ADDITIVE FUNCTIONS FOR NUMBER SYSTEMS IN FUNCTION FIELDS

MANFRED G. MADRITSCH AND JÖRG M. THUSWALDNER

ABSTRACT. Let $\mathbb{F}_q$ be a finite field with q elements and $p \in \mathbb{F}_q[X,Y]$. In this paper we study properties of additive functions with respect to number systems which are defined in the ring $\mathbb{F}_q[X,Y]/p\,\mathbb{F}_q[X,Y]$. Our results comprise distribution results, exponential sum estimations as well as a version of Waring's Problem restricted by such additive functions. Similar results have been shown for b-adic number systems as well as number systems in finite fields in the sense of Kovács and Pethő. In the proofs of the results contained in the present paper new difficulties occur because the "fundamental domains" associated to the number systems studied here have a complicated structure.

1. Introduction

In this paper we want to study additive functions. Before we start, however, we need an impression, what we mean by a number system and therefore by an additive function in this system. Therefore we start with the simplest case, a number system in the non-negative integers. Let $b \geq 2$ be a positive integer. Then every $g \in \mathbb{N}$ admits a unique and finite representation of the form

$$g = \sum_{k=0}^{\ell-1} d_k b^k \quad \text{with} \quad d_i \in \{0, \dots, b-1\} \quad \text{and} \quad d_{\ell-1} \neq 0 \quad \text{if} \quad g \neq 0.$$

We call a function $f : \mathbb{R} \to G$, with G an Abelian group, b-additive (in this number system) if

$$f(g) = \sum_{k=0}^{\ell-1} f(d_k b^k).$$

If f only acts on the digits d_i, *i.e.*, if

$$f(g) = \sum_{k=0}^{\ell-1} f(d_k)$$

we call f *strictly b-additive.* A simple example of a strictly b-additive function is the sum of digits function s_b, defined by

$$s_b(g) = \sum_{k=0}^{\ell-1} d_k.$$

There are many questions around these functions and one of the first answered is its distribution in residue classes.

Theorem (Kim [9]). *Let $b_1, \dots, b_r \geq 2$ be integers and $m_1, \dots, m_r$ be positive integers. Furthermore let $f_i : \mathbb{N} \to \mathbb{Z}$, $1 \leq i \leq r$, be a b_i-additive function.*

Set

$$H := \{(f_1(n) \bmod m_1, \dots, f_r(n) \bmod m_r) : n \geq 0\}.$$

Date: February 8, 2010.

2000 *Mathematics Subject Classification.* 11T23, 11A63.

Key words and phrases. Finite fields, digit expansions, distribution in residue classes, Weyl sums, Waring's Problem.

Supported by the French National Research Agency (ANR), Project LAREDA.

Supported by the Austrian Research Foundation (FWF), Project S9610, that is part of the Austrian Research Network "Analytic Combinatorics and Probabilistic Number Theory".

Then H is a subgroup of $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$ and for every $(a_1, \ldots, a_r) \in H$ we have

$$\#\left\{n < N : f_1(n) \equiv a_1 \bmod m_1, \ldots, f_r(n) \equiv a_r \bmod m_r\right\} = \frac{N}{|H|} + \mathcal{O}\left(N^{1-\delta}\right)$$

where $\delta = 1/(120r^2\overline{b}^3\overline{m}^2)$ with

$$\overline{b} = \max_{1\leq i\leq r} b_i \quad \text{and} \quad \overline{m} = \max_{1\leq i\leq r} m_i$$

and the $\mathcal{O}$-constant depends only on r and $b_1, \ldots, b_r$.

On the other hand one is also interested in the asymptotic distribution of the values of a b-additive function.

Theorem (Bassily and Katái [2]). *Let $f : \mathbb{N} \to \mathbb{R}$ be a b-additive function such that $f(ab^k) = \mathcal{O}(1)$ as $k \to \infty$ and $a \in \mathcal{N}$. Furthermore let*

$$m_{k,b} := \frac{1}{b}\sum_{a=0}^{b-1} f(ab^k), \quad \sigma_{k,b}^2 := \frac{1}{q}\sum_{a=0}^{b-1} f^2(ab^k) - m_{k,b}^2,$$

and

$$M_b(x) := \sum_{k=0}^{N} m_{k,b}, \quad D_b^2(x) = \sum_{k=0}^{N} \sigma_{k,b}^2$$

with $N = [\log_b x]$. Assume that $D_b(x)/(\log x)^{1/3} \to \infty$ as $x \to \infty$ and let $p(x)$ be a polynomial of degree d with integer coefficients and positive leading term. Then, as $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x : \frac{f(p(n)) - M_b(x^d)}{D_b(x^d)} < y\right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Generalizing these distribution results one can attack Waring's Problem with a digitally restricted set as base. In particular, Thuswaldner and Tichy [17] proved the following result.

Theorem. *Let b, k, a and m be integers. Then every sufficiently large integer N can be written as sum*

$$N = x_1^k + \cdots + x_s^k,$$

where $x_i \in \mathbb{N}$ and $s_b(x_i) \equiv a \bmod m$ for $i = 1, \ldots, s$ and s only depends on k. Moreover, the number of representations of N in this way obeys a Hardy-Littlewood type asymptotic formula.

A generalization of this theorem to arbitrary b-additive functions is due to Wagner [18].

In 1991 Kovács and Pethő [10] introduced number systems in the polynomial ring $\mathbb{F}_q[X]$ over a finite field $\mathbb{F}_q$. It is possible to define a generalization of b-additive functions with respect to such number systems. In particular, fix a polynomial $Q \in \mathbb{F}_q[X]$. Then every other polynomial $G \in \mathbb{F}_q[X]$ has a unique finite representation of the form

$$G = \sum_{k=0}^{\ell-1} D_k Q^k \quad \text{with} \quad \deg D_k < \deg Q$$

and $D_{\ell-1} \neq 0$ if $G \neq 0$.

Analogs of the two distribution theorems above were shown for this setting by Drmota and Gutenbrunner [6]. Waring's Problem with this digitally restricted set was solved by the first author [12] where the Weyl sum estimates came from the two authors of the present paper [13].

Recently, Scheicher and Thuswaldner [14] introduced a generalization of these number systems which live in certain function fields and will be defined below. In the present paper we will define and study analogues of b-additive functions in (slight generalizations of) these number systems. Compared with the case of number systems in $\mathbb{F}_q[X]$, new problems occur in this context. This is mainly due to the fact that the "fundamental domains" of these number systems, which have been studied by Beck *et al.* [3], have a nontrivial structure. Nevertheless we are not able to apply their results directly since we will work with a valuation instead of the degree function. Therefore we will develop our view of the fundamental domains in Section 3.

2. Definitions and results

The idea of number systems in function fields is based on the theory of number systems in algebraic number fields. Therefore we will first introduce number systems in these fields and then rewrite them for function fields. A number system in an algebraic number field is defined as follows. Let β be an algebraic integer. Let $b \in \mathbb{Z}[\beta]$ and $\mathcal{N} \subset \mathbb{Z}$, then we call the pair $(b, \mathcal{N})$ a number system in $\mathbb{Z}[\beta]$ if every $g \in \mathbb{Z}[\beta]$ admits a unique and finite representation of the form

$$g = \sum_{k=0}^{\ell-1} d_k b^k \quad \text{with} \quad d_k \in \mathcal{N}$$

and $d_{\ell-1} \neq 0$ if $g \neq 0$.

Now the idea is to replace $\mathbb{Z}$ by $\mathbb{F}_q[X]$ and consider the same construction. Thus let $\mathbb{F}_q[X]$ and $\mathbb{F}_q(X)$ be the ring of polynomials and the field of rational functions over a finite field $\mathbb{F}_q$, respectively. Furthermore let $p \in \mathbb{F}_q[X, Y]$ be a separable irreducible polynomial. Then we are interested in number systems in $\mathcal{S} = \mathbb{F}_q[X, Y]/p\mathbb{F}_q[X, Y]$. Let $B \in \mathcal{S}$ and $\mathcal{N} \subset \mathbb{F}_q[X]$, then we call the pair $(B, \mathcal{N})$ a number system in $\mathcal{S}$ if every $G \in \mathcal{S}$ admits a unique and finite representation of the form

$$G = \sum_{k=0}^{\ell-1} D_k B^k \quad \text{with} \quad D_k \in \mathcal{N} \tag{2.1}$$

and $D_{\ell-1} \neq 0$ if $G \neq 0$. We call this representation the B-digit representation of G and $L_B(G) = \ell$ its length and denote by $\mathcal{L}_B(m)$ the set of all $G \in \mathcal{S}$ whose B-adic length is less than m, *i.e.*,

$$\mathcal{L}_B(m) := \{Q \in \mathcal{S} \mid L_B(Q) < m\}.$$

Imitating the definitions above we call a function f *strictly B-additive* if it acts only on the digits of (2.1), *i.e.*, if

$$f(G) = \sum_{k=0}^{\ell-1} f(D_k)$$

with G as in (2.1). The definition of a B-additive function is done analogously. As mentioned above, number systems in $\mathcal{S}$ have been investigated by Scheicher and Thuswaldner [14] as well as Beck *et al.* [3]. They gained the following characterization.

Proposition 2.1. *Let $p \in \mathbb{F}_q[X, Y]$ be a polynomial such that*

$$p(Y) = Y^d + p_{d-1}Y^{d-1} + \cdots + p_1 Y + p_0.$$

Set $\mathcal{N} = \{D \in \mathbb{F}_q[X] : \deg D < \deg p_0\}$. Then $(Y, \mathcal{N})$ is a number system in $\mathbb{F}_q[X, Y]/p\mathbb{F}_q[X, Y]$ if and only if

$$\max_{i=1}^{d} \deg p_i < \deg p_0.$$

Indeed, in these papers only the case $B = Y$ has been considered. However, as we will see in Proposition 3.1 this restriction is not crucial.

We want to illustrate Proposition 2.1 by the following example.

Example 2.2. Let $p := Y^2 + XY + X^2$ then $p_2 = 1$, $p_1 = X$, and $p_0 = X^2$. Since $\deg p_2 < \deg p_1 < \deg p_0$ we get by an application of Proposition 2.1 that Y is a basis of a number system in $\mathbb{F}_q[X, Y]/p\mathbb{F}_q[X, Y]$.

We will use the following notations (we mainly follow those in [4] and [19]). It is well-known that $\mathbb{K}_\infty := \mathbb{F}_q((X^{-1}))$ is the completion of $\mathbb{K} := \mathbb{F}_q(X)$ for the valuation at ∞, *i.e.*, for every $\alpha = \frac{A}{B} \in \mathbb{K}$ let

$$\nu(\alpha) = \nu_\infty(\alpha) := \deg B - \deg A$$

be the valuation at ∞ (the inverse degree valuation). Let $\mathbb{L} = \mathbb{F}_q(X, Y)/p\mathbb{F}_q(X, Y)$ be an extension of degree n. We assume that $\mathcal{S}$ is the ring of integers of $\mathbb{L}$. We denote by ω the extension of ν to $\mathbb{L}$ and by $\mathbb{L}_\infty$ the completion of $\mathbb{L}$ for ω.

In order to get an extension of the degree in $\mathbb{L}$ we put for every $\alpha \in \mathbb{L}_\infty$,

$$d(\alpha) := -\omega(\alpha).$$

It is clear by the definition of d that $d(A) = \deg(A)$ for every $A \in \mathbb{F}_q[X]$.

For any positive integer m and a subset $\mathcal{T} \subset \mathbb{L}$ we define

$$\mathcal{T}(m) := \{A \in \mathcal{T} : d(A) \leq m\}. \tag{2.2}$$

Our first result is a generalization of Kim's result to these number systems.

Theorem 2.3. *For $i \in \{1, \ldots, r\}$ let $(B_i, \mathcal{N}_i)$ be number systems in $\mathcal{S}$. Let $f_i : \mathcal{S} \to \mathcal{S}$ be a B_i-additive function with coprime B_i for $i = 1, \ldots, r$. Furthermore let M_i be ideals in $\mathcal{S}$, $\mathcal{M}_i$ be any set of representatives of the congruence classes of M_i for $i \in \{1, \ldots, r\}$.*

Set

$$\mathcal{H} := \{(f_1(A) \bmod M_1, \ldots, f_r(A) \bmod M_r) : A \in \mathcal{S}\}.$$

Then $\mathcal{H}$ is isomorphic to a subgroup of $M_1 \times \cdots \times M_r$ and for every $(H_1, \ldots, H_r) \in \mathcal{H}$ we have

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{S}(n)} \# \{A \in \mathcal{S}(n) : f_1(A) \equiv H_1 \bmod M_1, \ldots, f_r(A) \equiv H_r \bmod M_r\} = \frac{1}{|\mathcal{H}|}.$$

Furthermore we get an equivalent result for the theorem of Bassily and Kátai.

Theorem 2.4. *Let $(B, \mathcal{N})$ be a number system in $\mathcal{S}$ with $d(B) = \frac{a}{b}$ and let $f : \mathcal{S} \to \mathbb{R}$ be a strictly B-additive function. Set*

$$\mu_f := \frac{1}{\#\mathcal{N}} \sum_{D \in \mathcal{N}} f(A) \quad \text{and} \quad \sigma_f^2 := \frac{1}{\#\mathcal{N}} \sum_{D\in\mathcal{N}} f(A)^2 - \mu_f^2.$$

Let $h \in \mathbb{L}_\infty[Z]$ be a polynomial of degree r. Suppose that $\sigma_f > 0$ and $\mathcal{S}$ is the ring of integers of $\mathbb{L}$, then for $n \to \infty$

$$\# \left\{ A \in \mathcal{S}(n) : \frac{f(h(A)) - \frac{nrb}{a}\mu_f}{\sqrt{\frac{nrb}{a}}\sigma_f} \leq x \right\} \to \Phi(x),$$

where Φ denotes the standard normal distribution function.

In the same way as above we want to apply this result in order to solve Waring's Problem. Therefore we first need a definition of Weyl sums in this setting.

Let Tr and N be the trace and the norm of an element in $\mathbb{L}_\infty$ over $\mathbb{K}_\infty$ and Res be the residue of an element of $\mathbb{F}_q((X^{-1}))$, *i.e.*,

$$\operatorname{Res}\left(\sum_{j\in\mathbb{Z}} a_j X^j\right) = a_{-1}.$$

In this paper exponential sums with digital restrictions form an important tool. To define such sums properly we need additive characters. Let ψ be a non-principal character on $\mathbb{F}_q$. Then we define a character E on $\mathbb{L}_\infty$ by

$$E(x) := \psi\left(\operatorname{Res} \circ \operatorname{Tr}(x)\right). \tag{2.3}$$

Now we can state the result concerning Weyl sums.

Theorem 2.5. *For $i \in \{1, \ldots, r\}$ let $(B_i, \mathcal{N}_i)$ be number systems in $\mathcal{S}$ with $d(B_i) = \frac{a_i}{b_i}$ and $\#\mathcal{N}_i = q^{d_i}$. Let $h \in \mathbb{L}_\infty[Z]$ be a polynomial of degree $k < \operatorname{char} \mathbb{F}_q$ and $f_i : \mathcal{S} \to \mathcal{S}$ be a B_i-additive function with coprime B_i for $i = 1, \ldots, r$. Furthermore let M_i be ideals in $\mathcal{S}$ and $\mathcal{M}_i$ be any set of representatives of the congruence classes of M_i for $i \in \{1, \ldots, r\}$.*

If there exist $\ell \in \{1, \ldots, r\}$ and $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ such that

$$\left| q^{-d_\ell b_\ell} \sum_{A \in \mathcal{L}_{B_\ell}(b_\ell)} E\left(\frac{R_\ell}{M_\ell} \Delta_k(f_\ell(A); \mathbf{H})\right) \right| < 1,$$

then there exists a constant $\gamma > 0$ depending only on f_ℓ and B_ℓ such that

$$\sum_{A\in\mathcal{S}(n)} E\left(h(A)+\sum_{i=1}^{r}\frac{R_i}{M_i}f_i(A)\right) \ll (\#\mathcal{S}(n))^{1-\frac{k+2}{2^{k+1}}-\gamma}.$$

With help of these estimates we can solve Waring's Problem in our setting.

Theorem 2.6. *For $i \in \{1,\dots,r\}$ let $(B_i,\mathcal{N}_i)$ be number systems in $\mathcal{S}$ with $d(B_i) = \frac{a_i}{b_i}$ and $\#\mathcal{N}_i = q^{d_i}$. Let $f_i : \mathcal{S} \to \mathcal{S}$ be B_i-additive functions for $i \in \{1,\dots,r\}$. Choose ideals M_i of $\mathcal{S}$ and let $\mathcal{M}_i$ be any set of representatives of the congruence classes of M_i for $i \in \{1,\dots,r\}$.*

Assume that $\mathcal{S}$ is the ring of integers of $\mathbb{L}$ and that for every $\mathbf{0} \neq \mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ there exist $\ell \in \{1,\dots,r\}$ and $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ such that

$$\left| q^{-d_\ell b_\ell} \sum_{A\in\mathcal{L}_{B_\ell}(b_\ell)} E\left(\sum_{i=1}^{r}\frac{R_i}{M_i}\Delta_k(f_i(A);\mathbf{H})\right)\right| < 1.$$

Let $0 < k < \operatorname{char}\mathbb{F}_q$ and s be an integer such that $s > 2^k$. Then every $N \in \mathcal{S}$, such that $d(N)$ is sufficiently large, admits a representation as sum of k-th powers of the form

$$N = P_1^k + \cdots + P_s^k$$

with $P_j \in \mathcal{S}(\lceil d(N)/k\rceil)$ and $f_i(P_j) \equiv J_i \bmod M_i$ for $i = 1,\dots,r$ and $j = 1,\dots,s$.

Remark 2.7. The restriction $s > 2^k$ originates from Waring's Problem without digital restrictions. In order to sharpen this bound, one needs a better understanding of the unrestricted problem.

The paper is organized as follows. In Section 3 we collect some basic facts about number systems in $\mathcal{S}$. Each of the subsequent sections will be devoted to the proof of one of our results. The proofs of our results are based on the proofs of the corresponding results for number systems in $\mathbb{F}_q[X]$ in the sense of Kovács and Pethő [10]. In particular, the proofs of Theorem 2.3 and Theorem 2.4 will follow Drmota and Gutenbrunner [6], the proof of Theorem 2.5 will follow Madritsch and Thuswaldner [13], and the proof of Theorem 2.6 will follow Madritsch [12]. New difficulties occur in our more general setting. For instance, the "fundamental domains" of the number systems in $\mathcal{S}$ are no longer trivial.

3. Properties of Number Systems in $\mathcal{S}$

Since the characterization of Scheicher and Thuswaldner (Proposition 2.1) deals only with the case of $B = Y$ we need to generalize this to arbitrary bases.

Proposition 3.1. *The pair $(B,\mathcal{N})$ is a number system in $\mathcal{S}$ if and only if there exists a polynomial $\tilde{p} \in \mathbb{F}_q[X,Z]$ and an $\mathbb{F}_q[X]$-isomorphism $\varphi : \mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z] \leftrightarrow \mathcal{S}$ such that Z is a basis of a number system in $\mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$ and $\varphi(Z) = B$.*

Proof. Let $(B,\mathcal{N})$ be a number system in $\mathcal{S}$. Then for $k = 1,\dots,d$ there exists r_k and $d_{i,j}$ with $i = 1,\dots,k$ and $j = 0,\dots,r_i$ such that

$$Y^k = d_{k,0} + d_{k,1}B + \cdots d_{k,r_k}B^{r_k}.$$

Since the $d_{i,j} \in \mathbb{F}_q[X]$ we get that there exists a polynomial $\tilde{p} \in \mathbb{F}_q[X,Z]$ such that $\tilde{p}(X,B) = p(X,Y)$. By setting $\varphi(Z) = B$ and $\varphi(d) = d$ for $d \in \mathcal{N}$ we get that φ is an isomorphism because of $(B,\mathcal{N})$ being a number system. In order to show that Z is also a basis we choose an element $s \in \mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$. Then

$$\varphi(s) = \sum_{k\geq 0} d_k B^k$$

which implies that

$$s = \sum_{k\geq 0} d_k Z^k.$$

Thus $(Z,\mathcal{N})$ is a number system in $\mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$.

For the contrary assume that there exists a polynomial $\tilde{p} \in \mathbb{F}_q[X,Z]$ together with an isomorphism φ and $(Z,\mathcal{N})$ is a number system in $\mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$. Set $B := \varphi(Z) \in \mathcal{S}$. Then every element $s \in \mathcal{S}$ gives rise to a representation

$$\varphi^{-1}(s) = \sum_{k\geq 0} d_k Z^k.$$

Following the isomorphism back we get that

$$s = \sum_{k\geq 0} d_k B^k.$$

□

Remark 3.2. It follows from the proof that the set of digits $\mathcal{N}$ is the same for both number systems. Thus in view of Proposition 2.1 we get that for every B there exists a d such that

$$\mathcal{N} := \{A \in \mathbb{F}_q[X] : d(A) < d\}$$

and the pairs $(B,\mathcal{N})$ and $(Z,\mathcal{N})$ are number systems in $\mathcal{S}$ and $\mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$, respectively.

Since this is very important for our considerations we want to illustrate this by the following example.

Example 3.3. Let $p := Y^2 + XY + X^4 + X^2$ and let $B = Y + X^2 + X$ be the basis of a number system in $\mathbb{F}_q[X,Y]/p\mathbb{F}_q[X,Y]$. Now by Proposition 3.1 it is sufficient to show that Z is the basis of a number system in $\tilde{p}$. Therefore we set as in the proposition $\varphi(B) = Z$ and get that $\tilde{p} = Z^2 + XZ + X^2$. By Example 2.2 we get that Z is a basis of a number system in $\mathbb{F}_q[X,Z]/\tilde{p}\mathbb{F}_q[X,Z]$.

In view of Remark 3.2 we get that for both number systems the set of digits is $\mathcal{N} = \{A \in \mathbb{F}_q[X] : d(A) < 4\}$.

The next thing we need in connection with the number systems is an estimation of the length of the expansion. Since our goal is to show distributional results, we have to be sure to count the elements in an appropriate way. Above in (2.2) we therefore defined the notation $\mathcal{S}(m)$, which will be justified by the following proposition.

Proposition 3.4. *Let $(B,\mathcal{N})$ be a number system in $\mathcal{S}$. Then for any $G \in \mathcal{S} \setminus \{0\}$ we have*

$$\left| L_B(G) - \frac{d(G)}{d(B)} \right| \leq c,$$

where c is a constant depending on B and $\mathcal{N}$.

Proof. The idea of this proof is based on the proof of the main result of [11], where the analogous result for number systems in algebraic number fields is shown.

Let $G \in \mathcal{S} \setminus \{0\}$ be arbitrary and let

$$G = D_0 + D_1 B + \cdots + D_k B^k \quad \text{with} \quad D_i \in \mathcal{N}$$

be its B-adic representation. Note that $d(B) > 0$ because otherwise by inspecting the B-adic representation of G we would have $d(G) \leq d$ where $d := \max_{D\in\mathcal{N}} d(D)$. Since G was chosen arbitrary, this is absurd.

As $d(B) > 0$ we get from the B-adic representation of G that

$$d(G) = \max_{i=0}^{k} d(D_i B^i) = \max_{i=0}^{k}(d(D_i) + i \cdot d(B)) \leq d + k\, d(B).$$

Thus

$$L_B(G) = k + 1 \geq \frac{d(G) - d}{d(B)},$$

which establishes the lower bound.

For the upper bound we let $G \in \mathcal{S} \setminus \{0\}$ and let $k \geq 1$ be such that

$$(k-1) \cdot d(B) \leq d(G) < k \cdot d(B). \tag{3.1}$$

Then there exists an $G' \in \mathcal{S}$ such that

$$G = \sum_{i=0}^{k-1} D_i B^i + G' B^k$$

with $D_i \in \mathcal{N}$ for $i = 0, \ldots, k-1$. Applying the degree function on both sides and using (3.1) yields

$$d(G') \leq d(G) - k \cdot d(B) + c \leq c,$$

where $c > 0$ is a constant depending on B and $\mathcal{N}$. Now let $L := \max_{A \in \mathcal{S}(c)} L_B(A)$ be the maximal length of elements of degree not bigger than c. Thus we have, using (3.1) again,

$$L_B(G) \leq (k-1) + 1 + L \leq \frac{d(G)}{d(B)} + L + 1.$$

□

4. Distribution in Residue Classes

In this section we want to prove Theorem 2.3. But before we get straight into it we have to state some preliminary lemmas.

4.1. **Preliminary Lemmas.**

Our first lemma is a consideration of so-called complete exponential sums in $\mathbb{L}$ where the character E is defined in (2.3).

Lemma 4.1 ([4, Corollary II.3.2]). *Let $R \in \mathcal{S}$. Furthermore let M be an ideal and $\mathcal{M}$ a complete set of residues modulo M. Then*

$$\sum_{A \in \mathcal{M}} E\left(\frac{R}{M} A\right) = \begin{cases} \mathrm{N}(M) & \text{if } R = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that N is the norm of an element of $\mathbb{L}$ over $\mathbb{K}$. For $k \geq 0$ we recursively define the k-times difference function Δ_k by

$$\Delta_0(f(A)) = f(A),$$
$$\Delta_{k+1}(f(A); H_1, \ldots, H_{k+1}) = \Delta_k(f(A + H_{k+1}); H_1, \ldots, H_k) - \Delta_k(f(A); H_1, \ldots, H_k)$$

The next lemma is a version of the Weyl-van der Corput inequality for the field $\mathbb{L}$.

Lemma 4.2. *Let k be a positive integer and $\mathcal{R}$ be a finite subset of $\mathcal{S}$. Then*

$$\left| \sum_{A \in \mathcal{R}} E(p(A)) \right|^{2^k} = (\#\mathcal{R})^{2^k - k - 1} \sum_{H_1 \in \mathcal{R}} \cdots \sum_{H_k \in \mathcal{R}} \sum_{A \in \mathcal{R}} E(\Delta_k(p(A); H_1, \ldots, H_k)).$$

Proof. The proof is the same as in the classical case (see for instance [1, Chapter IV, §5]). □

Finally we need a lemma to treat the different bases.

Lemma 4.3. *Let $(B, \mathcal{N})$ be a number system in $\mathcal{S}$ with $\#\mathcal{N} = q^d$. Let f be a completely B-additive function, and $t \in \mathbb{N}$, $K, R \in \mathcal{S}$ with $L_B(R), L_B(K) < t \cdot d$. Then for all $N \in \mathcal{S}$ satisfying $N \equiv R \bmod B^t$ we have*

$$f(N + K) - f(N) = f(R + K) - f(R).$$

Proof. This is analogous to the proof of [4, Lemma 3]. □

4.2. **The fundamental domain.** Let $(B,\mathcal{N})$ be a number system in $\mathcal{S}$. Then by the Theorem of Puiseux (*cf.* Theorem 4.1.1 of [5]) we get that there exist $a,b\in\mathbb{N}$ such that

$$d(B)=\frac{a}{b}.$$

Before we start proving our higher correlation result we have to consider the internal structure of $\mathcal{S}(n)$ in connection with the number system $(B,\mathcal{N})$. Assume that $\#\mathcal{N}=q^d$. If $R\in\mathcal{L}_B(b)$ we get

$$d(R)=d(D_{b-1}B^{b-1}+\cdots+D_1B+D_0)=\max_{i=0}^{b-1}\left(\deg(D_i)+i\frac{a}{b}\right)\leq(d-1)+a-\frac{a}{b}.$$

Assuming that $n\geq(d-1)+a-\frac{a}{b}$ this implies that

$$\mathcal{S}(n)=\{A\in\mathcal{S}:d(A)\leq n\}=\left\{PB^b+R\in\mathcal{S}\mid P\in\mathcal{S}(n-a),R\in\mathcal{L}_B(b)\right\}. \tag{4.1}$$

Remark 4.4. In our case the fundamental domain consists of all elements G with negative degree $d(G)$. In contrast Scheicher and Thuswaldner [15] let the fundamental domain consist of all elements G with only negative exponents in their B-adic representation. We will adopt their ideas in order to fit our circumstances.

Thus we define the fundamental domain $\mathcal{F}$ of a number system $(B,\mathcal{N})$ by

$$\mathcal{F}:=\{\alpha\in\mathbb{L}_\infty:d(\alpha)<0\}. \tag{4.2}$$

4.3. **Higher Correlation.** For the rest of this section let $(B_i,\mathcal{N}_i)$, $1\leq i\leq r$ with $d_i=1+\max_{D\in\mathcal{N}_i}\deg D$ be number systems in $\mathcal{S}$ with coprime bases and let f_i be B_i-additive functions. Let

$$d(B_i)=\frac{a_i}{b_i}\quad\text{for } i=1,\dots,r.$$

Furthermore let M_i, $1\leq i\leq r$ be ideals of $\mathcal{S}$ and let $\mathcal{M}_i$ be a complete set of residues modulo M_i, respectively. We define for $\mathbf{R}=(R_1,\dots,R_r)\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r$ and $\mathbf{H}\in\mathcal{S}^k$

$$\begin{aligned} g_{\mathbf{R},i,k}(A;\mathbf{H})=g_{i,k}(A;\mathbf{H})&:=E\left(\frac{R_i}{M_i}\Delta_k(f_i(A);\mathbf{H})\right),\\ g_{\mathbf{R},k}(A;\mathbf{H})=g_k(A;\mathbf{H})&:=\prod_{i=1}^r g_{i,k}(A;\mathbf{H}). \end{aligned} \tag{4.3}$$

We will omit the $\mathbf{R}$ (respectively the R_i) in the index of g if this omission causes no confusion.

In order to show our correlation results we define the following functions.

$$\begin{aligned} \Phi_{i,k}(\mathbf{H};n)&:=\frac{1}{\#\mathcal{S}(n)}\sum_{A\in\mathcal{S}(n)}g_{i,k}(A;\mathbf{H}),\\ \Psi_{i,k}(\mathbf{h};n)&:=\prod_{j=1}^k(\#\mathcal{S}(h_j))^{-1}\sum_{H_1\in\mathcal{S}(h_1)}\cdots\sum_{H_k\in\mathcal{S}(h_k)}\left|\Phi_{i,k}(\mathbf{H};n)\right|^2,\\ \Lambda_{i,k}(\mathbf{H})&:=q^{-d_ib_i}\sum_{A\in\mathcal{L}_{B_i}(b_i)}g_{i,k}(A;\mathbf{H}). \end{aligned} \tag{4.4}$$

Furthermore we denote by Φ_k and Ψ_k the corresponding correlations with $g_{i,k}$ replaced by g_k.

Note that $\Lambda_{i,k}$ is needed because the fundamental domains in our setting are non-trivial. This is reflected by (4.1) and (4.2).

We are now in a position to state our correlation result.

Proposition 4.5. *Let $h_1,\dots,h_k,n$ be positive integers. Then for every $0\neq\mathbf{R}\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r$ either*

$$\forall A\in\mathcal{S}:g_0(A)=E\left(\sum_{i=1}^r\frac{R_i}{M_i}f_i(A)\right)=1$$

or there exist $i \in \{1, \ldots, k\}$ *and an* $\mathbf{H} \in \mathcal{L}_{B_i}(b_i)^k$ *such that* $|\Lambda_{i,k}(\mathbf{H})| < 1$ *and*

$$\Psi_k(\mathbf{h}; n) \ll \exp\left(-\min(h_1, \ldots, h_k, n) \frac{1 - |\Lambda(\mathbf{H}; 1)|^2}{a_i q^{d_i b_i}}\right).$$

Before we start with the proof we want to take a closer look at those $\mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ such that $g_{\mathbf{R},0}(A) = 1$ for all $A \in \mathcal{S}$. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be such that $g_{\mathbf{R}_1,0}(A) = g_{\mathbf{R}_2,0}(A) = 1$. Then

$$\begin{aligned} g_{\mathbf{R}_1+\mathbf{R}_2,0}(A) &= E\left(\sum_{i=1}^{r} \frac{R_{1,i} + R_{2,i}}{M_i} f_i(A)\right) \\ &= E\left(\sum_{i=1}^{r} \frac{R_{1,i}}{M_i} f_i(A) + \frac{R_{2,i}}{M_i} f_i(A)\right) = g_{\mathbf{R}_1,0}(A) g_{\mathbf{R}_2,0}(A) = 1. \end{aligned}$$

Thus we get that together with the identity element 0 these $\mathbf{R}$ form a group by component addition. We denote this group by

$$\mathcal{G} := \{\mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r : \forall A \in \mathcal{S} : g_{\mathbf{R},0}(A) = 0\} \tag{4.5}$$

The proof of Proposition 4.5 is in two steps. First we assume that $r = 1$. Secondly we reduce the general case to the case $r = 1$.

Lemma 4.6. *Let* $k < \operatorname{char}(\mathbb{F}_q)$ *and* h *be positive integers. Fix an* $\ell \in \{1, \ldots, r\}$ *and* $R \in \mathcal{M}_\ell$. *If there exists an* $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ *such that* $|\Lambda_{\ell,k}(\mathbf{H})| < 1$, *then*

$$\Psi_{\ell,k}(\mathbf{h}; n) \ll \exp\left(-\min(h_1, \ldots, h_k, n) \frac{1 - |\Lambda_{\ell,k}(\mathbf{H})|^2}{a_\ell q^{d_\ell b_\ell}}\right).$$

Proof. We fix an $R \in \mathcal{M}_\ell$. As ℓ and k are fixed throughout the proof we drop the indices,*i.e.*, we set $B := B_\ell$, $\Psi := \Psi_{\ell,k}$, $\Phi := \Phi_{\ell,k}$, $\Lambda_{\ell,k} := \Lambda$, $g := g_{\ell,k}$, $f := f_\ell$, $d := d_\ell$, $a := a_\ell$, and $b := b_\ell$.

Following the proof of [6, Lemma 4] together with our observation in (4.1) we easily get that

$$\Phi(\mathbf{P}B^b + \mathbf{R}; n) = \Phi(\mathbf{P}; n - a)\Lambda(\mathbf{R}; b)$$

holds. We set

$$\Xi := q^{-kdb} \sum_{R_1 \in \mathcal{L}_B(b)} \cdots \sum_{R_k \in \mathcal{L}_B(b)} |\Lambda(\mathbf{R}, b)|^2.$$

This yields

$$\Psi(\mathbf{h}; n) = \Psi(\mathbf{h} - a; n - a)\Xi,$$

where $\mathbf{h} - a := (h_1 - a, \ldots, h_k - a)$.

By iteration we derive for $s \leq \min(h_1, \ldots, h_k, n)/a$

$$\Psi(\mathbf{h}; n) = \Psi(\mathbf{h} - sa; n - sa)\Xi^s.$$

By the trivial estimation of g we get that $|\Psi(\mathbf{h}; n)| \leq 1$ for all $\mathbf{h}$ and n. This implies that $|\Psi(\mathbf{h}; n)| \leq |\Xi|^s$. Therefore we are left with estimating $|\Xi|$. By hypothesis there exists an $\mathbf{H} \in \mathcal{L}_B(b)^k$ with $|\Lambda(\mathbf{H}, b)| < 1$, yielding

$$\Xi \leq 1 - \frac{1 - |\Lambda(\mathbf{H}, b)|^2}{q^{db}} \ll \exp\left(-\frac{1 - |\Lambda(\mathbf{H}, b)|^2}{q^{db}}\right).$$

Inserting this in (4.3) we get that

$$|\Psi(\mathbf{h}; n)| \leq |\Xi|^s \ll \exp\left(-\min(h_1, \ldots, h_k, n) \frac{1 - |\Phi(\mathbf{H}, 1)|^2}{aq^{db}}\right)$$

and the lemma is proven. □

Remark 4.7. $|\Psi_{\ell,k}(\mathbf{h};1)| = 1$ is uncommon. Indeed, we get

$$\begin{aligned}
&\forall \mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k : |\Phi_{\ell,k}(\mathbf{H})| = 1\\
&\quad\Leftrightarrow \forall \mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k \, \forall A \in \mathcal{L}_{B_\ell}(b_\ell)^k : g_{\ell,k}(A;\mathbf{H}) \text{ is constant}\\
&\quad\Leftrightarrow \forall \mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k \, \forall A,B \in \mathcal{L}_{B_\ell}(b_\ell) :\\
&\quad\overline{g_{\ell,k-1}(A;\mathbf{H})} g_{\ell,k-1}(A+H_k;\mathbf{H}) = \overline{g_{\ell,k-1}(B;\mathbf{H})} g_{\ell,k-1}(B+H_k;\mathbf{H})\\
&\quad\Leftrightarrow \forall \mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^{k-1} \, \forall A,B \in \mathcal{L}_{B_\ell}(b_\ell) : g_{\ell,k-1}(A+B;\mathbf{H}) = g_{\ell,k-1}(A;\mathbf{H}) g_{\ell,k-1}(B;\mathbf{H})\\
&\quad\Leftrightarrow \forall A,B \in \mathcal{L}_{B_\ell}(b_\ell) : g_{\ell,0}(A+B) = g_{\ell,0}(A) g_{\ell,0}(B).
\end{aligned}$$

Thus

$$\exists \mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k : |\Phi_{\ell,k}(\mathbf{H};1)| < 1$$
$$\Longleftrightarrow$$
$$\exists A,B \in \mathcal{L}_{B_\ell}(b_\ell) : g_{\ell,0}(A+B) \neq g_{\ell,0}(A) g_{\ell,0}(B).$$

In the next lemma we want to generalize to the case $r \neq 1$ and therefore replace the $\Psi_{\ell,k}$ from above by Ψ_k.

Lemma 4.8. *Let $k < \operatorname{char}(\mathbb{F}_q)$ and h be positive integers and fix an $\mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r$. If there exist an ℓ and an $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ such that $|\Lambda_{\ell,k}(\mathbf{H})| < 1$, then*

$$\Psi_k(\mathbf{h};n) \ll \exp\left(-\min\{h_1,\dots,h_k,n\} \frac{1 - |\Lambda_{\ell,k}(\mathbf{H})|^2}{a_i q^{d_\ell b_\ell}}\right).$$

Proof. We will follow the proof of [13, Lemma 3.6]. The main difference here is that the "degrees" of the bases need not be integers and therefore we have to use a special treatment for them.

Let $\ell \in \{1,\dots,r\}$ be such that $|\Lambda_{\ell,k}(\mathbf{H})| < 1$. Then we want to reduce the estimation of $\Phi_k(\mathbf{h};n)$ to that of $\Phi_{\ell,k}(\mathbf{h};n)$ by trivially estimating the rest. Let $s = \frac{n}{3r}$ and choose t_i ($i \in \{1,\dots,r\}$) in a way such that $s_i = t_i d_i b_i$ satisfies the inequality $s \leq s_i \leq 2s$. Now we split the sum over all $A \in \mathcal{S}(n)$ up according to the congruence classes modulo $B_1^{t_1},\dots,B_r^{t_r}$. Therefore let $\mathcal{B}_i$ be a complete set of residues modulo $\mathcal{S}B_i^{t_i}$ for $i = 1,\dots,r$.

Thus for a given $\mathbf{C} \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_r$ we define

$$N_{\mathbf{C}} := \left\{A \in \mathcal{S}(n) : A \equiv C_1 \bmod B_1^{t_1}, \dots, A \equiv C_r \bmod B_r^{t_r}\right\}.$$

For $n \geq$ we get by the Chinese Remainder Theorem that

$$|N_{\mathbf{C}}| = \frac{\#\mathcal{S}(n)}{\prod_{i=1}^r q^{d_i b_i t_i}}.$$

By our choice of the B_j we can apply Lemma 4.3 and get

$$\Phi_k(\mathbf{H};n) = \prod_{i=1}^r \left(\Lambda_{i,k}(\mathbf{H})\right)^{t_i}.$$

Now we take the modulus and estimate $\Lambda_{i,k}(\mathbf{H})$ for $i \neq \ell$ trivially. Thus

$$|\Phi_k(\mathbf{H};n)| \leq \prod_{i=1}^r |\Lambda_{i,k}(\mathbf{H})|^{t_i} \leq |\Lambda_{\ell,k}(\mathbf{H})|^{t_i}.$$

In the same way we can estimate Ψ_k by $\Lambda_{\ell,k}$. Noting that $s_\ell \ll n \ll s_\ell$ we get by an application of Lemma 4.6 that

$$\Psi_k(\mathbf{h};n) \leq \Psi_{\ell,k}(\mathbf{h};b_\ell) \ll \exp\left(-\min\{h_1,\dots,h_k,n\} \frac{1 - |\Lambda_{\ell,k}(\mathbf{H})|^2}{a_\ell q^{d_\ell b_\ell}}\right).$$

□

Now we are ready to state the proof of the higher correlation result.

Proof of Proposition 4.5. By the assumptions of Lemma 4.8 we split the proof into two cases.

Case 1: There exist an ℓ and $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ such that $|\Lambda_{\ell,k}(\mathbf{H})| < 1$. Then we get the result by an application of Lemma 4.8.

Case 2: If for all $\ell \in \{1, \dots, r\}$ and all $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ we have $|\Lambda_{\ell,k}(\mathbf{H})| = 1$ then we get by Remark 4.7 that $g_{\ell,k}(A+B;\mathbf{H}) = g_{\ell,k}(A;\mathbf{H})g_{\ell,k}(B;\mathbf{H})$ and consequently by the B_ℓ-additivity of the f_ℓ ($\ell = 1, \dots, r$) for $A, B \in \mathcal{S}$

$$g_k(A+B;\mathbf{H}) = g_k(A;\mathbf{H})g_k(B;\mathbf{H}). \tag{4.6}$$

We distinguish between two cases:

Case 2.1: $g_0(A) = 1$ for every $A \in \mathcal{S}$. This is the first alternative in the proposition.

Case 2.2: There exists $A \in \mathcal{S}$ such that $g_0(A) \neq 1$. In this case the proof is exactly the same as the proof of case 2.2 in [6, p.136] or [13, p.889]. □

4.4. **Distribution Result.** In order to show Theorem 2.3 we need a further lemma.

Lemma 4.9. *For every* $\mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ *either*

$$\forall A \in \mathcal{S} : g_0(A) = E\left(\sum_{i=1}^r \frac{R_i}{M_i} f_i(A)\right) = 1$$

or

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{S}(n)} \sum_{A\in\mathcal{S}(n)} g_0(A) = 0$$

holds.

Proof. We only consider the case that there exists an $\mathbf{R} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ with $g_0(A) \neq 1$ as otherwise there is nothing to show.

The idea is to apply Lemma 4.2 with $k = 1$. By this lemma we have

$$\left|\sum_{A\in\mathcal{S}(n)} g_0(A)\right|^2 \leq (\#\mathcal{S}(n))^2 \sum_{H\in\mathcal{S}(n)} \sum_{A\in\mathcal{S}(n)} E(\Delta_1(f(A);H)).$$

Taking the modulus and squaring again together with Cauchy's inequality yields

$$\left|\sum_{A\in\mathcal{S}(n)} g_0(A)\right|^4 \leq (\#\mathcal{S}(n))^3 \sum_{H\in\mathcal{S}(n)} |\Phi_1(H;n)|^2 = (\#\mathcal{S}(n))^4\,\Psi_1(n;n).$$

Now an application of Proposition 4.5 proves the lemma. □

Proof of Theorem 2.3. We define the additive group

$$\mathcal{H}_0 := \left\{\mathbf{C} \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_r : \forall \mathbf{R} \in \mathcal{G} : E\left(\sum_{i=1}^r -\frac{R_iC_i}{M_i}\right) = 1\right\} \tag{4.7}$$

where $\mathcal{G}$ is the group defined in (4.5).

Then we use Lemma 4.1 to rewrite the problem and get

$$\begin{aligned}
&\frac{1}{\#\mathcal{S}(n)}\#\left\{A \in \mathcal{S}(n) : f_1(A) \equiv C_1 \bmod M_1, \dots, f_r(A) \equiv C_r \bmod M_r\right\}\\
&= \frac{1}{\#\mathcal{S}(n)} \sum_{A\in\mathcal{S}(n)} \prod_{i=1}^r \frac{1}{\mathrm{N}(M_i)} \sum_{R_i\in\mathcal{M}_i} E\left(\frac{R_i}{M_i}(f_i(A) - C_i)\right)\\
&= \frac{1}{\prod_{i=1}^r \mathrm{N}(M_i)} \sum_{\mathbf{R}\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r} E\left(\sum_{i=1}^r -\frac{R_iC_i}{M_i}\right) \frac{1}{\#\mathcal{S}(n)} \sum_{A\in\mathcal{S}(n)} g_0(A)\\
&= \frac{1}{\prod_{i=1}^r \mathrm{N}(M_i)} \sum_{\mathbf{R}\in\mathcal{G}} E\left(\sum_{i=1}^r -\frac{R_iC_i}{M_i}\right) + o(1),
\end{aligned} \tag{4.8}$$

where we have applied Lemma 4.9.

By the definition of $\mathcal{H}_0$ in (4.7) and since $\mathcal{G}$ is a group we have

$$\sum_{\mathbf{R}\in\mathcal{G}} E\left(\sum_{i=1}^{r} -\frac{R_iC_i}{M_i}\right) = \begin{cases} \#\mathcal{H}_0 & \text{if } \mathbf{C}\in\mathcal{H}_0,\\ 0 & \text{otherwise.}\end{cases}$$

Plugging this into (4.8) yields

$$\frac{1}{\#\mathcal{S}(n)}\#\left\{A\in\mathcal{S}(n) : f_1(A)\equiv C_1 \bmod M_1,\dots,f_r(A)\equiv C_r \bmod M_r\right\} = \frac{1}{\mathcal{H}_0}+o(1)$$

if $\mathcal{S}\in\mathcal{H}_0$.

Thus we are left with showing that $\mathcal{H}=\mathcal{H}_0$. If $\mathcal{C}\in\mathcal{H}_0$ then clearly $\mathcal{C}\in\mathcal{H}$. Conversely, if $\mathcal{C}\in\mathcal{H}$, then there exists an $A\in\mathcal{S}$ such that $f_1(A)\equiv C_1 \bmod M_1,\dots,f_r(A)\equiv C_r \bmod M_r$. In particular

$$g_0(A) = E\left(\sum_{i=1}^{r}\frac{R_i}{M_i}f_i(A)\right) = E\left(\sum_{i=1}^{r}\frac{R_iC_i}{M_i}\right).$$

Moreover, by Proposition 4.5, for every $\mathbf{R}\in\mathcal{G}$ we have $g_0(A)=1$ which implies that $\mathbf{C}\in\mathcal{H}_0$ and the theorem is proven. □

5. Asymptotic Distribution

After showing the distribution into residue classes we want to consider the asymptotic distribution of the values of a single B-additive function. Therefore we fix a B-additive function $f:\mathcal{S}\to\mathbb{R}$ throughout the section.

In order to show Theorem 2.4 we need a refinement of a Weyl inequality. Therefore we have to introduce some notation in the function field $\mathbb{L}$.

5.1. **Definitions.** Since we need some geometry of numbers let $\mathcal{D}$ be the differential of the extension $\mathbb{L}$ over $\mathbb{F}_q(X)$. Set

$$S(m) = r\cdot m, \tag{5.1}$$

where r is the ramification index of the extension $\mathbb{L}$ over $\mathbb{F}_q(X)$. Finally we denote by g the genus of this extension.

For the proof of the Weyl inequality we will need Diophantine approximation in the field $\mathbb{L}_\infty$. We assume that $\mathcal{S}$ is the ring of integers in $\mathbb{L}$ and $\rho^{(1)},\dots,\rho^{(n)}$ be an $\mathbb{F}_q[X]$-basis (integer basis) of $\mathcal{S}$. Then we denote by

$$d^\star(\rho) := \max_{i=1,\dots,n} d(\rho^{(i)}).$$

To show Theorem 2.4 we start with some preliminaries and follow Drmota and Gutenbrunner [6].

5.2. **Preliminaries.** The first lemma will help us to extract one digit from the B-digit representation.

Lemma 5.1. *Let $\alpha\in\mathbb{L}_\infty$ such that*

$$\alpha = \sum_{k\in\mathbb{Z}} D_kB^k.$$

Let $\mathcal{B}$ be a complete set of residues modulo $\mathcal{S}B$ and $D\in\mathcal{N}$. For $R\in\mathcal{B}$ we set

$$c_{R,D} := \frac{1}{\mathrm{N}(\mathcal{S}B)}E\left(-\frac{DR}{B}\right).$$

Then for $j\in\mathbb{Z}$

$$\sum_{R\in\mathcal{B}} c_{R,D}E\left(\frac{R}{B^{j+1}}\alpha\right) = \begin{cases} 1 & \text{if } D_j = D,\\ 0 & \text{if } D_j\neq D.\end{cases}$$

Proof. Easily follows from the proof of [6, Lemma 7]. □

Since the coefficients of the polynomial need not be in $\mathcal{S}$ we have to consider how Diophantine approximation can be established in $\mathbb{L}_\infty$.

Lemma 5.2 ([4, Proposition I.2.2]). *Let a be a sufficiently large integer. Then for every $\alpha \in \mathbb{L}_\infty$ there exist $H \in \mathcal{S} \setminus \{0\}$ and $G \in \mathcal{D}^{-1}$, such that*

$$d(H) \leq a, \quad d(H\alpha - G) \leq -a - \epsilon,$$

where ϵ is a constant depending only on $\mathbb{L}$.

As we mentioned above we will need a refinement of Weyl's inequality of Car [4]. In order to establish this we follow an idea of Hua [8]. Therefore we need two further tools. The first deals with the number of representations of a number as a product.

Lemma 5.3 ([4, Proposition I.4.3]). *Let j be a positive integer, $N \in \mathbb{N}$ and $W \in \mathcal{S}(jN)$. Let $\tau(j,N,W)$ be the number of solutions $(W_1, ..., W_j) \in \mathcal{S}(N)^j$ of the equation*

$$W = W_1 \cdots W_j$$

Then, for every real number $\varepsilon > 0$, there exists a constant β (depending only on j and ε) such that for every non-zero element $W \in \mathcal{S}(jN)$ one has

$$\tau(j,N,W) \leq \beta q^{\varepsilon S(N)}.$$

Lemma 5.4 ([4, Proposition II.3.3]). *Let $H \in \mathcal{S} \setminus \{0\}$, $G \in \mathcal{D}^{-1}$, $b \in \mathbb{Z}$. Furthermore let $\mathcal{R}$ be a complete set of residues modulo $\mathcal{S}H$. Then*

$$\sum_{R \in \mathcal{R}} \sum_{A \in \mathcal{S}(b)} E\left(\frac{G}{H} AR\right) = \mathrm{N}(\mathcal{S}H) \#(\mathcal{S}H(b)),$$

where

$$\mathcal{S}H(b) = \{\alpha \in \mathcal{S}H : d(\alpha) < b\}.$$

Finally we need an estimation of the number of elements in an ideal $I(m)$.

Lemma 5.5 ([4, Equation I.2.6]). *Let I be an ideal of $\mathcal{S}$. Then for $m \in \mathbb{Z}$ such that $f \cdot m \geq 2g-2$ we have*

$$\#I(m) = \{A \in I : d(A) < m\} = q^{1-g+S(m)} \mathrm{N}(I)^{-1}. \tag{5.2}$$

5.3. **Main tool.** We now develop the main tool needed in order to properly prove the asymptotic distribution result.

Lemma 5.6. *Let $h \in \mathbb{L}_\infty[Z]$ be a polynomial of degree $k \geq 1$, i.e.,*

$$h(Z) = \alpha_k Z^k + \cdots + \alpha_1 Z + \alpha_0.$$

If there exist $G \in \mathcal{D}^{-1}$ and $H \in \mathcal{S} \setminus \{0\}$ such that

$$\omega(H\alpha_k - G) \geq kn - n^{1/3} + d^\star(\rho) + \varepsilon,$$
$$n^{1/3} - d^\star(\rho) + 1 \leq d(H) \leq kn - n^{1/3} + d^\star(\rho),$$

then there exists a constant $c > 0$ such that

$$\frac{1}{\#\mathcal{S}(n)} \sum_{A \in \mathcal{S}(n)} E(h(A)) \ll \exp\left(-cn^{1/3}\right).$$

Proof. The proof is based on the proof of [4, Proposition II.3.6]. Therefore we only emphasize on the differences occurring in our setting.

First of all we apply Lemma 4.2 and Lemma 5.3 to get

$$\begin{aligned}
\left| \sum_{A \in \mathcal{S}(n)} E(h(A)) \right|^{2^{k-1}} &\leq (\#\mathcal{S}(n))^{2^{k-1}-k} \sum_{W_1 \in \mathcal{S}(n)} \cdots \sum_{W_{k-1} \in \mathcal{S}(n)} \sum_{A \in \mathcal{S}(n)} E\left(\Delta_{k-1}(h(A), \mathbf{H})\right) \\
&\leq (\#\mathcal{S}(n))^{2^{k-1}-k} \sum_{W \in \mathcal{S}((k-1)n)} \sum_{A \in \mathcal{S}(n)} \tau(k-1,n,W) E(k!\alpha_k WA) \\
&\leq (\#\mathcal{S}(n))^{2^{k-1}-k} \beta q^{\varepsilon S(n)} \sum_{W \in \mathcal{S}((k-1)n)} \sum_{A \in \mathcal{S}(n)} E(k!\alpha_k WA).
\end{aligned} \tag{5.3}$$

Now by Lemma 5.2 there exist H and G such that

$$d(H) \leq kn - n^{1/3} + d^\star(\rho), \quad \omega(H\alpha_k - G) \geq kn - n^{1/3} + d^\star(\rho).$$

We set

$$m := \max((k-1)n, d^\star(\rho) - e + d(H)),$$
$$c := \min(\omega(H\alpha_k - G) + d(H) - m - \epsilon - 1, n).$$

Following the proof of Proposition II.3.5 of [4] we reach at

$$W^\star := \sum_{W \in \mathcal{S}((k-1)n)} \sum_{A \in \mathcal{S}(n)} E\left(k!\alpha_k WA\right) \leq q^{1-g+S(m)} \#(\mathcal{S}H(c)).$$

We distinguish three cases according to the size of $d(H)$.

Case 1: $n^{1/3} - d^\star(\rho) + 1 \leq d(H) \leq n$. We easily get that $m = (k-1)n$ and $c = n$. Thus

$$W^\star \leq q^{1-g+S((k-1)n)} q^{1+S(c-d(H))} \leq q^{2-g+S(d^\star(\rho)-1)} q^{S(kn-n^{1/3})}.$$

Case 2: $n < d(H) \leq (k-1)n - d^\star(\rho) + e$. Calculations give us that $m = (k-1)n$ and $c = n$. Since $c = n < d(H)$ we get that $\#(\mathcal{S}H(c)) = 1$ and therefore

$$W^\star \leq q^{1-g} q^{S(kn-n)}.$$

Case 3: $(k-1)n - d^\star(\rho) + e < d(H) \leq kn - n^{1/3} + d^\star(\rho)$. In this case $m = d^\star(\rho) - e + d(H)$ and $c = n$. Thus

$$W^\star \leq q^{1-g+S(2d^\star(\rho)-e)} q^{S(kn-n^{1/3})}.$$

Plugging this into (5.3) we get that

$$\left| \sum_{A \in \mathcal{S}(n)} E\left(h(A)\right) \right|^{2^{k-1}} \ll (\#\mathcal{S}(n))^{2^{k-1}-k} q^{S(kn-n^{1/3}+\varepsilon)},$$

which together with (5.2) proves the lemma. □

Now we can easily deduce the main proposition of this section.

Proposition 5.7. *Let ℓ be a positive integer and $\frac{n^{1/3}}{d(B)} \leq j_1 < \cdots < j_m \leq \frac{kn-n^{1/3}}{d(B)}$. Then*

$$\frac{1}{\#\mathcal{S}(n)} \#\left\{A \in \mathcal{S}(n) : D_{j_1}(h(A)) = D_1, \ldots, D_{j_m}(h(A)) = D_m\right\} = \frac{1}{|\mathcal{N}|^m} + \mathcal{O}\left(\exp\left(-cn^{1/3}\right)\right)$$

Proof. By Lemma 5.1 we get that

$$\begin{aligned}
&\#\left\{A \in \mathcal{S}(n) : D_{B,j_1}(h(A)) = D_1, \ldots, D_{B,j_m}(h(A)) = D_m\right\} \\
&= \sum_{A \in \mathcal{S}(n)} \left(\sum_{R_1 \in \mathcal{B}} c_{R_1,D_1} E\left(\frac{R_1}{B^{j_1+1}} h(A)\right)\right) \cdots \left(\sum_{R_m \in \mathcal{B}} c_{R_m,D_m} E\left(\frac{R_m}{B^{j_m+1}} h(A)\right)\right) \\
&= c_{0,D_1} \cdots c_{0,D_m} + \sum_{R_1,\ldots,R_m \in \mathcal{B}}{}' c_{R_1,D_1} \cdots c_{R_m,D_m} \sum_{A \in \mathcal{S}(n)} E\left(\left(\frac{R_1}{B^{j_1+1}} + \cdots + \frac{R_m}{B^{j_m+1}}\right) h(A)\right),
\end{aligned}$$

where $\sum'$ denotes the sum over all elements $(R_1, \ldots, R_m) \neq \mathbf{0}$.

Now we fix $(R_1, \ldots, R_m) \neq \mathbf{0}$ and set

$$R = R_1 + R_2 B^{j_2 - j_1} + \cdots + R_m B^{j_m - j_1}.$$

Thus we have to estimate

$$\sum_{A \in \mathcal{S}(n)} E\left(\frac{R}{B^{j_1+1}} P(A)\right).$$

We want to apply Lemma 5.6 and therefore write ξ for the leading coeficient of $RP(A)$. Then by an application of Lemma 5.2 we get that there exist $A \in \mathcal{D}^{-1}$ and $Q \in \mathcal{S}$ such that

$$d(Q) \le (k-1)n - n^{1/3} + d^\star(\rho),$$
$$d\left(\frac{\xi}{B^{j_1+1}}Q - A\right) \le -(k-1)n + n^{1/3} - d^\star(\rho) - \epsilon.$$

Now we distinguish two cases according to the size of $d(Q)$.

Case 1: $n^{1/3} - d^\star(\rho) - 1 \le d(Q) \le (k-1)n - n^{1/3} + d^\star(\rho)$. In this case we can apply Lemma 5.6 and get

$$\sum_{A\in\mathcal{S}(n)} E\left(\frac{R}{B^{j_1+1}}P(A)\right) \ll (\#\mathcal{S}(n))\exp\left(-cn^{1/3}\right).$$

Case 2: $0 \le d(Q) \le n^{1/3} - d^\star(\rho) - 1$. We want to show that this is actually not possible. Therefore we further distinguish two cases according to the size of $d(\xi)-(j_1+1)d(B)+d(Q)$.

Case 2.1: $d(\xi) - (j_1+1)d(B) + d(Q) \ge D$. In this case we get

$$j_1 + 1 \le \frac{d(\xi) + d(Q) - D}{d(B)} \ll \frac{d(Q)}{d(B)} \le \frac{n^{1/3}}{d(B)}$$

contradicting the lower bound.

Case 2.2: $d(\xi) - (j_1+1)d(B) + d(Q) < D$. Now we immedeately get that A must be 0. Thus we have

$$d\left(\frac{\xi}{B^{j_1+1}}Q\right) \le -(k-1)n + n^{1/3} - d^\star(\rho) - \epsilon$$

which implies

$$j_1 + 1 \ge \frac{(k-1)n - n^{1/3} + d^\star(\rho) + \epsilon + d(\xi) + d(Q)}{d(B)} \gg \frac{(k-1)n - n^{1/3}}{d(B)}$$

contradicting the upper bound.

Therefore we only may apply Lemma 5.6 and derive the desired result. □

5.4. **Weak Convergence.** We want to show Theorem 2.4 by comparing the gained distribution with the one of independent identically distributed random variables. Let $Y_0, Y_1, \ldots$ be iid random variables on $\mathcal{N}$ such that $\mathbb{P}[Y_i = D] = |\mathcal{N}|^{-1}$. Thus Proposition 5.7 can be seen as

$$\frac{1}{|\mathcal{N}|^m}\#\left\{A\in\mathcal{S}(n) : D_{Q,j_1}(h(A)) = D_1, \ldots, D_{Q,j_m}(h(A)) = D_m\right\}$$
$$= \mathbb{P}\left[Y_{j_1} = D_1, \ldots, Y_{j_m} = D_m\right] + \mathcal{O}\left(\exp\left(-cn^{1/3}\right)\right).$$

In fact we want to show that the moments are the same and have to consider that we shrank our scope to $\frac{n^{1/3}}{d(B)} \le j_1 < \cdots < j_\ell \le \frac{kn-n^{1/3}}{d(B)}$. Thus we need to show that the moment method holds also for our truncated version. This will be provided by the following lemma.

Lemma 5.8. *Let $(B,\mathcal{N})$ be a number system in $\mathcal{S}$ and g be a B-additive function. Set*

$$\mu = \frac{1}{|\mathcal{N}|}\sum_{D\in\mathcal{N}} g(D) = \mathbb{E}g(Y_j).$$

Then the m-th (central) moment of $\tilde{g}(P(A))$ is given by

$$\frac{1}{|\mathcal{S}(n)|}\sum_{A\in\mathcal{S}(n)}\left(\tilde{g}(P(A)) - \left(\frac{kn-n^{1/3}}{d(B)}\right)\mu\right)^m$$
$$= \mathbb{E}\left(\sum_{\frac{n^{1/3}}{d(B)}\le j\le\frac{kn-n^{1/3}}{d(B)}}(g(Y_j)-\mu)\right)^m + \mathcal{O}\left(n^m\exp\left(-cn^{1/3}\right)\right).$$

We truncate our B-additive function f as follows.

$$\tilde{f}(h(A)) := \sum_{\frac{n^{1/3}}{d(B)} \leq k \leq \frac{kn-n^{1/3}}{d(B)}} f(D_k(h(A))).$$

Thus it follows from Lemma 5.8 that

$$\frac{1}{\#\mathcal{S}(n)} \# \left\{ A \in \mathcal{S}(n) : \frac{\tilde{f}(h(A)) - \frac{kn-2n^{1/3}}{d(B)}\mu}{\sqrt{\frac{kn-2n^{1/3}}{d(B)}}\sigma} \leq x \right\} = \Phi(x) + o(1).$$

Since

$$\left| \tilde{f}(h(A)) - f(h(A)) \right| \ll n^{1/3}$$

we also get that

$$\frac{1}{\#\mathcal{S}(n)} \# \left\{ A \in \mathcal{S}(n) : \frac{f(h(A)) - \frac{kn}{d(B)}\mu}{\sqrt{\frac{kn}{d(B)}}\sigma} \leq x \right\} = \Phi(x) + o(1).$$

6. Weyl Sums with Digital Restrictions

In this section we want to prove Theorem 2.5. The idea is to do Weyl differentiation and apply Proposition 4.5. Our aim is to estimate

$$S_n(h) := \sum_{A \in \mathcal{S}(n)} E\left(h(A) + \sum_{i=1}^{r} \frac{R_i}{M_i} f_i(A) \right),$$

where $h \in \mathbb{L}_\infty[Z]$ is a polynomial of degree $k < \mathrm{char}\mathbb{F}_q$.

By hypotheses there exist an ℓ and $\mathbf{H} \in \mathcal{L}_{B_\ell}(b_\ell)^k$ with $|\Lambda_{\ell,k}(\mathbf{H})| < 1$. We set

$$\varphi(A) := h(A) + \sum_{i=1}^{r} \frac{R_i}{M_i} f_i(A). \tag{6.1}$$

Then we apply Weyl's method (Lemma 4.2) to get the following estimation.

$$|S_n(h)|^{2^k} \leq (\#\mathcal{S}(n))^{2^k-k-1} \sum_{P_1 \in \mathcal{S}(n)} \cdots \sum_{P_k \in \mathcal{S}(n)} \sum_{A \in \mathcal{S}(n)} E(\Delta_k(\varphi(A); \mathbf{P}))$$

We have to consider the k-th difference operator of φ. By linearity of the difference operator and the definitions of φ in (6.1) and $g_{\mathbf{R},k}$ in (4.3) we get

$$\begin{aligned} E(\Delta_k(\varphi(A); \mathbf{P})) &= E\left(\Delta_k(h(A); \mathbf{P}) + \Delta_k\left(\sum_{i=1}^{r} \frac{R_i}{M_i} f_i(A); \mathbf{P} \right) \right) \\ &= E\left(k! \alpha_k P_1 \cdots P_k\right) g_{\mathbf{R},k}(A; \mathbf{P}), \end{aligned}$$

where α_k is the leading coefficient of h. Thus

$$|S_n(\alpha)|^{2^k} \leq (\#\mathcal{S}(n))^{2^k-k-1} \sum_{P_1 \in \mathcal{S}(n)} \cdots \sum_{P_k \in \mathcal{S}(n)} E\left(k! \alpha_k P_1 \cdots P_k\right) \sum_{A \in \mathcal{S}(n)} g_{\mathbf{R},k}(A; \mathbf{P}).$$

Taking the modulus and shifting to the innermost sum yields together with the definition of Φ_k in (4.4)

$$|S_n(h)|^{2^k} \leq (\#\mathcal{S}(n))^{2^k-k-1} \sum_{P_1 \in \mathcal{S}(n)} \cdots \sum_{P_k \in \mathcal{S}(n)} |\Phi_k(\mathbf{P}; n)|.$$

We apply Cauchy's inequality to get the modulus squared

$$|S_n(h)|^{2^{k+1}} \leq (\#\mathcal{S}(n))^{2^{k+1}-k-2} \sum_{P_1 \in \mathcal{S}(n)} \cdots \sum_{P_k \in \mathcal{S}(n)} |\Phi_k(\mathbf{P}; n)|^2 = (\#\mathcal{S}(n))^{2^{k+1}-k-2} \Psi_k(\mathbf{n}; n).$$

Finally we apply Proposition 4.5 to estimate $\Psi_k(\mathbf{n};n)$. Thus

$$|S_n(h)|^{2^{k+1}} \ll (\#\mathcal{S}(n))^{2^{k+1}-k-2}\exp\left(-\frac{n}{a_\ell}\frac{1-|\Lambda_{\ell,k}(\mathbf{H})|^2}{q^{d_\ell b_\ell}}\right)$$

and therefore

$$S_n(h) \ll (\#\mathcal{S}(n))^{1-\frac{k+2}{2^{k+1}}-\gamma},$$

where $\gamma > 0$ is defined by

$$(\#\mathcal{S}(n))^{-2^{k+1}\gamma} = \exp\left(-\frac{n}{a_\ell}\frac{1-|\Lambda_{\ell,k}(\mathbf{H})|^2}{q^{d_\ell b_\ell}}\right). \tag{6.2}$$

7. Waring's Problem with Digital Restrictions

This section is devoted to the proof of Theorem 2.6. Therefore we first state the corresponding result without digital restrictions.

We say that a polynomial $N \in \mathcal{S}$ is the *strict* sum of k-th powers if it has a representation of the form

$$N = X_1^k + \cdots + X_s^k \quad (X_1,\ldots,X_s \in \mathcal{S}(m)), \tag{7.1}$$

where m is defined by

$$k(m-1) < d(N) \le km. \tag{7.2}$$

By $R(N,s,k)$ we denote the number of solutions of (7.1). Then Car [4] was able to show the following.

Proposition 7.1 ([4, Theorem])**.** *Let s be an integer such that $s \ge 1+2^k$. Then every $N \in \mathcal{S}$, such that $d(N)$ is sufficiently large, admits a strict representation as in* (7.1)*. Moreover one has an asymptotic estimate for the number $R(N,s,k)$ of these representations.*

$$R(N,s,k) = \mathfrak{S}_s(N)q^{(s-k)S(m)} + o(q^{(s-k)S(m)}),$$

where m is as in (7.2)*, $0 < \mathfrak{S}_s(N) \ll 1$ and s is defined in* (5.1)*.*

In our case we are interested in the number of solution of

$$N = X_1^k + \cdots + X_s^k \quad (X_1,\ldots,X_s \in \mathcal{S}(m)), \tag{7.3}$$

with $f_i(X_j) \equiv J_i \bmod M_i$ for $i = 1,\ldots,r$ and $j = 1,\ldots,s$. We denote the number of solutions of (7.3) by $R(N,s,k,\mathbf{f},\mathbf{J},\mathbf{M})$. The idea will be the reduction of this special case to the general one.

As in [4] we denote by $\mathfrak{P}$ the valuation ideal of ν and by $\mathfrak{M}$ the valuation ideal of ω. Furthermore we write $\mathfrak{P}^{\otimes n} := \mathfrak{P}\times\cdots\times\mathfrak{P}$, with $\mathfrak{P}$ repeated n times. Let $\rho := (\rho_1,\ldots,\rho_n)$ be an integral $\mathbb{F}_q[X]$-basis and $\gamma = (\gamma_1,\ldots,\gamma_n)$ its dual basis. Then γ is a basis for $\mathcal{D}^{-1}$ (*cf.* [16, Chapter III,§3]). We define $h\gamma$ to be the isomorphism

$$h\gamma(t_1,\ldots,t_n) = (t_1\gamma_1\rho_1,\ldots,t_n\gamma_n\rho_n).$$

We choose the Haar measures on $\mathbb{K}_\infty$ and $\mathbb{L}_\infty$ to be such that the values of the valuation ideals $\mathfrak{P}$ and $\mathfrak{M}$ equals 1, *i.e.* $\rho = \mathrm{d}x$ on $\mathbb{K}_\infty$ and μ on $\mathbb{L}_\infty$. We will always denote by $t = (t_1,\ldots,t_n)$ and element of $\mathbb{K}_\infty^n$ and by x one of $\mathbb{L}_\infty$. Finally on $\mathbb{K}_\infty^n$ we have the product measure $\rho^{\otimes n} = \mathrm{d}t_1\times\cdots\times\mathrm{d}t_n = \mathrm{d}t$.

In order to count the solutions we will use the following Lemma.

Lemma 7.2 ([4, Proposition I.3.1])**.** *Let $N \in \mathcal{S}$. Then*

$$\int_{\mathfrak{P}^{\otimes n}} E(h\gamma(t)\cdot N)\mathrm{d}t = \begin{cases} 1 & \textit{if } N = 0,\\ 0 & \textit{else.}\end{cases}$$

For short we set for $z\in\mathbb{L}_\infty$, $m\geq 0$, $1\leq i\leq s$, and $R\in\mathcal{D}^{-1}$

$$F(z,m)=\sum_{W\in\mathcal{S}(m)}E\left(zW^k\right),$$
$$S(z,m)=\sum_{\substack{W\in\mathcal{S}(m)\\ f_i(W)\equiv J_i \bmod M_i}}E\left(zW^k\right),$$

Thus we get the following integral representation for $R(N,s,k)$.

Lemma 7.3 ([4, Proposition II.1.2])**.**

$$R(N,s,k)=c_I\int_{h\gamma(\mathfrak{P}^{\otimes n})}F(z,m)^sE(-zN)\mathrm{d}z,$$

where c_I is a constant depending only on $\mathbb{L}$.

We want to rewrite $S(z,m)$ to $F(z,m)$. Therefore we apply a trick which goes back to Gelfond [7] to connect the second and third sum

$$S(z,m)=\prod_{i=1}^r(\mathrm{N}(\mathcal{S}M_i))^{-1}\sum_{\mathbf{R}\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r}\sum_{W\in\mathcal{S}(m)}E\left(zW^k+\sum_{i=1}^r\frac{R_i}{M_i}(f_i(W)-J_i)\right).$$

In view of Lemma 7.3 we get that

$$\begin{aligned}R(N,s,k,\mathbf{f},\mathbf{J},\mathbf{M})=R'(N,s,k)&=c_I\int_{h\gamma(\mathfrak{P}^{\otimes n})}S(z,m)^sE(-zN)\mathrm{d}z\\&=c_I\prod_{i=1}^r(\mathrm{N}(\mathcal{S}M_i))^{-s}\int_{h\gamma(\mathfrak{P}^{\otimes n})}\sum_{P_1\in\mathcal{S}(m)}\cdots\sum_{P_s\in\mathcal{S}(m)}\sum_{\mathbf{R}\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r}\\&\quad\times E\left(\sum_{i=1}^r\frac{R_i}{M_i}(f_i(P_1)-J_i)\right)\cdots E\left(\sum_{i=1}^r\frac{R_i}{M_i}(f_i(P_s)-J_i)\right)\\&\quad\times E(z(P_1^k+\cdots+P_s^k-N))\mathrm{d}z.\end{aligned}$$

We split the integral up into two parts according to whether $\mathbf{R}=0$ or not. Thus

$$R'(N,s,k)=c_I\prod_{i=1}^r(\mathrm{N}(\mathcal{S}M_i))^{-s}\left(I_1+I_2\right),$$

where

$$I_1=\int_{h\gamma(\mathfrak{P}^{\otimes n})}E(z(P_1^k+\cdots+P_s^k-N))\mathrm{d}z=\int_{h\gamma(\mathfrak{P}^{\otimes n})}F(z,m)^sE(-zN)\mathrm{d}z$$
$$I_2=\int_{h\gamma(\mathfrak{P}^{\otimes n})}\sum_{\mathbf{0}\neq\mathbf{R}\in\mathcal{M}^s}\prod_{i=1}^sH_{R_i}(z,m)E\left(-\sum_{i=1}^s\frac{R_iJ}{M}-zN\right)\mathrm{d}z.$$

In order to estimate the first integral we apply Proposition 7.1 and get

$$I_1=\mathfrak{S}_s(N)q^{(s-k)S(m)}+o(q^{(s-k)S(m)}).$$

In order to prove our theorem we need to show that $I_2=o(q^{(s-k)S(m)})$, *i.e.*, I_2 only contributes to the error term. Therefore we split the second integral I_2 up again according to the different values of $\mathbf{R}$. Thus

$$I_2=\sum_{\mathbf{0}\neq\mathbf{R}\in\mathcal{M}_1\times\cdots\times\mathcal{M}_r}I_{\mathbf{R}},$$

where

$$I_{\mathbf{R}}=\int_{h\gamma(\mathfrak{P}^{\otimes n})}H_{\mathbf{R}}(z,m)^sE\left(-zN\right)\mathrm{d}z$$

with

$$H_{\mathbf{R}}(z,m) = \sum_{P\in\mathcal{S}(m)} E\left(zP^k - \sum_{i=1}^{r} \frac{R_i}{M_i}\left(f_i(P) - J_i\right)\right).$$

We split this integral up into two parts. Thus

$$|I_{\mathbf{R}}| \le \sup_{\mathbf{R},z} |H_{\mathbf{R}}(z,m)|^{s-2^k} \max_{\mathbf{R}} \int_{h\gamma(\mathfrak{P}^{\otimes n})} H_{\mathbf{R}}(z,m)^{2^k} \mathrm{d}z. \tag{7.4}$$

For the supremum we apply Theorem 2.5 to get

$$\sup_{\mathbf{R},z} |H_{\mathbf{R}}(z,m)|^{s-2^k} \ll (\#\mathcal{S}(m))^{\left(s-2^k\right)\left(1-\frac{k+2}{2^{k+1}}+\gamma\right)}, \tag{7.5}$$

where γ is defined in (6.2).

In order to estimate the integral we will apply Hua's Lemma. Therefore we need the following lemma.

Lemma 7.4 ([4, Proposition II.5.2])**.** *Let c be any integer such that $1 \le c \le k$. Let $\varepsilon > 0$. Then*

$$\int_{h\gamma(\mathfrak{P}^{\otimes n})} F(z,m)^{2^c} E(-zN)\mathrm{d}z \ll (\#\mathcal{S}(m))^{2^c-c+\varepsilon}.$$

Thus we get

$$\max_{\mathbf{R}} \int_{h\gamma(\mathfrak{P}^{\otimes n})} H_{\mathbf{R}}(z,m)^{2^k} \mathrm{d}z \ll \max_{\mathbf{R}} \int_{h\gamma(\mathfrak{P}^{\otimes n})} F(z,m)^{2^k} \mathrm{d}z \ll (\#\mathcal{S}(m))^{2^k-k+\varepsilon}. \tag{7.6}$$

Now plugging (7.5) and (7.6) into (7.4) yields

$$|I_{\mathbf{R}}| \ll (\#\mathcal{S}(m))^{\left(s-2^k\right)\left(1-\frac{k+2+\gamma}{2^{k+1}}\right)} (\#\mathcal{S}(m))^{2^k-k+\varepsilon} \ll (\#\mathcal{S}(m))^{s-k-\delta},$$

where ε has to be chosen such that

$$(s-2^k)\left(\frac{k+2}{2^{k+1}}+\gamma\right) - \varepsilon =: \delta > 0$$

which is possible since $s > 2^k$.

Thus a final application of (5.2) yields

$$I_2 = o\left((\#\mathcal{S}(m))^{(s-k)}\right) = o\left(q^{(s-k)S(m)}\right)$$

and the theorem is proven.

Remark 7.5. It is easy to generalize this result to the investigation of the following case

$$N = P_1^k + \cdots + P_s^k \quad (f_{i_j}(P_j) \equiv J_{i_j} \bmod M_{i_j}),$$

where every summand has its own set of B_{i_j}-additive functions f_{i_j} together with his own congruence relation $\equiv J_{i_j} \bmod M_{i_j}$. This can be done in quite the same way and is therefore left to the reader.

References

[1] R. Ayoub. *An introduction to the analytic theory of numbers.* Mathematical Surveys, No. 10. American Mathematical Society, Providence, R.I., 1963.

[2] N. L. Bassily and I. Kátai. Distribution of the values of q-additive functions on polynomial sequences. *Acta Math. Hungar.*, 68(4):353–361, 1995.

[3] T. Beck, H. Brunotte, K. Scheicher, and J. M. Thuswaldner. Number systems and tilings over Laurent series. *Math. Proc. Cambridge. Philos. Soc.*, 147(1):9–29, 2009.

[4] M. Car. Waring's problem in function fields. *Proc. London Math. Soc. (3)*, 68(1):1–30, 1994.

[5] P. M. Cohn. *Algebraic numbers and algebraic functions.* Chapman and Hall Mathematics Series. Chapman & Hall, London, 1991.

[6] M. Drmota and G. Gutenbrunner. The joint distribution of Q-additive functions on polynomials over finite fields. *J. Théor. Nombres Bordeaux*, 17(1):125–150, 2005.

[7] A. O. Gel′fond. Sur les nombres qui ont des propriétés additives et multiplicatives données. *Acta Arith.*, 13:259–265, 1967/1968.

[8] L.-K. Hua. *Additive Primzahltheorie.* B. G. Teubner Verlagsgesellschaft, Leipzig, 1959.

[9] D.-H. Kim. On the joint distribution of q-additive functions in residue classes. *J. Number Theory*, 74(2):307–336, 1999.
[10] B. Kovács and A. Pethő. Number systems in integral domains, especially in orders of algebraic number fields. *Acta Sci. Math. (Szeged)*, 55(3-4):287–299, 1991.
[11] B. Kovács and A. Pethő. On a representation of algebraic integers. *Studia Sci. Math. Hungar.*, 27(1-2):169–172, 1992.
[12] M. G. Madritsch. Waring's problem with digital restrictions in $\mathbb{F}_q[x]$. *Mathematica Slovaca*, 2009. to appear.
[13] M. G. Madritsch and J. M. Thuswaldner. Weyl sums in $\mathbb{F}_q[x]$ with digital restrictions. *Finite Fields Appl.*, 14(4):877–896, 2008.
[14] K. Scheicher and J. M. Thuswaldner. Digit systems in polynomial rings over finite fields. *Finite Fields Appl.*, 9(3):322–333, 2003.
[15] K. Scheicher and J. M. Thuswaldner. On the characterization of canonical number systems. *Osaka J. Math.*, 41(2):327–351, 2004.
[16] J.-P. Serre. *Corps locaux*. Hermann, Paris, 1968. Deuxième édition, Publications de l'Université de Nancago, No. VIII.
[17] J. M. Thuswaldner and R. F. Tichy. Waring's problem with digital restrictions. *Israel J. Math.*, 149:317–344, 2005. Probability in mathematics.
[18] S. Wagner. Waring's problem with restrictions on q-additive functions. *Math. Slovaca.*, 2008. to appear.
[19] A. Weil. *Basic number theory*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition.

(M.G. Madritsch) Laboratoire GREYC, Université de Caen and ENSICAEN, F-14032 Caen, France
E-mail address: Manfred.Madritsch@info.unicaen.fr

(J.M. Thuswaldner) Department of Mathematics and Information Technology, University of Leoben, A-8700 Leoben, Austria
E-mail address: Joerg.Thuswaldner@mu-leoben.at