutions of the equation $f(x)=a$ where $a\in\mathbb{C}$ (cf. [3], [11, Section 8.21]) for entire functions of zero order.
- Following the methods of Nakai and Shiokawa [7, 8] we reformulate the problem in an estimation of exponential sums.
- Finally, the resulting exponential sums are treated by an exponential sum estimate of Baker [1], which was originally used to show that the sequences

$$(f(n))_{n\geq 1} \qquad \text{and} \qquad (f(p))_{p\text{ prime}}$$

are uniformly distributed modulo 1 for f an entire function with logarithmic order $1<\alpha<\frac{4}{3}$.

The main results of our papers are as follows.

Theorem 1. *Let $f(x)$ be a transcendental entire function which takes real values on the real line. Suppose that the logarithmic order $\alpha=\alpha(f)$ of f satisfies $1<\alpha<\frac{4}{3}$. Then for any block $d_1\dots d_l\in\{0,1,\dots,q-1\}^l$, we have*

$$\mathcal{N}(\theta_q(f);d_1\dots d_l;N)=\frac{1}{q^l}N+o(N)$$

as N tends to ∞. The implied constant depends only on f, q, and l.

For primes we show that $\tau_q(f)$ is normal in the following theorem.

Theorem 2. *Let $f(x)$ be a transcendental entire function which takes real values on the real line. Suppose that the logarithmic order $\alpha=\alpha(f)$ of f satisfies $1<\alpha<\frac{4}{3}$. Then for any block $d_1\dots d_l\in\{0,1,\dots,q-1\}$, we have*

$$\mathcal{N}(\tau_q(f);d_1\dots d_l,N)=\frac{1}{q^l}N+o(N)$$

as N tends to ∞. The implied constant depends only on f, q, and l.

2. Notation

Throughout the paper let f be a transcendental entire function of logarithmic order α satisfying $1<\alpha<\frac{4}{3}$ and taking real values on the real line. Let

$$f(x)=\sum_{k=1}^{\infty}a_kx^k$$

be the power series expansion of f. By $\log x$ and $\log_q x$ we denote the natural logarithm and the logarithm with respect to base q, respectively. Moreover, we set $e(\beta):=\exp(2\pi i\beta)$.

Let p always denote a prime and $\sum'$ be a sum over primes. By an integer interval I we mean a set of the form $I=\{a,a+1,\dots,b-1,b\}$ for arbitrary integers a and b.

Furthermore, we denote by $n(r,f)$ the number of zeros of $f(x)$ for $|x|\leq r$.

3. Lemmas

First we state the above-mentioned result of Baker that will permit us to estimate exponential sums over entire functions with small logarithmic order by choosing the occurring parameters appropriately.

Lemma 3.1 ([1, Theorem 4]). *Let d and h be integers, with $8 \leq h \leq d$. Let $a_1, \dots, a_d$ be real numbers and suppose that*

$$N^{-h}\exp\left(20\frac{\log N}{(\log\log N)^2}\right) < |a_h| < \exp(-10^3h^2), \tag{3.1}$$

$$|a_k| \leq \exp\left(-20\frac{\log N}{(\log\log N)^2}\right) \quad (h < k \leq d). \tag{3.2}$$

Suppose further that

$$\log N \geq 10^5 d^3 (\log d)^5. \tag{3.3}$$

Then, writing $g(x) = a_d x^d + \cdots + a_1 x$, we have

$$S = \sum_{n\leq N} e(g(n)) \ll N\exp\left(-\frac{1}{2}(\log N)^{\frac{1}{3}}\right) + N\left|a_h\right|^{1/(10h)}. \tag{3.4}$$

Lemma 3.2 ([1, Theorem 3]). *Under the hypotheses of Lemma 3.1 we have*

$$S = {\sum_{p\leq P}}' e(g(p)) \ll P\exp(-c(\log\log P)^2) + P(\log P)^{-1}\left|a_h\right|^{1/(10h)},$$

where c is a constant depending on g.

The following lemma due to Vinogradov provides an estimate of the Fourier coefficients of certain Urysohn functions.

Lemma 3.3 ([12, Lemma 12]). *Let α, β, Δ be real numbers satisfying*

$$0 < \Delta < \frac{1}{2}, \quad \Delta \leq \beta - \alpha \leq 1 - \Delta.$$

Then there exists a periodic function $\psi(x)$ with period 1, satisfying

1. *$\psi(x) = 1$ in the interval $\alpha + \frac{1}{2}\Delta \leq x \leq \beta - \frac{1}{2}\Delta$,*
2. *$\psi(x) = 0$ in the interval $\beta + \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta$,*
3. *$0 \leq \psi(x) \leq 1$ in the remainder of the interval $\alpha - \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta$,*
4. *$\psi(x)$ has a Fourier series expansion of the form*

$$\psi(x) = \beta - \alpha + \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} A(\nu)e(\nu x),$$

where

$$|A(\nu)| \ll \min\left(\frac{1}{\nu}, \beta - \alpha, \frac{1}{\nu^2\Delta}\right).$$

Finally, we give an easy result on the limit of quotients of sequences that will be used in our proof.

Lemma 3.4. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences with $0 < a_n \leq b_n$ for all n and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0. \tag{3.5}$$

Then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then by (3.5) there exists an n_0 such that

$$\frac{a_n}{b_n} < \varepsilon/2 \tag{3.6}$$

for $n > n_0$. Let $A(N) := \sum_{n=1}^{N} a_n$ and $B(N) := \sum_{n=1}^{N} b_n$. We show that there exists a n_1 such that $A(n)/B(n) < \varepsilon$ for $n > n_1$. Therefore we define $C(N) := \sum_{n=n_0+1}^{N} b_n$. As (3.6) implies that $a_n < \frac{\varepsilon}{2} b_n$ for $n > n_0$ we get

$$\frac{A(n)}{B(n)} = \frac{A(n_0) + \sum_{i=n_0+1}^{n} a_i}{B(n_0) + \sum_{i=n_0+1}^{n} b_i} < \frac{A(n_0) + \frac{\varepsilon}{2} C(n)}{B(n_0) + C(n)}.$$

As $b_n > 0$ we have that $C(n) \to \infty$ for $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{A(n_0) + \frac{\varepsilon}{2} C(n)}{B(n_0) + C(n)} = \frac{\varepsilon}{2}.$$

Therefore there is a $n_1 \geq n_0$ such that $A(n)/B(n) \leq \varepsilon$ for $n > n_1$ which proves the lemma. □

4. Value Distribution of Entire Functions

Before we start with the proof of the theorems, we need an estimation of the number of solutions for the equation $f(x) = a$ with f a transcendental entire function and $a \in \mathbb{C}$.

In this section we want to show the following result.

Proposition 1. *Let f be a transcendental entire function of logarithmic order α. Then for the number of solutions of the equation $f(x) = a$ the following estimate holds.*

$$n(r, f - a) \ll (\log r)^{\alpha - 1}. \tag{4.1}$$

As usual in Nevanlinna Theory we do not deal with $n(r, f - a)$ directly but use a strongly related function, which is defined by

$$N(r, f) = \int_1^r \frac{n(t, f) - n(0, f)}{t} dt - n(0, f) \log r \tag{4.2}$$

in order to prove the proposition. The connection between $n(r, f-a)$ and $N(r, f-a)$ is illustrated in the following lemma.

Lemma 4.1 ([3, Theorem 4.1])**.** *Let $f(x)$ be a non-constant meromorphic function in $\mathbb{C}$. For each $a \in \mathbb{C}$, $N(r, f - a)$ is of logarithmic order $\lambda + 1$, where λ is the logarithmic order of $n(r, f - a)$.*

The next lemma provides us with a very good estimation of the order of $N(r, f - a)$.

Lemma 4.2 ([9, Theorem])**.** *If f is an entire function of logarithmic order α where $1 < \alpha \leq 2$, then for all values $a \in \mathbb{C}$*

$$\log M(r, f) \sim N(r, f - a) \sim \log M\left(r(\log r)^{2-\alpha}\right) \sim N\left(r(\log r)^{2-\alpha}\right).$$

Now it is easy to prove Proposition 1.

Proof of Proposition 1. As f fulfills the assumptions of Lemma 4.2 we have that

$$N(r, f - a) \sim M(r, f) \ll (\log r)^{\alpha}. \tag{4.3}$$

Thus we have that $N(r, f - a)$ is of logarithmic order α and therefore by Lemma 4.1 we get that $n(r, f - a)$ is of logarithmic order $\alpha - 1$. □

5. Proof of Theorem 1

We fix the block $d_1 \dots d_l$ throughout the proof. Moreover, we adopt the following notation. Let $\mathcal{N}(f(n))$ be the number of occurrences of the block $d_1 \dots d_l$ in the q-ary expansion of the integer part $\lfloor f(n) \rfloor$. Furthermore, denote by $\ell(m)$ the length of the q-ary expansion of the integer m, i.e., $\ell(m) = \lfloor \log_q m \rfloor + 1$. Define M by

$$\sum_{n=1}^{M-1} \ell(f(n)) < N \leq \sum_{n=1}^{M} \ell(f(n)). \tag{5.1}$$

Because f is of logarithmic order $\alpha < \frac{4}{3}$ we easily see that

$$\ell(f(n)) \ll (\log M)^\alpha \qquad (1 \leq n \leq M).$$

Thus

$$\left| \mathcal{N}(\theta_q(f); d_1 \dots d_l; N) - \sum_{n=1}^{M} \mathcal{N}(f(n)) \right| \ll lM$$

We denote by J and $\bar{J}$ the maximum length and the average length of $\lfloor f(n) \rfloor$ for $n \in \{1, \dots, N\}$, respectively, i.e.,

$$\begin{aligned} J &:= \max_{1 \leq n \leq M} \ell(\lfloor f(n) \rfloor) \ll\gg (\log M)^\alpha, \\ \bar{J} &:= \frac{1}{M} \sum_{n=1}^{M} \ell(\lfloor f(n) \rfloor) \ll\gg (\log M)^\alpha, \end{aligned} \tag{5.2}$$

where $\ll\gg$ stands for both $\ll$ and $\gg$. Note that from these definitions we immediately see that

$$N = M\bar{J} + \mathcal{O}((\log M)^\alpha). \tag{5.3}$$

Thus in order to prove the theorem it suffices to show

$$\sum_{n=1}^{M} \mathcal{N}(f(n)) = \frac{1}{q^l} N + o(N). \tag{5.4}$$

In order to count the occurrences of the block $d_1 \dots d_l$ in the q-ary expansion of $\lfloor f(n) \rfloor$ ($1 \leq n \leq M$) we define the indicator function

$$\mathcal{I}(t) = \begin{cases} 1 & \text{if } \sum_{i=1}^{l} d_i q^{-i} \leq t - \lfloor t \rfloor < \sum_{i=1}^{l} d_i q^{-i} + q^{-l}, \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

which is an 1-periodic function. Indeed, write $f(n)$ in q-ary expansion for every $n \in \{1, \dots, M\}$, i.e.,

$$f(n) = b_r q^r + b_{r-1} q^{r-1} + \dots b_1 q + b_0 + b_{-1} q^{-1} + \dots,$$

then the function $\mathcal{I}(t)$ is defined in a way that

$$\mathcal{I}(q^{-j} f(n)) = 1 \iff d_1 \dots d_l = b_{j-1} \dots b_{j-l}.$$

In order to write $\sum_{n \leq M} \mathcal{N}(f(n))$ properly in terms of $\mathcal{I}$ we define the subsets $I_l, \dots, I_J$ of $\{1, \dots, M\}$ by

$$n \in I_j \Leftrightarrow f(n) \geq q^j \qquad (l \leq j \leq J).$$

Every I_j consists of those $n \in \{1, \dots, M\}$ for which we can shift the q-ary expansion of $\lfloor f(n) \rfloor$ at least j digits to the right to count the occurrences of the block $d_1 \dots d_l$. Using these sets we get

$$\sum_{n \leq M} \mathcal{N}(f(n)) = \sum_{j=l}^{J} \sum_{n \in I_j} \mathcal{I}\left(\frac{f(n)}{q^j} \right). \tag{5.6}$$

In the next step we fix j and show that $I_j = I_j(M)$ consists of integer intervals which are of asymptotically increasing length for M increasing. As I_j consists of all n such that $f(n) \geq q^j$ these n have to be between two zeros of the equation $f(x) = q^j$. By Proposition 1 the number of solutions for this equation is $n(M,f-q^j)\ll(\log M)^{\alpha-1}$. Therefore we can split I_j into k_j integer subintervals

$$I_j=\bigcup_{i=1}^{k_j}\{n_{ji},\dots,n_{ji}+m_{ji}-1\}$$

where m_{ji} is the length of the integer interval and $k_j\ll(\log M)^{\alpha-1}$. Thus the length of the integer intervals is increasing, i.e., $M(\log M)^{1-\alpha}\ll m_{ji}\ll M$. Thus we get that

$$\sum_{n\leq M}\mathcal{N}(f(n))=\sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}\mathcal{I}\left(\frac{f(n)}{q^j}\right). \tag{5.7}$$

Following Nakai and Shiokawa [7, 8] we want to approximate $\mathcal{I}$ from above and from below by two 1-periodic functions having small Fourier coefficients. In particular, we set

$$\begin{aligned}\alpha_-&=\sum_{\lambda=1}^{l}d_\lambda q^{-\lambda}+(2\delta_i)^{-1}, &\beta_-&=\sum_{\lambda=1}^{l}d_\lambda q^{-\lambda}+q^{-l}-(2\delta_i)^{-1}, &\Delta_-&=\delta_i^{-1},\\ \alpha_+&=\sum_{\lambda=1}^{l}d_\lambda q^{-\lambda}-(2\delta_i)^{-1}, &\beta_+&=\sum_{\lambda=1}^{l}d_\lambda q^{-\lambda}+q^{-l}+(2\delta_i)^{-1}, &\Delta_+&=\delta_i^{-1}.\end{aligned} \tag{5.8}$$

We apply Lemma 3.3 with $(\alpha,\beta,\Delta)=(\alpha_-,\beta_-,\Delta_-)$ and $(\alpha,\beta,\Delta)=(\alpha_+,\beta_+,\Delta_+)$, respectively, in order to get two functions $\mathcal{I}_-$ and $\mathcal{I}_+$. By the choices of $(\alpha_\pm,\beta_\pm,\Delta_\pm)$ it is immediate that

$$\mathcal{I}_-(t)\leq\mathcal{I}(t)\leq\mathcal{I}_+(t)\qquad(t\in\mathbb{R}). \tag{5.9}$$

Lemma 3.3 also implies that these two functions have Fourier expansions

$$\mathcal{I}_\pm(t)=q^{-l}\pm\delta_i^{-1}+\sum_{\substack{\nu=-\infty\\ \nu\neq0}}^{\infty}A_\pm(\nu)e(\nu t) \tag{5.10}$$

satisfying

$$\left|A_\pm(\nu)\right|\ll\min(\left|\nu\right|^{-1},\delta_i\left|\nu\right|^{-2}). \tag{5.11}$$

In a next step we want to replace $\mathcal{I}$ by $\mathcal{I}_+$ in (5.6). To this matter we observe, using (5.9), that

$$\left|\mathcal{I}(t)-\mathcal{I}_+(t)\right|\leq\left|\mathcal{I}_+(t)-\mathcal{I}_-(t)\right|\ll\delta_i^{-1}+\sum_{\substack{\nu=-\infty\\ \nu\neq0}}^{\infty}A_\pm(\nu)e(\nu t).$$

Together with (5.6) this implies that

$$\sum_{n\leq M}\mathcal{N}(f(n))=\sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}\left(\mathcal{I}_+\left(\frac{f(n)}{q^j}\right)+\mathcal{O}\left(\delta_i^{-1}+\sum_{\substack{\nu=-\infty\\ \nu\neq0}}^{\infty}A_\pm(\nu)e\left(\nu\frac{f(n)}{q^j}\right)\right)\right).$$

Inserting the Fourier expansion of $\mathcal{I}_+$ this yields

$$\sum_{n\leq M}\mathcal{N}(f(n))=\sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}\left(\frac{1}{q^l}+\mathcal{O}\left(\delta_i^{-1}+\sum_{\substack{\nu=-\infty\\ \nu\neq0}}^{\infty}A_\pm(\nu)e\left(\nu\frac{f(n)}{q^j}\right)\right)\right). \tag{5.12}$$

Because of the definition of M and $\bar{J}$ in (5.1) and (5.2), respectively, and the estimate in (5.3) we get that

$$\sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}1=\bar{J}M+\mathcal{O}(lM)=N+\mathcal{O}(lM). \tag{5.13}$$

Inserting this in (5.12) and subtracting the main part Nq^{-l} we obtain

$$\left|\sum_{n\leq M}\mathcal{N}(f(n))-\frac{N}{q^l}\right| \ll \sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}\left(\delta_i^{-1}+\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}A_{\pm}(\nu)e\left(\frac{\nu}{q^j}f(n)\right)\right)+lM. \tag{5.14}$$

Now we consider the coefficients $A_{\pm}(\nu)$. Noting (5.11) one sees that

$$A_{\pm}(\nu)\ll\begin{cases}\nu^{-1} & \text{for } |\nu|\leq\delta_i,\\ \delta_i\nu^{-2} & \text{for } |\nu|>\delta_i.\end{cases}$$

Estimating trivially all summands with $|\nu|>\delta$ we get

$$\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}A_{\pm}(\nu)e\left(\frac{\nu}{q^j}f(n)\right)\ll\sum_{\nu=1}^{\delta_i}\nu^{-1}e\left(\frac{\nu}{q^j}f(n)\right)+\delta_i^{-1}. \tag{5.15}$$

Using this in (5.14) and changing the order of summation yields

$$\left|\sum_{n\leq M}\mathcal{N}(f(n))-\frac{N}{q^l}\right| \ll \sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\leq n<n_{ji}+m_{ji}}\left(\delta_i^{-1}+\sum_{\nu=1}^{\delta_i}\nu^{-1}e\left(\frac{\nu}{q^j}f(n)\right)\right)+lM. \tag{5.16}$$

The crucial part is now to estimate the exponential sum containing the entire function f. Define

$$S(X):=\sum_{n\leq X}e\left(\frac{\nu}{q^j}f(n)\right). \tag{5.17}$$

We now treat the sum $S(X)$ by a similar reasoning as in the proof of Baker [1, Theorem 2]. We will show that the sum only depends on f and X.

To this matter we let the parameter d occurring in Lemma 3.1 be a function of X, in particular, we set

$$d=d(X)=\lfloor 10^{-2}(\log X)^{1/3}(\log\log X)^{-2}\rfloor, \tag{5.18}$$

which tends to infinity with X (see equation (11) of [1]). Moreover, we define the polynomial

$$g_j(x)=\frac{\nu}{q^j}(a_1x+\cdots+a_dx^d)$$

by the first d summands of the power series of $\frac{\nu}{q^j}f$. The parameter h of Lemma 3.1 will also be a function of X. In particular, we set $h=h(X)$ to be the largest positive integer such that $h\leq d$ and

$$X^{-h+\frac{1}{2}}<\left|\frac{\nu}{q^j}a_h\right|. \tag{5.19}$$

As shown in [1], h also tends to infinity with X.

Up to now we have not chosen a value for δ_i. For the moment, we just assume that $\delta_i\leq h$ because this choice implies that the summation index ν varies only over positive integers that are less than h. Thus the logarithmic order of $\frac{\nu}{q^j}f(n)$ is less than $\frac{4}{3}$. Indeed,

$$\log\left(\frac{\nu}{q^j}f(n)\right)<\log h-j\log q+\log f(n)<\log\log X+(\log X)^{\alpha}<(\log X)^{\bar{\alpha}} \tag{5.20}$$

where $\bar{\alpha}=\alpha+\varepsilon<\frac{4}{3}$. Note that g_j satisfies the conditions of Lemma 3.1. The estimate for the logarithmic order of $\frac{\nu}{q^j}f(n)$ will enable us to replace f by g_j in (5.17) causing only a small error term. This will then permit us to apply Lemma 3.1 in order to estimate $S(X)$.

By (5.20), equation (15) of [1] implies that for d as in (5.18)

$$\sum_{t>d}\left|\frac{\nu}{q^j}a_t\right|X^t<(2X)^{-1} \tag{5.21}$$

and therefore (see [1])

$$\left|\sum_{n\le X} e\left(\frac{\nu}{q^j}f(n)\right)\right| \le \left|\sum_{n\le X} e(g_j(n))\right| + \pi.$$

By this we can use Baker's estimations for exponential sums over entire functions contained in Lemma 3.1 and get with $d=d(X)$ and $h=h(X)$ defined in (5.18) and (5.19), respectively,

$$S(X) \ll X\exp(-\tfrac{1}{2}(\log X)^{\frac{1}{3}}) + X\exp(-h). \tag{5.22}$$

Now it is time to set δ_i for every i. As ν changes the coefficients of the function under consideration we calculate for every $\nu=1,\dots,d(m_{ji})$ the corresponding $h_\nu(m_{ji})$. In order to fulfill the constraint on the logarithmic order we need to chose δ_i smaller than the smallest $h_\nu(m_{ji})$ with $\nu\le\delta_i$. Thus we set

$$\delta_i := \max\{r\le d(m_{ji}) : r \le \min\{h_\nu(m_{ji}) : \nu\le r\}\}. \tag{5.23}$$

This is always possible since $h_\nu(m_{ji})\ge 1$. For this choice we also have $\delta_i\le h_\nu(m_{ji})$ and $\delta_i\to\infty$ as $m_{ji}\to\infty$ because the minimum of the $h_\nu(m_{ji})$ tends to infinity for $m_{ji}\to\infty$. Doing this for every $i=1,\dots,k$ (i.e., for every integer interval comprising the set I_j) we can apply (5.22) with $X=m_{ji}$ and use the fact that δ_i is the smallest $h_\nu(m_{ji})$ for i. This yields

$$\begin{aligned}\sum_{i=1}^{k_j}\sum_{\nu=1}^{\delta_i}\nu^{-1}\sum_{n_{ji}\le n<n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j}f(n)\right) &\ll \sum_{i=1}^{k_j}\sum_{\nu=1}^{\delta_i}\nu^{-1}S(m_{ji})\\ &\ll \sum_{i=1}^{k_j}\sum_{\nu=1}^{\delta_i}\nu^{-1}m_{ji}\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}}) + m_{ji}\exp(-\delta_i)\\ &\ll \sum_{i=1}^{k_j}\Big(m_{ji}\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}}) + m_{ji}\exp(-\delta_i)\Big)\log\delta_i.\end{aligned}$$

As we do not know the asymptotic behavior of δ_i we have to distinguish the cases whether $\exp(-\delta_i)$ is greater or smaller than $\exp(-\frac{1}{2}(\log m_{ji})^{\frac{1}{3}})$. In both cases we can assume that m_{ji} is sufficiently large.

- Suppose first that $\exp(-\frac{1}{2}(\log m_{ji})^{\frac{1}{3}}) > \exp(-\delta_i)$ holds. As $\delta_i\le d(m_{ji})\le(\log m_{ji})^{1/3}$ we get

  $$\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}})\log\delta_i \ll \exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}})(\log\log m_{ji}) \ll \exp(-\tfrac{1}{3}(\log m_{ji})^{\frac{1}{3}})$$

  and thus

  $$\Big(\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i)\Big)\log\delta_i \ll \exp(-\tfrac{1}{3}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i/2).$$

- For the second case assume that $\exp(-\frac{1}{2}(\log m_{ji})^{\frac{1}{3}}) \le \exp(-\delta_i)$ holds. This implies that $\log\delta_i \ll \log\log m_{ji}$ and we get

  $$\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}})\log\delta_i \ll \exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}})(\log\log m_{ji}) \ll \exp(-\tfrac{1}{3}(\log m_{ji})^{\frac{1}{3}}).$$

  Therefore we also have

  $$\Big(\exp(-\tfrac{1}{2}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i)\Big)\log\delta_i \ll \exp(-\tfrac{1}{3}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i/2).$$

By this we have the estimation

$$\sum_{\nu=1}^{\delta_i}\nu^{-1}\sum_{n_{ji}\le n<n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j}f(n)\right) \ll \sum_{i=1}^{k} m_{ji}\left(\exp(-\tfrac{1}{3}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i/2)\right). \tag{5.24}$$

By (5.16) we get that

$$\left|\sum_{n\le M}\mathcal{N}(f(n)) - \frac{N}{q^l}\right| \ll \sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum_{n_{ji}\le n<n_{ji}+m_{ji}}\left(\delta_i^{-1} + \sum_{\nu=1}^{\delta}\nu^{-1}e\left(\frac{\nu}{q^j}f(n)\right)\right) + lM$$

Thus it remains to show that

$$\sum_{i=1}^{k_j} \sum_{n_{ji} \le n < n_{ji}+m_{ji}} \delta^{-1} = \sum_{i=1}^{k_j} \frac{m_{ji}}{\delta_i} = o(|I_j|). \tag{5.25}$$

and

$$\sum_{i=1}^{k_j} \sum_{n_{ji} \le n < n_{ji}+m_{ji}} \sum_{\nu=1}^{\delta_i} \nu^{-1} e\left(\frac{\nu}{q^j} f(n)\right) = o\left(|I_j|\right), \tag{5.26}$$

where $|I_j| = \sum_{i=1}^{k_j} m_{ji}$ the sum of the lengths of the integer intervals.

First we consider (5.25). Therefore we set $a_i = \frac{m_{ji}}{\delta_i}$ and $b_i = m_{ji}$. By noting that $\frac{a_i}{b_i} = \delta_i^{-1} \to 0$ we are able to apply Lemma 3.4 and get

$$0 \le \frac{\sum_{i=1}^{k} \frac{m_{ji}}{\delta_i}}{\sum_{i=1}^{k} m_{ji}} \to 0.$$

Finally we have to show (5.26). We again want to apply Lemma 3.4 by setting

$$\begin{aligned} a_i &:= m_{ji} \exp(-\tfrac{1}{3} (\log m_{ji})^{\frac{1}{3}}) + m_{ji} \exp(-\delta_i/2), \\ b_i &:= m_{ji}. \end{aligned}$$

As $M(\log M)^{1-\alpha} \ll m_{ji} \ll M$ we get that both $\exp(-\frac{1}{3}(\log m_{ji})^{\frac{1}{3}})$ and $\exp(-\delta_i/2)$ tend to zero. Thus we have that $\frac{a_i}{b_i} \to 0$ for $M \to \infty$. An application of Lemma 3.4 together with (5.24) gives

$$0 \le \frac{\sum_{\nu=1}^{\delta} \nu^{-1} \sum_{n_{ji} \le n < n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j} f(n)\right)}{|I_j|} \ll \frac{\sum_{i=1}^{k} m_{ji} \left(\exp(-\frac{1}{3}(\log m_{ji})^{\frac{1}{3}}) + \exp(-\delta_i/2)\right)}{\sum_{i=1}^{k} m_{ji}} \to 0$$

for $M \to \infty$ and thus (5.26) holds.

We put (5.25) and (5.26) in our estimate (5.16) and get together with (5.13) that

$$\begin{aligned} \left| \sum_{n \le M} \mathcal{N}(f(n)) - \frac{N}{q^l} \right| &\ll \sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji} \le n < n_{ji}+m_{ji}} \left(\delta_i^{-1} + \sum_{\nu=1}^{\delta} \nu^{-1} e\left(\frac{\nu}{q^j} f(n)\right)\right) + lM \\ &\ll \sum_{j=l}^{J} o(|I_j|) + lM = o(\bar{J}M) = o(N). \end{aligned}$$

Thus by (5.4) the theorem is proven.

6. Proof of Theorem 2

Throughout the proof p will always denote a prime and $\pi(x)$ will denote the number of primes less than or equal to x. As in the proof of Theorem 1 we fix the block $d_1 \dots d_l$ and write $\mathcal{N}(f(p))$ for the number of occurrences of this block in the q-ary expansion of $\lfloor f(p) \rfloor$. By $\ell(m)$ we denote the length of the q-ary expansion of an integer m. We define an integer P by

$$\sum_{p \le P-1}{}' \ell\left(\lfloor f(p) \rfloor\right) < N \le \sum_{p \le P}{}' \ell\left(\lfloor f(p) \rfloor\right). \tag{6.1}$$

As above we get that

$$\ell(\lfloor f(p) \rfloor) \le (\log P)^\alpha \quad (2 \le p \le P).$$

Again we set J the greatest and $\bar{J}$ the average length of the q-ary expansions over the primes. Thus

$$J := \max_{p \le P \text{ prime}} \ell(\lfloor f(p) \rfloor) \ll\gg (\log P)^\alpha \tag{6.2}$$

$$\bar{J} := \frac{1}{\pi(P)} \sum_{p \le P}{}' \ell(\lfloor f(p) \rfloor) \ll\gg (\log P)^\alpha. \tag{6.3}$$

Note that by these definitions we have

$$N = \bar{J}P + \mathcal{O}((\log P)^{\alpha}). \tag{6.4}$$

Thus by the same reasoning as in the proof of Theorem 1 it sufficies to show that

$$\sum_{p\le P}{}' \mathcal{N}(f(p)) = \frac{N}{q^l} + o(N). \tag{6.5}$$

We define the indicator function as in (5.5) and also the subsets $I_l, \dots, I_J$ of $\{2, \dots, P\}$ by

$$n \in I_j \Leftrightarrow f(n) \ge q^j \qquad (l \le j \le J).$$

Following the proof of Theorem 1 we see that

$$\sum_{p\le P}{}' \mathcal{N}(f(p)) = \sum_{j=l}^{J} \sum_{p\in I_j}{}' \mathcal{I}\left(\frac{f(p)}{q^j}\right) + \mathcal{O}\left(l\pi(P)\right). \tag{6.6}$$

Now we fix j and split I_j into k_j integer intervals of length m_{ji} for $i = 1, \dots, k$. Thus

$$I_j = \bigcup_{i=1}^{k_j} \{n_{ji}, n_{ji} + 1, \dots, n_{ji} + m_{ji} - 1\}$$

By Proposition 1 we again get that $k_j \ll (\log P)^{\alpha-1}$. Thus the length of the m_{ji} is asymptotically increasing for P, indeed, we have $P(\log P)^{1-\alpha} \ll m_{ji} \ll P$. Now we can rewrite (6.6) by

$$\sum_{p\le P}{}' \mathcal{N}(f(p)) = \sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji}\le p< n_{ji}+m_{ji}}{}' \mathcal{I}\left(\frac{f(p)}{q^j}\right) + \mathcal{O}\left(l\pi(P)\right). \tag{6.7}$$

Following Nakai and Shiokawa [7, 8] again we get as in the proof of Theorem 1 that there exist two functions $\mathcal{I}_-$ and $\mathcal{I}_+$. We replace $\mathcal{I}$ by $\mathcal{I}_+$ in (6.7) and together with the Fourier expansion of $\mathcal{I}_+$ in (5.10) we get in the same manner as in (5.12) that

$$\sum_{p\le P}{}' \mathcal{N}(f(p)) = \sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji}\le p< n_{ji}+m_{ji}}{}' \left(\frac{1}{q^j} + \mathcal{O}\left(\delta_i^{-1} + \sum_{\substack{\nu=-\infty\\ \nu\ne 0}}^{\infty} A_{\pm}(\nu) e\left(\nu \frac{f(n)}{q^j}\right)\right)\right). \tag{6.8}$$

By (6.1) and (6.2) together with (6.4) we have

$$\sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji}\le p< n_{ji}+m_{ji}}{}' 1 = \bar{J}\pi(P) + \mathcal{O}(l\pi(P)) = N + \mathcal{O}(l\pi(P)). \tag{6.9}$$

We subtract the main part Nq^{-l} in (6.8) and get by (6.9)

$$\left|\sum_{p\le P}{}' \mathcal{N}(f(p)) - \frac{N}{q^l}\right| \ll \sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji}\le p< n_{ji}+m_{ji}}{}' \left(\delta_i^{-1} + \sum_{\substack{\nu=-\infty\\ \nu\ne 0}}^{\infty} A_{\pm}(\nu) e\left(\frac{\nu}{q^j} f(n)\right)\right) + l\pi(P). \tag{6.10}$$

We estimate the coefficients $A_{\pm}(\nu)$ in the same way as in (5.15). Then (6.10) simplifies to

$$\left|\sum_{p\le P}{}' \mathcal{N}(f(p)) - \frac{N}{q^l}\right| \ll \sum_{j=l}^{J} \sum_{i=1}^{k_j} \sum_{n_{ji}\le p< n_{ji}+m_{ji}}{}' \left(\delta_i^{-1} + \sum_{\nu=1}^{\delta_i} \nu^{-1} e\left(\frac{\nu}{q^j} f(p)\right)\right) + l\pi(P). \tag{6.11}$$

Again the crucial part is the estimation of an exponential sum over the primes. We apply quite the same reasoning as in the proof of Theorem 1. We set

$$S'(X) := \sum_{p\le X}{}' e\left(\frac{\nu}{q^j} f(p)\right). \tag{6.12}$$

and use the functions $d(X)$ and $h(X)$ defined in (5.18) and (5.19), respectively. If we assume that $\delta_i \le h(X)$ then we get that the logarithmic order of $\frac{\nu}{q^j} f(x)$ is less than $\frac{4}{3}$ as in (5.20). We set

$$g_j(x) = \frac{\nu}{q^j}(a_d x^d + \cdots + a_1 x).$$

By (5.21) we also get that

$$\left| \sum_{p \le X}{}' e\left(\frac{\nu}{q^j} f(p)\right) \right| \le \left| \sum_{p \le X}{}' e\left(g_j(p)\right) \right| + \pi.$$

We can apply Lemma 3.2 to get the estimate

$$S'(X) \ll X \exp(-c_\nu (\log\log X)^2) + \frac{X}{\log X} \exp(-h), \tag{6.13}$$

where c_ν is a constant depending on ν and $h = h(X)$ is the function defined in (5.19).

Now we fix i and for every $\nu = 1, \ldots, d(m_{ji})$ we calculate the corresponding $h_\nu(m_{ji})$ and c_ν. We set

$$\begin{aligned} \delta_i &:= \max\{r \le d(m_{ji}) : r \le \min\{h_\nu(m_{ji}) : \nu \le r\}\}, \\ \bar{c}_i &:= \min\{c_\nu : \nu = 1, \ldots, \delta_i\}. \end{aligned} \tag{6.14}$$

By the above reasoning we have that $\delta_i \to \infty$ for m_{ji} and therefore for P.

By this we get a δ_i for every $i = 1, \ldots, k$ and we can estimate the exponential sum in (6.11) with help of (6.13) and the definitions of δ_i and $\bar{c}_i$ in (6.14) to get

$$\begin{aligned} \sum_{i=1}^{k_j} \sum_{n_{ji} \le p < n_{ji} + m_{ji}}{}' \sum_{\nu=1}^{\delta_i} \nu^{-1} e\left(\frac{\nu}{q^j} f(p)\right) &\ll \sum_{i=1}^{k_j} \sum_{\nu=1}^{\delta_i} \nu^{-1} S'(m_{ji}) \\ &\ll \sum_{i=1}^{k_j} \sum_{\nu=1}^{\delta_i} \nu^{-1} m_{ji} \left(\exp(-\bar{c}_i (\log\log m_{ji})^2) + \frac{\exp(-\delta_i)}{\log m_{ji}} \right) \\ &\ll \sum_{i=1}^{k_j} m_{ji} \left(\exp(-\bar{c}_i (\log\log m_{ji})^2) + \frac{\exp(-\delta_i)}{\log m_{ji}} \right) \log \delta_i. \end{aligned} \tag{6.15}$$

As we do not know the asymptotic behavior of δ_i we want to merge it with the expression in the parathesis and therefore have to distinguish two cases according whether $\exp(-\delta_i)(\log m_{ji})^{-1}$ is greater or smaller than $\exp(-\bar{c}_i(\log\log m_{ji})^2)$.

- If $\exp(-\bar{c}_i(\log\log m_{ji})^2) > \exp(-\delta_i)(\log m_{ji})^{-1}$ then as $\delta_i \le (\log P)^{1/3}$ we have that

$$\exp(-\bar{c}_i(\log\log m_{ji})^2) \log \delta_i \le \exp(-\bar{c}_i(\log\log m_{ji})^2) \log\log m_{ji} < \exp(-\bar{c}_i/2(\log\log m_{ji})^2).$$

  Thus

$$\begin{aligned} &\left(\exp(-\bar{c}_i(\log\log m_{ji})^2) + \exp(-\delta_i)(\log m_{ji})^{-1}\right) \log \delta_i \\ &\qquad \ll \exp(-\bar{c}_i/2(\log\log m_{ji})^2) + \exp(-\delta_i/2)(\log m_{ji})^{-1}. \end{aligned}$$

- On the contrary we have $\exp(-\bar{c}_i(\log\log m_{ji})^2) \le \exp(-\delta_i)(\log m_{ji})^{-1}$ and this implies $\delta_i \le c(\log\log m_{ji})^2$ for a positive constant c. Therefore we get

$$\exp(-\bar{c}_i(\log\log m_{ji})^2) \log \delta_i \le \exp(-\bar{c}_i(\log\log m_{ji})^2) c(\log\log m_{ji})^2 < \exp(-\bar{c}_i/2(\log\log m_{ji})^2).$$

  We again have

$$\begin{aligned} &\left(\exp(-\bar{c}_i(\log\log m_{ji})^2) + \exp(-\delta_i)(\log m_{ji})^{-1}\right) \log \delta_i \\ &\qquad \ll \exp(-\bar{c}_i/2(\log\log m_{ji})^2) + \exp(-\delta_i/2)(\log m_{ji})^{-1}. \end{aligned}$$

By this we have

$$\sum_{i=1}^{k_j}\sum_{\nu=1}^{\delta_i}\nu^{-1}\sideset{}{'}\sum_{n_{ji}\leq p<n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j}f(p)\right) \ll \sum_{i=1}^{k_j} m_{ji}\left(\exp(-\bar{c}_i/2(\log\log m_{ji})^2)+\exp(-\delta_i/2)(\log m_{ji})^{-1}\right). \tag{6.16}$$

The considerations above can be used in (6.11) in order to obtain

$$\left|\sideset{}{'}\sum_{p\leq P}\mathcal{N}(f(p))-\frac{N}{q^l}\right| \ll \sum_{j=l}^{J}\sum_{i=1}^{k_j}\sideset{}{'}\sum_{n_{ji}\leq p<n_{ji}+m_{ji}}\left(\delta_i^{-1}+\sum_{\nu=1}^{\delta_i}\nu^{-1}e\left(\frac{\nu}{q^j}f(p)\right)\right)+l\pi(P).$$

Thus it remains to show that

$$\sum_{i=1}^{k_j}\sideset{}{'}\sum_{n_i\leq p<n_i+m_{ji}}\delta_i^{-1}=o(\pi(I_j)) \tag{6.17}$$

and

$$\sum_{\nu=1}^{\delta_i}\nu^{-1}\sum_{i=1}^{k_j}\sideset{}{'}\sum_{n_{ji}\leq p<n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j}f(p)\right)=o(\pi(I_j)), \tag{6.18}$$

where $\pi(I_j)$ stands for the number of primes in the interval I_j.

First we have to estimate the number of primes in I_j for every j. Therefore we set $m'_{ji} := \pi\left(\{n_{ji},\dots,n_{ji}+m_{ji}-1\}\right)$. Thus the number of primes in I_j is the sum of the m'_{ji}, i.e. $\pi(I_j) = \sum_{i=1}^{k_j} m'_{ji}$. As

$$P(\log P)^{1-\alpha}\ll m_{ji}\ll P\quad (i=1,\dots,k_j) \tag{6.19}$$

holds we consider an integer interval $[x-y,x]\cap\mathbb{Z}$ with $x(\log x)^{1-\alpha}\leq y<x$. We set $y := x\beta^{-1}$ and get

$$1<\beta\leq(\log x)^{\alpha-1}. \tag{6.20}$$

To estimate the number of primes we apply the Prime Number Theorem in the following form (which is a weaker result than in Chapter 11 of [6]).

$$\pi(x)=\frac{x}{\log x}+\mathcal{O}\left(\frac{x}{(\log x)^2}\right). \tag{6.21}$$

Thus we get with (6.20) and (6.21)

$$\begin{aligned}\pi\left([x-y,x]\cap\mathbb{Z}\right) &= \pi(x)-\pi(x-y)\\ &= \frac{x}{\log x}-\frac{x-x\beta^{-1}}{\log(x-x\beta^{-1})}+\mathcal{O}\left(\frac{x}{(\log x)^2}\right)\\ &= \frac{x}{\log x}-\frac{x-x\beta^{-1}}{\log x+\mathcal{O}\left(\beta^{-1}\right)}+\mathcal{O}\left(\frac{x}{(\log x)^2}\right)\\ &= \frac{x}{\log x}-\frac{x-x\beta^{-1}}{\log x}\left(1+\mathcal{O}\left(\beta^{-1}(\log x)^{-1}\right)\right)+\mathcal{O}\left(\frac{x}{(\log x)^2}\right)\\ &= \frac{y}{\log x}+\mathcal{O}\left(\frac{x}{(\log x)^2}\right).\end{aligned} \tag{6.22}$$

Now we reformulate (6.22) by setting $x=P$ and $y=m_{ji}$ and get with (6.19)

$$m'_{ji}=\pi\left(\{n_i,\dots,n_i+m_{ji}-1\}\right)=\frac{m_{ji}}{\log P}+\mathcal{O}\left(\frac{P}{(\log P)^2}\right). \tag{6.23}$$

Now we use the estimation (6.23) in order to show (6.17). By setting $a_i = \frac{m'_{ji}}{\delta_i}$ and $b_i = m'_{ji}$ we note that as $m'_{ji} \to \infty$ we get that $m_{ji} \to \infty$ which implies $\frac{a_i}{b_i} \to 0$. Therefore we can apply Lemma 3.4 and get

$$0 \leq \frac{\sum'_{p\in I_j} \delta^{-1}}{\pi(I_j)} = \frac{\sum_{i=1}^{k} \frac{m_{ji}}{\delta_i}}{\sum_{i=1}^{k} m'_{ji}} \to 0.$$

Finally we show that (6.18) holds. We set

$$a_i = m_{ji}(\exp(-\bar{c}_i/2(\log\log m_{ji})^2) + \exp(-\delta_i/2)(\log m_{ji})^{-1}),$$
$$b_i = m'_{ji}.$$

By the estimation in (6.23) we get that $\frac{a_i}{b_i} \to 0$ for $P \to \infty$ and we are able to apply Lemma 3.4. Thus with (6.16) we get

$$0 \leq \frac{\sum_{i=1}^{k_j} \sum_{\nu=1}^{\delta} \nu^{-1} \sum'_{n_{ji}\leq p<n_{ji}+m_{ji}} e\left(\frac{\nu}{q^j} f(p)\right)}{\pi(I_j)}$$
$$\ll \frac{\sum_{i=1}^{k_j} m_{ji}(\exp(-\bar{c}_i/2(\log\log m_{ji})^2) + \exp(-\delta_i/2)(\log m_{ji})^{-1})}{\sum_{i=1}^{k_j} m'_{ji}} \to 0.$$

Thus by putting (6.11), (6.18), and (6.17) together we get

$$\left|\sum'_{p\leq P} \mathcal{N}(f(p)) - \frac{N}{q^l}\right| \ll \sum_{j=l}^{J}\sum_{i=1}^{k_j}\sum'_{n_{ji}\leq p<n_{ji}+m_{ji}} \left(\delta_i^{-1} + \sum_{\nu=1}^{\delta_i} \nu^{-1} e\left(\frac{\nu}{q^j} f(p)\right)\right) + l\pi(P)$$
$$\ll \sum_{j=l}^{J} o(\pi(I_j)) + l\pi(P) \ll o(\bar{J}P) \ll o(N),$$

which, together with (6.5), proves Theorem 2.

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(M. G. Madritsch) Departments of Mathematics A, Graz University of Technology
E-mail address: `madritsch@finanz.math.tugraz.at`

(J. M. Thuswaldner) Department Mathematik und Informationstechnologie, MU Leoben
E-mail address: `Joerg.Thuswaldner@mu-leoben.at`

(R. F. Tichy) Departments of Mathematics A, Graz University of Technology
E-mail address: `tichy@tugraz.at`