

TILINGS INDUCED BY A CLASS OF CUBIC RAUZY FRACTALS

BENOÎT LORIDANT, ALI MESSAOUDI, PAUL SURER, AND JÖRG M. THUSWALDNER

ABSTRACT. We study aperiodic and periodic tilings induced by the Rauzy fractal and its subtiles associated to beta-substitutions related to the polynomial $x^3 - ax^2 - bx - 1$ for $a \geq b \geq 1$. In particular, we compute the corresponding boundary graphs, describing the adjacencies in the tilings. These graphs are a valuable tool for more advanced studies of the topological properties of Rauzy fractals. The methods presented here may be used to obtain similar results for other classes of substitutions.

1. INTRODUCTION

In 1982 Gérard Rauzy [21] studied the symbolic dynamical system over the alphabet $\{1, 2, 3\}$ induced by the substitution

$$1 \mapsto 12, \quad 2 \mapsto 13, \quad 3 \mapsto 1$$

and associated to it a set known as Rauzy fractal. It is a compact set equal to the closure of its interior and it decomposes naturally into three subsets (subtiles). Moreover, the Rauzy fractal induces two types of tilings: a periodic tiling whose central tile is the Rauzy fractal, and an aperiodic tiling generated by the three subtiles. In [16, 18, 19], topological properties of the Rauzy fractal were studied and the Hausdorff dimension of its boundary was computed.

Generalisations of this dynamical system and results concerning the associated fractal sets can be found in the literature. In [3], the considerations of Rauzy are formulated in a general way for primitive Pisot substitutions. The interiors of the subtiles associated to a primitive unimodular Pisot substitution do not overlap provided that the substitution satisfies the so called strong coincidence condition [3, 13]. Several classes of substitutions were shown to satisfy this condition. For example, in [5] it was proven that every primitive irreducible Pisot substitution over an alphabet consisting of two letters satisfies it. It is conjectured that this is true for alphabets of arbitrary size but a general proof is still outstanding.

Rauzy fractals associated to primitive unimodular Pisot substitutions have been studied in various articles [7, 9, 11, 12, 15, 20, 23, 24]. They appear naturally in connection to many topics as numeration systems, geometrical representation of symbolic dynamical systems, multidimensional continued fractions and simultaneous approximations, self-similar tilings and Markov partitions of hyperbolic automorphisms of the torus.

In [13, 17] it was shown that the subtiles induce an aperiodic multiple tiling of the space, called self-replicating multiple tiling. If the substitution is irreducible, the Rauzy fractals also provide a periodic (or lattice) multiple tiling (see [3, 11]). Actually, a lattice multiple tiling even exists in some reducible cases. A necessary and sufficient condition can be found in [22]. For large classes of substitutions these multiple tilings are shown to be proper tilings, *i.e.*, two different tiles have disjoint interiors. Even if there is no known counterexample, it is up to now not possible to prove this *tiling property* in general without requiring additional conditions like the super coincidence condition or the finiteness property.

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for all $w \in \mathcal{A}^*$. Thus we have

$$\mathbf{M} = \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

\mathbf{M} is a primitive Matrix, *i.e.*, \mathbf{M}^k has only strictly positive entries for some power $k \in \mathbb{N}$; we denote by β the corresponding dominant Perron-Frobenius eigenvalue, satisfying $\beta^3 = a\beta^2 + b\beta + 1$. The substitution σ has the following properties. It is

- *primitive*: the incidence matrix \mathbf{M} is a primitive matrix;
- *unimodular*: β is an algebraic unit;
- *irreducible*: the algebraic degree of β is exactly $|\mathcal{A}| = 3$;
- *Pisot*: the Galois conjugates α_1, α_2 of β have modulus strictly smaller than 1.

Observe that the substitutions $\sigma_{a,b}$ are so-called *beta-substitutions*, that is, the induced dynamical system is intimately related to beta-expansions. Details can be found, for example, in [7].

2.2. Associated Rauzy fractals. There are several equivalent ways of constructing the Rauzy fractal. For an overview of the different methods we refer to [6]. Here we will use the way via the so-called prefix-suffix-automaton presented in [11].

Let \mathbf{u}_β be a strictly positive right eigenvector and \mathbf{v}_β a strictly positive left eigenvector of \mathbf{M} that correspond to the dominant eigenvalue β such that $\langle \mathbf{u}_\beta, \mathbf{v}_\beta \rangle = 1$. We set

$$\begin{aligned} \mathbf{v}_\beta &= (v_1, v_2, v_3) = (1, \beta - a, \beta^2 - a\beta - b) = (1, \beta^{-2} + b\beta^{-1}, \beta^{-1}), \\ \mathbf{u}_\beta &= \frac{1}{3\beta^2 - 2a\beta - b} (\beta^2, \beta, 1). \end{aligned}$$

Note that

$$(2.1) \quad 1 = v_1 > v_2 > v_3 > 0.$$

Moreover, let \mathbf{u}_{α_i} be the eigenvectors for the Galois conjugates obtained by replacing β by α_i in the coordinates of the vector \mathbf{u}_β . We obtain the decomposition

$$\mathbb{R}^3 = \mathbb{H}_e \oplus \mathbb{H}_c,$$

where

- \mathbb{H}_e is the *expanding line*, generated by \mathbf{u}_β .
- \mathbb{H}_c is the *contracting space*, generated by $\mathbf{u}_{\alpha_1} + \mathbf{u}_{\alpha_2}$ and $-\alpha_2 \mathbf{u}_{\alpha_1} - \alpha_1 \mathbf{u}_{\alpha_2}$.

We denote by $\pi : \mathbb{R}^3 \rightarrow \mathbb{H}_c$ the projection onto \mathbb{H}_c along \mathbb{H}_e and by \mathbf{h} the restriction of \mathbf{M} on the contractive space \mathbb{H}_c . Note that if we define the norm

$$\|\mathbf{x}\| = \max\{|\langle \mathbf{x}, \mathbf{v}_{\alpha_1} \rangle|, |\langle \mathbf{x}, \mathbf{v}_{\alpha_2} \rangle|\},$$

then \mathbf{h} is a contraction with $\|\mathbf{h}\mathbf{x}\| \leq \max\{|\alpha_1|, |\alpha_2|\} \|\mathbf{x}\| < \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{H}_c$. Furthermore, we have

$$(2.2) \quad \forall w \in \mathcal{A}^*, \quad \mathbf{h}(\pi(\mathbf{l}(w))) = \pi(\mathbf{M}\mathbf{l}(w)) = \pi(\mathbf{l}(\sigma(w))).$$

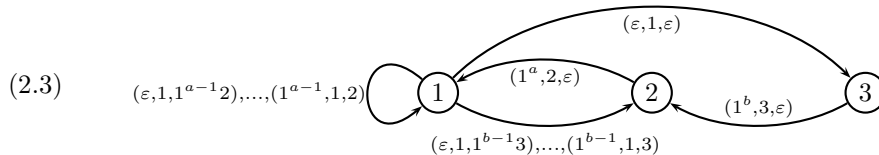
The *prefix-suffix automaton* Γ_σ is defined as follows (see also [10]). Let \mathcal{P} be the finite set

$$\mathcal{P} = \{(p, i, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^* \mid \exists j \in \mathcal{A}, \sigma(j) = pis\}.$$

Then Γ_σ is the directed graph with

- vertices : the letters of the alphabet \mathcal{A} ;
- edges : $i \xrightarrow{(p,i,s)} j$ if and only if $\sigma(j) = pis$, where $(p, i, s) \in \mathcal{P}$.

The prefix-suffix automaton $\Gamma_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is



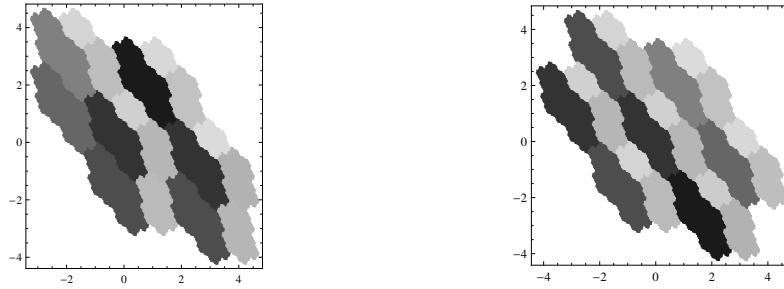


FIGURE 1. Clippings around the origin of the self-replicating tiling (left) and the lattice tiling (right) associated to the substitution $\sigma_{1,1}$.

Here, for a letter $i \in \mathcal{A}$, i^k stands for $\underbrace{i \dots i}_{k \text{ times}}$.

The Rauzy fractal and its subtiles are geometric representations of the infinite walks in the prefix-suffix automaton [11]:

$$\mathcal{T} = \left\{ \sum_{k \geq 0} \mathbf{h}^k \pi(\mathbf{l}(p_k)) \mid \begin{array}{l} j_0 \xrightarrow{(p_0, j_0, s_0)} j_1 \xrightarrow{(p_1, j_1, s_1)} j_2 \xrightarrow{(p_2, j_2, s_2)} \dots \\ \text{is an infinite path of } \Gamma_\sigma \end{array} \right\}$$

and for $j \in \mathcal{A}$

$$(2.4) \quad \mathcal{T}(j) = \left\{ \sum_{k \geq 0} \mathbf{h}^k \pi(\mathbf{l}(p_k)) \mid \begin{array}{l} j_0 = j \xrightarrow{(p_0, j_0, s_0)} j_1 \xrightarrow{(p_1, j_1, s_1)} j_2 \xrightarrow{(p_2, j_2, s_2)} \dots \\ \text{is an infinite path of } \Gamma_\sigma \end{array} \right\}.$$

Since σ is a primitive unimodular Pisot substitution satisfying the strong coincidence condition, the subtiles have disjoint interiors (*e.g.*, see [7]). Moreover, by [23] each subtile is the closure of its interior.

Due to the connection with beta-expansions, the Rauzy fractals for our class coincide with beta-tiles treated, for example, in [1, 8].

2.3. Tilings. For a substitution $\sigma_{a,b}$ of our class, the Rauzy fractal gives rise to two types tilings of the contracting space \mathbb{H}_c : an aperiodic tiling and a periodic tiling, obtained as follows.

The *self-replicating translation set* is

$$(2.5) \quad \Gamma_{srs} := \{[\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^3) \times \mathcal{A} \mid 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_i\}.$$

Then $\{\mathcal{T}(i) + \gamma \mid [\gamma, i] \in \Gamma_{srs}\}$ is the *self-replicating tiling* of the contracting space (see [17]).

The *lattice translation set* is

$$\Gamma_{lat} = \{[\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^3) \times \mathcal{A} \mid \mathbf{x} = (x_1, x_2, x_3), x_1 + x_2 + x_3 = 0\}.$$

Then $\{\mathcal{T}(i) + \gamma \mid [\gamma, i] \in \Gamma_{lat}\}$ is the *lattice tiling* of the contracting space (see [3, 11]).

By [2] the tilings are proper tiling, *i.e.*, the tiles have disjoint interior. In particular, we have the following properties:

- *covering property:* $\mathbb{H}_c = \bigcup_{[\gamma, i] \in \Gamma_{srs}} \mathcal{T}(i) + \gamma = \bigcup_{[\gamma, i] \in \Gamma_{lat}} \mathcal{T}(i) + \gamma$;
- *tiling property:* the interiors of two tiles $\mathcal{T}(i) + \gamma$, $\mathcal{T}(j) + \gamma'$ with $[\gamma, i] \neq [\gamma', j] \in \Gamma_{srs}$ or $[\gamma, i] \neq [\gamma', j] \in \Gamma_{lat}$ are disjoint;
- *local finiteness:* for each compact subset B of \mathbb{H}_c , the subsets $\{[\gamma, i] \in \Gamma_{srs} \mid (\mathcal{T}(i) + \gamma) \cap B \neq \emptyset\}$ and $\{[\gamma, i] \in \Gamma_{lat} \mid (\mathcal{T}(i) + \gamma) \cap B \neq \emptyset\}$ are finite.

Figure 1 shows the self-replicating tiling (left) and the lattice tiling (right) for the Tribonacci substitution $\sigma_{1,1}$. The lattice tiling and topological properties of \mathcal{T} have been already studied in [18, 19].

2.4. Boundary graphs. Graphs that describe the intersection of two tiles in the above tilings were introduced by Siegel and Thuswaldner [22]. The aim of this paper is the computation of these graphs for the whole class $\sigma_{a,b}$ introduced in Subsection 2.1. We recall briefly their definitions in terms of our class $\sigma_{a,b}$.

We call *neighbours* two subtiles of the self-replicating (or lattice) tiling if their intersection is non-empty. The intersection $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma)$ will be described by the vertex $[i, \gamma, j]$ in the boundary graph. Since $[j, -\gamma, i]$ would correspond to the same intersection translated by $-\gamma$, we impose the vertices to belong to

$$\mathcal{D} = \{[i, \gamma, j] \in \mathcal{A} \times \pi(\mathbb{Z}^3) \times \mathcal{A} \mid \gamma = \pi(\mathbf{x}), \langle \mathbf{x}, \mathbf{v}_\beta \rangle > 0 \text{ or } (\gamma = \mathbf{0} \text{ and } i < j)\}.$$

Definition 2.1 (cf. [22]). The *self-replicating boundary graph* $\mathcal{G}_{srs}^{(B)}$ (*lattice boundary graph* $\mathcal{G}_{lat}^{(B)}$, respectively) is the largest graph with the following properties.

(1) The vertices $[i, \gamma, j]$ are elements of \mathcal{D} such that

$$(2.6) \quad \|\gamma\| \leq \frac{2 \max_{(p,j,s) \in \mathcal{P}_\sigma} \|\pi \mathbf{l}(p)\|}{1 - \max\{|\alpha_1|, |\alpha_2|\}}.$$

(2) There is an edge from $[i, \gamma, j]$ to $[i', \gamma', j']$ if and only if there exist $[\bar{i}, \bar{\gamma}, \bar{j}] \in \mathcal{A} \times \pi(\mathbb{Z}^3) \times \mathcal{A}$ and $(p_1, a_1, s_1), (p_2, a_2, s_2) \in \mathcal{P}$ such that

$$\begin{cases} [i', \gamma', j'] = [\bar{i}, \bar{\gamma}, \bar{j}] \text{ (Type 1) or } [i', \gamma', j'] = [\bar{j}, -\bar{\gamma}, \bar{i}] \text{ (Type 2),} \\ a_1 = i \text{ and } p_1 a_1 s_1 = \sigma(\bar{i}), \\ a_2 = j \text{ and } p_2 a_2 s_2 = \sigma(\bar{j}), \\ \mathbf{h}\bar{\gamma} = \gamma + \pi(\mathbf{l}(p_2) - \mathbf{l}(p_1)). \end{cases}$$

The edge is labelled by

$$\eta = \begin{cases} \pi \mathbf{l}(p_1), & \langle \mathbf{l}(p_1), \mathbf{v}_\beta \rangle \leq \langle \mathbf{l}(p_2) + \mathbf{x}, \mathbf{v}_\beta \rangle, \\ \pi \mathbf{l}(p_2) + \gamma, & \text{otherwise;} \end{cases}$$

where $\mathbf{x} \in \mathbb{Z}^3$ such that $\pi(\mathbf{x}) = \gamma$.

(3) Each vertex belongs to an infinite walk starting from a vertex $[i, \gamma, j]$ with $[\gamma, j] \in \Gamma_{srs}$ ($[\gamma, j] \in \Gamma_{lat} \setminus (\{\mathbf{0}\} \times \mathcal{A})$, respectively).

There exist algorithms to compute $\mathcal{G}_{srs}^{(B)}$ and $\mathcal{G}_{lat}^{(B)}$ for any given substitution (see [22]). However, the bound (2.6) in Definition 2.1 is inconvenient when working with a whole class like $\sigma_{a,b}$. We will formulate an equivalent definition for $\mathcal{G}_{srs}^{(B)}$ in Section 4 without this bound (see Theorem 4.1).

The following three propositions contain information on the structure and the use of the boundary graphs. The proofs can be found in [22].

Proposition 2.2 (cf. [22, Proposition 5.5]). *The self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ and the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ are well defined and finite.*

Proposition 2.3 (cf. [22, Theorem 5.7]). *Let $[i, \gamma, j]$ be a vertex in the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. Then $[\gamma, j] \in \Gamma_{srs}$.*

Unfortunately, an analogue assertion for the lattice boundary graph does not hold. For this reason, our results concerning the lattice boundary graph will be weaker than for the self-replicating boundary graph.

Proposition 2.4 (cf. [22, Corollary 5.9]). *Let $[i, \gamma, j] \in \mathcal{D}$. A point $\xi \in \mathbb{H}_c$ belongs to the intersection $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma)$ with $[\gamma, j] \in \Gamma_{srs}$ if and only if there exists an infinite walk in $\mathcal{G}_{srs}^{(B)}$ starting from $[i, \gamma, j]$ and labelled by $(\eta^{(k)})_{k \geq 0}$ such that*

$$\xi = \sum_{k \geq 0} \mathbf{h}^k \eta^{(k)}.$$

Vertex	Edge(s) to	Label(s)	Type	Condition
$[1, \pi(0, 0, 0), 2]$	$[1, \pi(0, 0, 1), 1]$	$\{\pi(a-1, 0, 0)\}$	1	
$[1, \pi(0, 0, 0), 3]$	$[1, \pi(0, 0, 0), 2]$	$\{\pi(b, 0, 0)\}$	1	$a \neq b$
	$[1, \pi(0, 0, 1), 2]$	$\{\pi(b-1, 0, 0)\}$	1	
	$[2, \pi(0, 0, 1), 2]$	$\{\pi(b-1, 0, 0)\}$	1	$b = 1$
$[1, \pi(0, 0, 1), 1]$	$[1, \pi(0, 1, 0), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-b-1\}$	1	$a \neq b$
	$[1, \pi(0, 1, -1), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-b\}$	1	
	$[1, \pi(0, 1, -1), 2]$	$\{\pi(0, 0, 0)\}$	1	
	$[3, \pi(0, 1, -1), 2]$	$\{\pi(0, 0, 0)\}$	1	
$[1, \pi(0, 0, 1), 2]$	$[1, \pi(0, 1, 0), 1]$	$\{\pi(a-b, 0, 0)\}$	1	
	$[1, \pi(0, 1, -1), 1]$	$\{\pi(a-b+1, 0, 0)\}$	1	$b \neq 1$
$[1, \pi(0, 1, 0), 1]$	$[1, \pi(1, 0, -1), 1]$	$\{\pi(0, 0, 0)\}$	1	$a \neq b$
	$[2, \pi(1, 0, -1), 1]$	$\{\pi(0, 0, 0)\}$	1	
	$[3, \pi(1, 0, -1), 1]$	$\{\pi(0, 0, 0)\}$	1	
$[1, \pi(0, 1, -1), 1]$	$[2, \pi(1, -1, 0), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-1\}$	1	
	$[1, \pi(1, -1, 1), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
	$[2, \pi(1, -1, 1), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
$[1, \pi(0, 1, -1), 2]$	$[1, \pi(1, -1, 1), 1]$	$\{\pi(b-1, 0, 0)\}$	1	$b \geq 2$
	$[2, \pi(1, -1, 1), 1]$	$\{\pi(b-1, 0, 0)\}$	1	
$[3, \pi(0, 1, -1), 2]$	$[2, \pi(1, -1, 0), 1]$	$\{\pi(b, 0, 0)\}$	1	
$[2, \pi(1, 0, -1), 1]$	$[1, \pi(0, 1, 0), 1]$	$\{\pi(a-b, 0, -1)\}$	2	$a \neq b$
	$[1, \pi(0, 1, -1), 1]$	$\{\pi(a-b+1, 0, -1)\}$	2	
$[3, \pi(1, 0, -1), 1]$	$[1, \pi(0, 1, -1), 2]$	$\{\pi(1, 0, -1)\}$	2	
	$[3, \pi(0, 1, -1), 2]$	$\{\pi(1, 0, -1)\}$	2	
$[2, \pi(1, -1, 0), 1]$	$[1, \pi(1, 0, -1), 1]$	$\{\pi(1, -1, 0)\}$	2	$a \neq b$ or $a \geq 4$
	$[2, \pi(1, 0, -1), 1]$	$\{\pi(1, -1, 0)\}$	2	
	$[3, \pi(1, 0, -1), 1]$	$\{\pi(1, -1, 0)\}$	2	
$[2, \pi(1, -1, 1), 1]$	$[2, \pi(1, -1, 0), 1]$	$\{\pi(b, -1, 1)\}$	2	
	$[1, \pi(1, -1, 1), 1]$	$\{\pi(b-1, -1, 1)\}$	2	$b \geq 2$
	$[2, \pi(1, -1, 1), 1]$	$\{\pi(b-1, -1, 1)\}$	2	$b \geq 2$
Vertex exists if $b = 1$				
$[2, \pi(0, 0, 1), 2]$	$[1, \pi(0, 1, -1), 1]$	$\{\pi(a, 0, 0)\}$	1	$b = 1$
Vertex exists if $a = b$				
$[2, \pi(0, 0, 0), 3]$	$[1, \pi(0, 0, 0), 2]$	$\{\pi(a, 0, 0)\}$	1	$a = b$
Vertex exists if $a \neq b$ or $a \geq 4$				
$[1, \pi(1, 0, -1), 1]$	$[1, \pi(0, 1, 0), 1]$	$\{\pi(e, 0, -1) \mid 1 \leq e \leq a-b-1\}$	2	$a \geq b+2$
	$[1, \pi(0, 1, -1), 1]$	$\{\pi(e, 0, -1) \mid 1 \leq e \leq a-b\}$	2	$a \neq b$
Vertex exists if $b \geq 2$				
$[1, \pi(1, -1, 1), 1]$	$[2, \pi(1, -1, 0), 1]$	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-1\}$	2	$b \geq 2$
	$[1, \pi(1, -1, 1), 1]$	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$
	$[2, \pi(1, -1, 1), 1]$	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$

ADJACENCY TABLE 1. The subgraph $\mathcal{S}(1)$ of the self-replicating boundary graph.

3. MAIN THEOREMS

The main results of this paper consist in a description of the boundary graphs associated to the substitutions of the class defined in Subsection 2.1. For convenience, we set

$$m(a, b) := \max \left\{ 1, \left\lfloor \frac{a}{a-b+2} \right\rfloor \right\}.$$

Note that $m(a, b) = 1$ if and only if $a \geq 2b - 3$.

3.1. The self-replicating boundary graph. For $1 \leq t \leq m(a, b)$, we call $\mathcal{S}(t)$ the graph whose nodes and edges are given in Adjacency Table 1 ($t = 1$) and Adjacency Table 2 ($t \geq 2$). For $1 \leq t \leq m(a, b) - 1$, we denote by $\mathcal{S}'(t)$ the graph described by Adjacency Table 3. Finally, let \mathcal{S} denote the union of these graphs:

$$\mathcal{S} = \bigcup_{t=1}^{m(a,b)} \mathcal{S}(t) \cup \bigcup_{t=1}^{m(a,b)-1} \mathcal{S}'(t).$$

Theorem 3.1. *The self-replicating boundary graph $\mathcal{G}_{sr_s}^{(B)}$ related to the substitution $\sigma_{a,b}$ is equal to the graph \mathcal{S} . Its nodes and edges can be read off from Adjacency Tables 1, 2 and 3.*

We will prove the theorem in Section 5. The subdivision into several subgraphs has technical reasons that will become apparent in the proof.

Vertex	Edge(s) to	Label(s)	T	Condition
[1, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 3\}$	2	
	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_{t-1} - 1\}$	2	
[2, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(\delta_t - t, t-1, -t)\}$	2	
[3, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(\delta_t - a + b - t, t-1, -t)\}$	2	
[3, $\pi(t-1, 1-t, t-1), 1$]	[1, $\pi(t-1, 1-t, t), 2$]	$\{\pi(a - \delta_t + t, 1-t, t-1)\}$	2	
[1, $\pi(t-1, 1-t, t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t - 2\}$	1	
	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t + b - a - 2\}$	1	
[1, $\pi(t-1, 1-t, t), 2$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(\delta_t - 1, 0, 0)\}$	1	
[1, $\pi(1-t, t, -t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a - \delta_t\}$	1	$a \geq \delta_t + 1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a - \delta_t\}$	1	
[1, $\pi(1-t, t, -t), 2$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(a - \delta_t + 1, 0, 0)\}$	1	$a \geq \delta_t + 1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(a - \delta_t + 1, 0, 0)\}$	1	
[2, $\pi(t, -t, t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(a - \delta_t + t, -t, t)\}$	2	$a \geq \delta_t + 1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(a - \delta_t + t, -t, t)\}$	2	
Vertex exists if $a \geq \delta_t + 1$				
[1, $\pi(t, -t, t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 1\}$	2	$a \geq \delta_t + 1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 1\}$	2	$a \geq \delta_t + 1$

ADJACENCY TABLE 2. The subgraph $\mathcal{S}(t)$ of the self-replicating boundary graph, where $2 \leq t \leq m(a, b)$ ($\delta_t = t(a - b + 2)$).

Vertex	Edge(s) to	Label(s)	T	Condition
[1, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t-1), 1$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 2\}$	2	
	[2, $\pi(1-t, t, -t-1), 1$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t - 2\}$	2	
	[3, $\pi(1-t, t, -t-1), 1$]	$\{\pi(2-t, t-1, -t)\}$	2	
[2, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t-1), 1$]	$\{\pi(\delta_t + 1 - t, t-1, -t)\}$	2	
	[2, $\pi(1-t, t, -t-1), 1$]	$\{\pi(\delta_t + 1 - t, t-1, -t)\}$	2	
[1, $\pi(t-1, 1-t, t), 1$]	[1, $\pi(1-t, t, -t-1), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t - 1\}$	1	
	[2, $\pi(1-t, t, -t-1), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t - 1\}$	1	
	[3, $\pi(1-t, t, -t-1), 1$]	$\{\pi(0, 0, 0)\}$	1	
[1, $\pi(t-1, 1-t, t), 2$]	[1, $\pi(1-t, t, -t-1), 1$]	$\{\pi(\delta_t, 0, 0)\}$	1	
	[2, $\pi(1-t, t, -t-1), 1$]	$\{\pi(\delta_t, 0, 0)\}$	1	
[1, $\pi(1-t, t, -t), 1$]	[3, $\pi(t, -t, t), 1$]	$\{\pi(0, 0, 0)\}$	1	
	[1, $\pi(t, -t, t+1), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a - \delta_t - 1\}$	1	
	[1, $\pi(t, -t, t+1), 2$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b - \delta_t - 1\}$	1	
[1, $\pi(1-t, t, -t), 2$]	[1, $\pi(t, -t, t+1), 1$]	$\{\pi(a - \delta_t, 0, 0)\}$	1	
[2, $\pi(t, -t, t), 1$]	[1, $\pi(t, -t, t+1), 1$]	$\{\pi(a - \delta_t - 1 + t, -t, t)\}$	2	
[1, $\pi(t, -t, t), 1$]	[3, $\pi(t, -t, t), 1$]	$\{\pi(t, -t, t)\}$	2	
	[1, $\pi(t, -t, t+1), 1$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a - \delta_t - 2\}$	2	
	[1, $\pi(t, -t, t+1), 2$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq b - \delta_t - 2\}$	2	

ADJACENCY TABLE 3. The subgraph $\mathcal{S}'(t)$ of the self-replicating boundary graph, where $1 \leq t \leq m(a, b) - 1$ ($\delta_t = t(a - b + 2)$).

3.2. The lattice boundary graph. As already mentioned, we are not able to characterize $\mathcal{G}_{lat}^{(B)}$ completely for all possible values of a and b . We give a complete description of the lattice boundary graph for the case $m(a, b) = 1$ and conjecture the shape of $\mathcal{G}_{lat}^{(B)}$ for the other cases. For $1 \leq t \leq m(a, b)$, we call $\mathcal{L}(t)$ the graph whose nodes and edges are given in Adjacency Table 4 ($t = 1$) and Adjacency Table 5 ($t \geq 2$). For $1 \leq t \leq m(a, b) - 1$, we denote by $\mathcal{L}'(t)$ the graph described by Adjacency Table 6. Note that in these tables the vertices that correspond to elements of the lattice translation set Γ_{lat} are highlighted in grey. We denote by \mathcal{L} the union of all these graphs:

$$\mathcal{L} = \bigcup_{t=1}^{m(a,b)} \mathcal{L}(t) \cup \bigcup_{t=1}^{m(a,b)-1} \mathcal{L}'(t).$$

Theorem 3.2. *For all a, b , the graph \mathcal{L} is a subgraph of the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to $\sigma_{a,b}$.*

For $m(a, b) = 1$, we can prove the reverse inclusion.

Theorem 3.3. *Let $a \geq 2b - 3$. Then the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to the substitution $\sigma_{a,b}$ equals \mathcal{L} .*

We will prove these theorems in Section 6. The following conjecture remains.

Vertex	Edge(s) to	Label(s)	Type	Condition
[1, $\pi(0, 1, -1), 1$]	[2, $\pi(1, -1, 0), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-1\}$	1	
	[1, $\pi(1, -1, 1), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
	[2, $\pi(1, -1, 1), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq b-2\}$	1	$b \geq 2$
[1, $\pi(0, 1, -1), 2$]	[1, $\pi(1, -1, 1), 1$]	$\{\pi(b-1, 0, 0)\}$	1	$b \geq 2$
	[2, $\pi(1, -1, 1), 1$]	$\{\pi(b-1, 0, 0)\}$	1	
[3, $\pi(0, 1, -1), 2$]	[2, $\pi(1, -1, 0), 1$]	$\{\pi(b, 0, 0)\}$	1	
[2, $\pi(1, 0, -1), 1$]	[1, $\pi(0, 1, 0), 1$]	$\{\pi(a-b, 0, -1)\}$	2	$a \neq b$
	[1, $\pi(0, 1, -1), 1$]	$\{\pi(a-b+1, 0, -1)\}$	2	
[3, $\pi(1, 0, -1), 1$]	[1, $\pi(0, 1, -1), 2$]	$\{\pi(1, 0, -1)\}$	2	
	[3, $\pi(0, 1, -1), 2$]	$\{\pi(1, 0, -1)\}$	2	
	[1, $\pi(1, 0, -1), 1$]	$\{\pi(1, -1, 0)\}$	2	$a \neq b$ or $a \geq 4$
[2, $\pi(1, -1, 0), 1$]	[2, $\pi(1, 0, -1), 1$]	$\{\pi(1, -1, 0)\}$	2	
	[3, $\pi(1, 0, -1), 1$]	$\{\pi(1, -1, 0)\}$	2	
	[2, $\pi(1, -1, 0), 1$]	$\{\pi(1, -1, 0)\}$	2	
[2, $\pi(1, -1, 1), 1$]	[2, $\pi(1, -1, 0), 1$]	$\{\pi(b, -1, 1)\}$	2	
	[1, $\pi(1, -1, 1), 1$]	$\{\pi(b-1, -1, 1)\}$	2	$b \geq 2$
	[2, $\pi(1, -1, 1), 1$]	$\{\pi(b-1, -1, 1)\}$	2	$b \geq 2$
Vertex exists if $a \neq b$				
[1, $\pi(0, 1, 0), 1$]	[1, $\pi(1, 0, -1), 1$]	$\{\pi(0, 0, 0)\}$	1	$a \neq b$
	[2, $\pi(1, 0, -1), 1$]	$\{\pi(0, 0, 0)\}$	1	
	[3, $\pi(1, 0, -1), 1$]	$\{\pi(0, 0, 0)\}$	1	
Vertex exists if $a \neq b$ or $a \geq 4$				
[1, $\pi(1, 0, -1), 1$]	[1, $\pi(0, 1, 0), 1$]	$\{\pi(e, 0, -1) \mid 1 \leq e \leq a-b-1\}$	2	$a \geq b+2$
	[1, $\pi(0, 1, -1), 1$]	$\{\pi(e, 0, -1) \mid 1 \leq e \leq a-b\}$	2	$a \neq b$
Vertex exists if $b \geq 2$				
[1, $\pi(1, -1, 1), 1$]	[2, $\pi(1, -1, 0), 1$]	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-1\}$	2	$b \geq 2$
	[1, $\pi(1, -1, 1), 1$]	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$
	[2, $\pi(1, -1, 1), 1$]	$\{\pi(e, -1, 1) \mid 1 \leq e \leq b-2\}$	2	$b \geq 3$

ADJACENCY TABLE 4. The subgraph $\mathcal{L}(1)$ of the lattice boundary graph.

Vertex	Edge(s) to	Label(s)	T	Condition
[1, $\pi(0, t-1, 1-t), 3$]	[1, $\pi(t-1, 1-t, t), 2$]	$\{\pi(b-3-\delta_{t-1}+t, 0, 0)\}$	1	
[1, $\pi(0, t, -t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-1-\delta_t+t\}$	1	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-1-\delta_t+t\}$	1	
[1, $\pi(0, t, -t), 2$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+t, 0, 0)\}$	1	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+t, 0, 0)\}$	1	
[1, $\pi(1, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(e+1, t-1, -t) \mid 0 \leq e \leq \delta_t-t-2\}$	2	
	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(e+1, t-1, -t) \mid 0 \leq e \leq \delta_{t-1}-t\}$	2	
[2, $\pi(1, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(\delta_t-t, t-1, -t)\}$	2	
[3, $\pi(1, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(\delta_{t-1}-t+2, t-1, -t)\}$	2	
[1, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_t-3\}$	2	
	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(e+2-t, t-1, -t) \mid 0 \leq e \leq \delta_{t-1}-1\}$	2	
[2, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(\delta_t-t, t-1, -t)\}$	2	
[3, $\pi(2-t, t-1, -t), 1$]	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(\delta_t-a+b-t, t-1, -t)\}$	2	
[3, $\pi(t-1, 1-t, t-1), 1$]	[1, $\pi(t-1, 1-t, t), 2$]	$\{\pi(a-\delta_t+t, 1-t, t-1)\}$	2	
[1, $\pi(t-1, 1-t, t), 1$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t-2\}$	1	
	[1, $\pi(1-t, t, -t), 2$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t+b-a-2\}$	1	
[1, $\pi(t-1, 1-t, t), 2$]	[1, $\pi(1-t, t, -t), 1$]	$\{\pi(\delta_t-1, 0, 0)\}$	1	
[1, $\pi(1-t, t, -t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-\delta_t\}$	1	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(e, 0, 0) \mid 0 \leq e \leq a-\delta_t\}$	1	
[1, $\pi(1-t, t, -t), 2$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+1, 0, 0)\}$	1	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+1, 0, 0)\}$	1	
[2, $\pi(t, -t, t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+t, -t, t)\}$	2	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(a-\delta_t+t, -t, t)\}$	2	
Vertex exists if $a \geq \delta_t+1$				
[1, $\pi(t, -t, t), 1$]	[1, $\pi(t, -t, t), 1$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a-\delta_t-1\}$	2	$a \geq \delta_t+1$
	[2, $\pi(t, -t, t), 1$]	$\{\pi(e+t, -t, t) \mid 0 \leq e \leq a-\delta_t-1\}$	2	$a \geq \delta_t+1$

ADJACENCY TABLE 5. The subgraph $\mathcal{L}(t)$ of the lattice boundary graph, where $2 \leq t \leq m(a, b)$ ($\delta_t = t(a-b+2)$).

Conjecture 3.4. \mathcal{L} coincides with the lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ related to the substitution $\sigma_{a,b}$ for all $a \geq b \geq 1$.

Theorem 3.2 shows that for $a \leq 2b-4$, i.e., $m(a, b) \geq 2$, the each tile in the lattice tiling has 10 or more neighbours. Using a classical result concerning lattice tilings (see, for example, [4, Lemma 5.1] or [14]), we conclude that the Rauzy fractal in these cases are not disk-like.

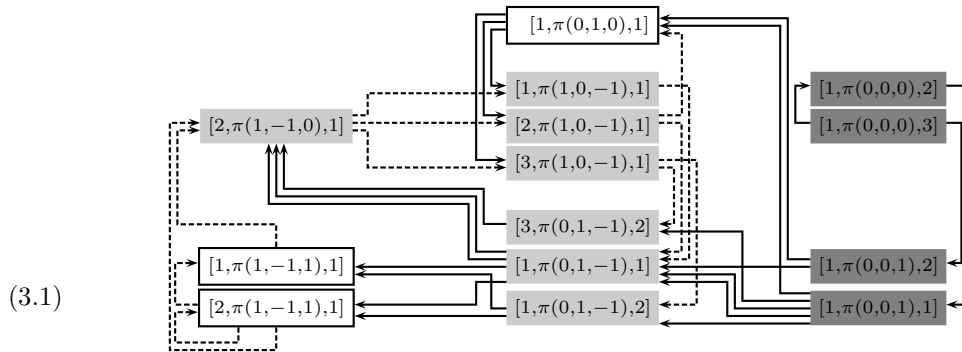
Corollary 3.5. *If $a \leq 2b-4$ then the Rauzy fractal \mathcal{T} induced by the substitution $\sigma_{a,b}$ is not homeomorphic to a topological disk.*

Vertex	Edge(s) to	Label(s)	T	Condition
$[1, \pi(0, t, -t), 1]$	$[3, \pi(t, -t, t), 1]$ $[1, \pi(t, -t, t+1), 1]$ $[1, \pi(t, -t, t+1), 2]$	$\{\pi(\delta_t - t, 0, 0)\}$ $\{\pi(e, 0, 0) \mid 0 \leq e \leq a - 2 - \delta_t + t\}$ $\{\pi(e, 0, 0) \mid 0 \leq e \leq b - 2 - \delta_t + t\}$	1 1 1	
$[1, \pi(0, t, -t), 2]$	$[1, \pi(t, -t, t+1), 1]$	$\{\pi(a - 1 - \delta_t + t, 0, 0)\}$	1	
$[1, \pi(1, t - 1, -t), 1]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$ $[3, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(e + 1, t - 1, -t) \mid 0 \leq e \leq \delta_t - t - 1\}$ $\{\pi(e + 1, t - 1, -t) \mid 0 \leq e \leq \delta_t - t - 1\}$ $\{\pi(1, t - 1, -t)\}$	2 2 2	
$[2, \pi(1, t - 1, -t), 1]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(\delta_t - t + 1, t - 1, -t)\}$ $\{\pi(\delta_t - t + 1, t - 1, -t)\}$	2 2	
$[1, \pi(2 - t, t - 1, -t), 1]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$ $[3, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(e + 2 - t, t - 1, -t) \mid 0 \leq e \leq \delta_t - 2\}$ $\{\pi(e + 2 - t, t - 1, -t) \mid 0 \leq e \leq \delta_t - 2\}$ $\{\pi(2 - t, t - 1, -t)\}$	2 2 2	
$[2, \pi(2 - t, t - 1, -t), 1]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(\delta_t + 1 - t, t - 1, -t)\}$ $\{\pi(\delta_t + 1 - t, t - 1, -t)\}$	2 2	
$[1, \pi(t - 1, 1 - t, t), 1]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$ $[3, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t - 1\}$ $\{\pi(e, 0, 0) \mid 0 \leq e \leq \delta_t - 1\}$ $\{\pi(0, 0, 0)\}$	1 1 1	
$[1, \pi(t - 1, 1 - t, t), 2]$	$[1, \pi(1 - t, t, -t - 1), 1]$ $[2, \pi(1 - t, t, -t - 1), 1]$	$\{\pi(\delta_t, 0, 0)\}$ $\{\pi(\delta_t, 0, 0)\}$	1 1	
$[1, \pi(1 - t, t, -t), 1]$	$[3, \pi(t, -t, t), 1]$ $[1, \pi(t, -t, t+1), 1]$ $[1, \pi(t, -t, t+1), 2]$	$\{\pi(0, 0, 0)\}$ $\{\pi(e, 0, 0) \mid 0 \leq e \leq a - \delta_t - 1\}$ $\{\pi(e, 0, 0) \mid 0 \leq e \leq b - \delta_t - 1\}$	1 1 1	
$[1, \pi(1 - t, t, -t), 2]$	$[1, \pi(t, -t, t+1), 1]$	$\{\pi(a - \delta_t, 0, 0)\}$	1	
$[2, \pi(t, -t, t), 1]$	$[1, \pi(t, -t, t+1), 1]$	$\{\pi(a - \delta_t - 1 + t, -t, t)\}$	2	
$[1, \pi(t, -t, t), 1]$	$[3, \pi(t, -t, t), 1]$ $[1, \pi(t, -t, t+1), 1]$ $[1, \pi(t, -t, t+1), 2]$	$\{\pi(t, -t, t)\}$ $\{\pi(e + t, -t, t) \mid 0 \leq e \leq a - \delta_t - 2\}$ $\{\pi(e + t, -t, t) \mid 0 \leq e \leq b - \delta_t - 2\}$	2 2 2	

ADJACENCY TABLE 6. The subgraph $\mathcal{L}'(t)$ of the lattice boundary graph, where $1 \leq t \leq m(a, b) - 1$ ($\delta_t = t(a - b + 2)$).

3.3. Examples.

Example 3.6. Let $a = 3$ and $b = 2$. The self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ of $\sigma_{3,2}$ consists of 14 vertices. The graph is shown in (3.1). Edges of Type 1 are drawn solid while those of Type 2 are dashed. The labels can be obtained from Adjacency Table 1. The lattice boundary graph $\mathcal{G}_{lat}^{(B)}$ has a similar shape. Indeed, it can be obtained from (3.1) by removing the darkgrey vertices (and the associated edges). The vertices that correspond to elements of Γ_{lat} are the lightgrey ones. Figure 2 shows the Rauzy fractal and its neighbours in the self-replicating tiling (left) and the lattice tiling (right). $\mathcal{T}(1)$ is the biggest subtile, followed by $\mathcal{T}(2)$ and $\mathcal{T}(3)$. The numbers inside show the corresponding translation. The boundaries between subtiles with respect to the same translation are grey.



Example 3.7. We consider the substitution $\sigma_{4,4}$. We use Theorem 3.1 to obtain its self-replicating boundary graph. It consists of 24 vertices. The graph is sketched in [22, Figure 5.7] (without labels). The labels can be taken from Adjacency Tables 1, 2 and 3. By Theorem 3.2, the lattice boundary graph for this example has at least 24 vertices and the Rauzy fractal has 10 neighbours in the lattice tiling. Thus, it is not homeomorphic to a disk. The two tilings can be found in [22, Figure 3.2]. The actual graph is depicted in [22, Figure 5.3]. We see that in this case, Conjecture 3.4 holds.

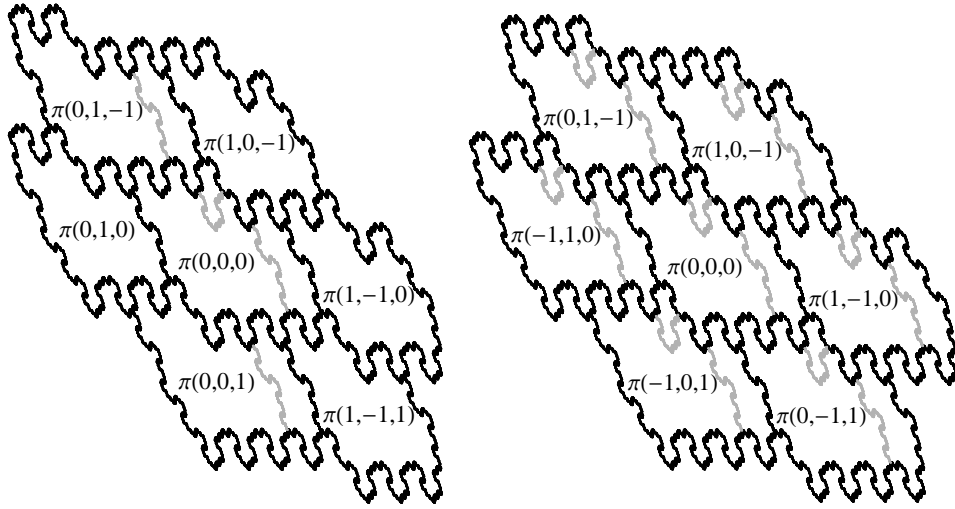


FIGURE 2. The neighbours of the Rauzy fractal in the self-replicated (left) and the lattice (right) tiling for $\sigma_{3,2}$.

4. SOME PREPARATIONS

We are collecting here several theorems and lemmas needed for the proof of the main results of the paper. In fact, they could be derived for other types of substitutions. As already mentioned at the end of Section 2, we give an alternative definition of the self-replicating boundary graph. Actually, the following theorem can be directly generalised to each primitive unimodular Pisot substitution.

Theorem 4.1. *The self replicated boundary graph $\mathcal{G}_{srs}^{(B)}$ equals the largest graph with*

- (1) *vertex set that consists of elements $[i, \gamma, j] \in \mathcal{D}$ with $[\gamma, j] \in \Gamma_{srs}$;*
- (2) *an edge from $[i, \gamma, j]$ to $[i', \gamma', j']$ if and only if there exist $[\bar{i}, \bar{\gamma}, \bar{j}] \in \mathcal{A} \times \pi(\mathbb{Z}^3) \times \mathcal{A}$ and $(p_1, a_1, s_1), (p_2, a_2, s_2) \in \mathcal{P}$ such that*

$$\begin{cases} [i', \gamma', j'] = [\bar{i}, \bar{\gamma}, \bar{j}] \text{ (Type 1) or } [i', \gamma', j'] = [\bar{j}, -\bar{\gamma}, \bar{i}] \text{ (Type 2),} \\ a_1 = i \text{ and } p_1 a_1 s_1 = \sigma(\bar{i}), \\ a_2 = j \text{ and } p_2 a_2 s_2 = \sigma(\bar{j}), \\ \mathbf{h}\bar{\gamma} = \gamma + \pi(\mathbf{l}(p_2) - \mathbf{l}(p_1)). \end{cases}$$

The edge is labelled by

$$\eta = \begin{cases} \pi \mathbf{l}(p_1), & \langle \mathbf{l}(p_1), \mathbf{v}_\beta \rangle \leq \langle \mathbf{l}(p_2) + \mathbf{x}, \mathbf{v}_\beta \rangle, \\ \pi \mathbf{l}(p_2) + \gamma, & \text{otherwise;} \end{cases}$$

where $\mathbf{x} \in \mathbb{Z}^3$ such that $\pi(\mathbf{x}) = \gamma$;

- (3) *every vertex lies on a path that ends up in a strongly connected component.*

Proof. Denote by G the largest graph fulfilling (1), (2) and (3) from above. We want to prove that $G = \mathcal{G}_{srs}^{(B)}$. The edges are defined in the same way, hence, it suffices to prove that both graphs have the same set of vertices.

By Proposition 2.3 for every vertex $[i, \gamma, j]$ of $\mathcal{G}_{srs}^{(B)}$ we have $[\gamma, j] \in \Gamma_{srs}$. Furthermore, $\mathcal{G}_{srs}^{(B)}$ is finite by Proposition 2.2 and every vertex of $\mathcal{G}_{srs}^{(B)}$ lies on an infinite walk by Definition 2.1. This implies that every vertex, in fact, lies on a path ending up in a strongly connected component. Hence, $\mathcal{G}_{srs}^{(B)}$ is a subgraph of G .

Now consider $[i, \gamma, j] \in G$. Obviously, $[i, \gamma, j] \in \mathcal{G}_{srs}^{(B)}$ as soon as γ satisfies (2.6). Indeed, the other points of Definition 2.1 are easily seen to be fulfilled.

By (3) there exists a (finite) path from $[i, \gamma, j]$ to a vertex belonging to a strongly connected component of G . Therefore, there is an infinite path

$$[i, \gamma, j] \rightarrow [i_1, \gamma_1, j_1] \rightarrow [i_2, \gamma_2, j_2] \rightarrow \dots$$

in G ending in a cycle, thus going through finitely many vertices. Using the relation $\mathbf{h}\gamma_{k+1} = \pm\gamma_k \pm \pi(\mathbf{l}(p_2^{(k)}) - \mathbf{l}(p_1^{(k)}))$ that holds for each edge of this walk and the fact that \mathbf{h} is a contraction, one obtains that γ satisfies (2.6). \square

In the following lemma we estimate the number and shape of the predecessors of a given vertex in the boundary graph. The computations could be derived for every irreducible beta-substitution.

Lemma 4.2. *Consider an edge from $[i, \pi(\mathbf{x}), j]$ to $[i', \pi(x', y', z'), j']$ in the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. Then*

$$(4.1) \quad \mathbf{x} = (-\lfloor x'(\beta^{-2} + b\beta^{-1}) + y'\beta^{-1} \rfloor, x', y')$$

if the edge is of Type 1 and

$$(4.2) \quad \mathbf{x} = (\lceil x'(\beta^{-2} + b\beta^{-1}) + y'\beta^{-1} \rceil, -x', -y')$$

if the edge is of Type 2.

Proof. If the edge is of Type 1, by the definition of $\mathcal{G}_{srs}^{(B)}$ and (2.2), we have

$$(4.3) \quad \mathbf{h}(\pi(x', y', z')) = \pi(\mathbf{M}(x', y', z')) = \pi((ax' + by' + z', x', y')) = \pi(\mathbf{x}) - \pi(\mathbf{l}(p_1)) + \pi(\mathbf{l}(p_2))$$

with $\sigma(i') = p_1 i s_2$ and $\sigma(j') = p_2 j s_2$. Now observe that for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Q}^3$ we have

$$\pi(\mathbf{x}_1) = \pi(\mathbf{x}_2) \iff \langle \mathbf{x}_1, \mathbf{v}_\beta \rangle = \langle \mathbf{x}_2, \mathbf{v}_\beta \rangle \iff \mathbf{x}_1 = \mathbf{x}_2.$$

The first equivalence can be obtained by considering Galois conjugates (cf. [22, Equation (2.5)]), the second one is a consequence of the irreducibility of the substitution. In particular, this shows that π is injective for integer vectors. Therefore (4.3) can only hold if

$$(4.4) \quad \mathbf{x} = (ax' + by' + z' - e_2 + e_1, x', y')$$

where $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$ are integer vectors (by the shape of $\sigma_{a,b}$ the prefixes p_1 and p_2 consist of the symbols ε or 1 only). Since $[i, \pi(\mathbf{x}), j]$ is a vertex of $\mathcal{G}_{srs}^{(B)}$ we have, by Proposition 2.3, $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_j < 1$. Applying this on (4.4) gives

$$\begin{aligned} 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle &= \langle (ax' + by' + z' - e_2 + e_1, x', y'), \mathbf{v}_\beta \rangle \\ &= ax' + by' + z' - e_2 + e_1 + x'(\beta^{-2} + b\beta^{-1}) + y'\beta^{-1} < 1. \end{aligned}$$

Since $ax' + by' + z' - e_2 + e_1$ is an integer we immediately obtain $ax' + by' + z' - e_2 + e_1 = -\lfloor x'(\beta^{-2} + b\beta^{-1}) + y'\beta^{-1} \rfloor$. Inserting this into (4.4) yields the assertion.

If the edge is of Type 2 we obtain, analogously to (4.4),

$$(4.5) \quad \mathbf{x} = (-ax' - by' - z' - e_2 + e_1, -x', -y')$$

with $\sigma(j') = p_1 i s_2$, $\sigma(i') = p_2 j s_2$, $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$. The same considerations as above yield

$$-ax' - by' - z' - e_2 + e_1 = -\lfloor -x'(\beta^{-2} + b\beta^{-1}) - y'\beta^{-1} \rfloor = \lceil x'(\beta^{-2} + b\beta^{-1}) + y'\beta^{-1} \rceil$$

which gives (4.2). \square

We see that for a predecessor $[i, \pi(\mathbf{x}), j]$ of a vertex $[i', \gamma', j']$ in the self-replicating boundary graph there are at most 2 possible choices for \mathbf{x} , one connected via an edge of Type 1, another connected via an edge of Type 2. We also see that, if there is a predecessor of the form $[i, \gamma, j]$ with $\gamma = \mathbf{0}$, then all predecessors are of this shape. In the proof of Theorem 3.1 we will frequently make use of this fact. Again, an analogue to Lemma 4.2 for $\mathcal{G}_{lat}^{(B)}$ does not exist and makes the proof of Theorem 3.2 more complicated.

Notation 4.3. Given an edge from $[i, \pi(x, y, z), j]$ to $[i', \pi(x', y', z'), j']$ we call the term $ax' + by' + z' - x$ (when the edge is of Type 1) or the term $ax' + by' + z' + x$ (when the edge is of Type 2), respectively, the *significant difference*.

We are interested in the pairs (i, j) , (i', j') inducing significant differences. This is shown in Table 7. There is an entry in the cell in row (i, j) and column (i', j') if there are $p_1, p_2, s_1, s_2 \in \mathcal{A}^*$ such that $\sigma(i') = p_1 i s_1$ and $\sigma(j') = p_2 j s_2$, i.e., if there are edges $i \xrightarrow{(p_1, i, s_1)} i'$ and $j \xrightarrow{(p_2, j, s_2)} j'$ in Γ_σ . The corresponding entry is then a list of all possibilities for $e_2 - e_1$ with $(e_1, 0, 0) = \mathbf{l}(p_1)$ and $(e_2, 0, 0) = \mathbf{l}(p_2)$. Note that, by construction, n is an element of the list in row (i, j) and column (i', j') if and only if $-n$ is an element of the list in row (j, i) and column (j', i') .

	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)	(3, 1)	(3, 2)	(3, 3)
(1, 1)	$-a+1, \dots, a-1$	$-a+1, \dots, b-1$	$-a+1, \dots, 0$	$-b+1, \dots, a-1$	$-b+1, \dots, b-1$	$-b+1, \dots, 0$	$0, \dots, a-1$	$0, \dots, b-1$	0
(1, 2)	$1, \dots, a$			$a-b+1, \dots, a$			a		
(1, 3)		$b-a+1, \dots, b$			$1, \dots, b$			b	
(2, 1)	$-a, \dots, -1$	$-a, \dots, -a+b-1$	$-a$						
(2, 2)	0								
(2, 3)		$b-a$							
(3, 1)				$-b, \dots, a-b-1$	$-b, \dots, -1$	$-b$			
(3, 2)				$a-b$					
(3, 3)					0				

TABLE 7. The Table shows all possible differences for the prefixes of the labels of the product automaton $\Gamma_\sigma \times \Gamma_\sigma$

The use of this table is enlightened in the following lemma.

Lemma 4.4. *Consider an edge from $[i, \pi((x, y, z)), j]$ to $[i', \pi((x', y', z')), j']$ in the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. If the edge is of Type 1 then $ax' + by' + z' - x$ is contained in the list in row (i, j) and column (i', j') of Table 7. If the edge is of Type 2 then $ax' + by' + z' + x$ is contained in the list in row (j, i) and column (i', j') of Table 7.*

Proof. Let p_1, p_2 as in (2) of Definition 2.1. By the shape of the substitution we have $\mathbf{l}(p_1) = (e_1, 0, 0)$ and $\mathbf{l}(p_2) = (e_2, 0, 0)$ with $e_1, e_2 \geq 0$. Suppose the edge is of Type 1. By construction of Table 7, $e_2 - e_1$ is an element of the list in row (i, j) and column (i', j') and by (4.4) we have that $e_2 - e_1 = ax' + by' + z' - x$.

If the edge is of Type 2, $e_1 - e_2$ is element of the list in row (j, i) and column (i', j') . On the other hand, (4.5) shows that $ax' + by' + z' + x = e_1 - e_2$. \square

5. PROOF OF THEOREM 3.1

The present section is devoted to the proof of Theorem 3.1. The first lemma (Lemma 5.1) shows that the vertices of \mathcal{S} defined in Section 3.1 are really related to the self-replicating translation set Γ_{srs} , i.e., that \mathcal{S} satisfies Proposition 2.3. Afterwards, in Lemma 5.2, we characterise the strongly connected component of the self-replicating boundary graph. Finally we will use Theorem 4.1 to prove Theorem 3.1.

Lemma 5.1. *For each vertex $[i, \pi(\mathbf{x}), j]$ of \mathcal{S} , $[\pi(\mathbf{x}), j] \in \Gamma_{srs}$.*

Proof. By definition, $[\pi(\mathbf{x}), j] \in \Gamma_{srs}$ if and only if $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_j$. Recall that $\mathbf{v}_\beta = (v_1, v_2, v_3) = (1, \beta - a, \beta^2 - a\beta - b)$. For $\mathbf{x} = \mathbf{0}$ the statement is trivial. If $\mathbf{x} \neq \mathbf{0}$ we consider six cases.

Case 1. $\mathbf{x} = (0, 1, 0)$: using (2.1) we see

$$0 \leq \langle (0, 1, 0), \mathbf{v}_\beta \rangle = v_2 < v_1.$$

Thus $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 2. $\mathbf{x} = (1, -1, 0)$: again, (2.1) immediately yields

$$0 \leq \langle (1, -1, 0), \mathbf{v}_\beta \rangle = v_1 - v_2 < v_1,$$

hence $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 3. $\mathbf{x} = (t, -t, t)$ ($1 \leq t \leq m(a, b)$): we have $\langle (t, -t, t), \mathbf{v}_\beta \rangle = tc$ with

$$c := \langle (1, -1, 1), \mathbf{v}_\beta \rangle = 1 - (\beta^{-2} + b\beta^{-1}) + \beta^{-1} = \frac{a-b+2}{\beta+1}.$$

Hence,

$$\langle (t, -t, t), \mathbf{v}_\beta \rangle = tc = t \frac{a-b+2}{\beta+1} \geq 0$$

for each $t \geq 0$. On the other hand, if $\frac{a}{a-b+2} \geq 1$, we have $t \leq \frac{a}{a-b+2}$. Then

$$\langle (t, -t, t), \mathbf{v}_\beta \rangle \leq \frac{a}{a-b+2} c = \frac{a}{\beta+1} < 1 = v_1$$

since $a < \beta$. If $\frac{a}{a-b+2} < 1$ (and therefore $t = 1$) we have $b < 2$ and, hence, $b = 1$. Thus

$$\langle (t, -t, t), \mathbf{v}_\beta \rangle = c = \frac{a-b+2}{\beta+1} = \frac{a+1}{\beta+1} < 1.$$

In both cases, $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 4. $\mathbf{x} = (2-t, t-1, -t)$ ($1 \leq t \leq m(a, b)$):

$$\langle (2-t, t-1, -t), \mathbf{v}_\beta \rangle = 1 - (t-1) \frac{a-b+2}{\beta+1} - \beta^{-1} < 1.$$

On the other hand,

$$\begin{aligned} \langle (2-t, t-1, -t), \mathbf{v}_\beta \rangle &= 2 - b\beta^{-1} - \beta^{-2} - t \frac{a-b+2}{\beta+1} \\ &= \underbrace{(1 - b\beta^{-1} - \beta^{-2})}_{>0} + \underbrace{(1 - t \frac{a-b+2}{\beta+1})}_{>0} > 0. \end{aligned}$$

Thus $[\pi(\mathbf{x}), 1] \in \Gamma_{srs}$.

Case 5. $\mathbf{x} = (t-1, 1-t, t)$ ($1 \leq t \leq m(a, b)$): we use the previous case to estimate

$$\begin{aligned} (5.1) \quad 0 < v_3 = \langle (0, 0, 1), \mathbf{v}_\beta \rangle &\leq \underbrace{\langle (t-1, 1-t, t-1), \mathbf{v}_\beta \rangle}_{\geq 0 \text{ by Case 3}} + \langle (0, 0, 1), \mathbf{v}_\beta \rangle = \langle (t-1, 1-t, t), \mathbf{v}_\beta \rangle \\ &= -1 + \langle (t, -t, t), \mathbf{v}_\beta \rangle + \langle (0, 1, 0), \mathbf{v}_\beta \rangle < \langle (0, 1, 0), \mathbf{v}_\beta \rangle = v_2. \end{aligned}$$

Hence $[\pi(\mathbf{x}), 1], [\pi(\mathbf{x}), 2] \in \Gamma_{srs}$.

Case 6. $\mathbf{x} = (1-t, t, -t)$ ($1 \leq t \leq m(a, b)$): by Case 3,

$$\langle (1-t, t, -t), \mathbf{v}_\beta \rangle = 1 - \langle (t, -t, t), \mathbf{v}_\beta \rangle > 0.$$

Lower estimation yields

$$\begin{aligned} \langle (1-t, t, -t), \mathbf{v}_\beta \rangle &= -(t-1) \langle (1, -1, 1), \mathbf{v}_\beta \rangle + \langle (0, 1, 0), \mathbf{v}_\beta \rangle - \langle (0, 0, 1), \mathbf{v}_\beta \rangle \\ &< \langle (0, 1, 0), \mathbf{v}_\beta \rangle = v_2. \end{aligned}$$

This shows that $[\pi(\mathbf{x}), 1], [\pi(\mathbf{x}), 2] \in \Gamma_{srs}$.

□

In the following lemma we will characterize the strongly connected components of the self-replicating boundary graph. Let

$$\begin{aligned} C(1) := & \{[1, \pi(0, 1, -1), 1], [1, \pi(0, 1, -1), 2], [3, \pi(0, 1, -1), 2], [2, \pi(1, 0, -1), 1], \\ & [3, \pi(1, 0, -1), 1], [2, \pi(1, -1, 0), 1], [2, \pi(1, -1, 1), 1]\} \\ & \cup \begin{cases} \{[1, \pi(0, 1, 0), 1], [1, \pi(1, 0, -1), 1]\} & \text{if } a \neq b \\ \emptyset & \text{otherwise} \end{cases} \\ & \cup \begin{cases} \{[1, \pi(1, -1, 1), 1]\} & \text{if } b \geq 2 \\ \emptyset & \text{otherwise} \end{cases}. \end{aligned}$$

and, for $t \in \{2, \dots, m(a, b)\}$, set

$$C(t) = \{[2, \pi(t, -t, t), 1]\} \cup \begin{cases} \{[1, \pi(t, -t, t), 1]\} & \text{if } a > t(a - b + 2) \\ \emptyset & \text{otherwise} \end{cases}.$$

We will show that the vertices of the strongly connected components of $\mathcal{G}_{srs}^{(B)}$ are contained in the sets $C(t)$. For our aims this is enough here. In fact, using Theorem 3.1, one can easily verify that there are exactly $m(a, b)$ strongly connected components whose vertices coincide with $C(t)$ for $t = 1, \dots, m(a, b)$.

Lemma 5.2. *The vertices of the strongly connected components of the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ are contained in $\bigcup_{t=1}^{m(a,b)} C(t)$.*

Proof. The vertices of the strongly connected components are exactly those vertices that are contained in cycles. Therefore, consider a cycle of the self-replicating boundary graph passing the vertices $[i_n, \pi(x_n, y_n, z_n), j_n]$, $n \in \{0, \dots, q-1\}$. By Proposition 2.3, for every $n \in \{0, \dots, q-1\}$ we have $[\pi(x_n, y_n, z_n), j_n] \in \Gamma_{srs}$ and, by definition, $[i_n, \pi(x_n, y_n, z_n), j_n]$ and $[i_{n+1}, \pi(x_{n+1}, y_{n+1}, z_{n+1}), j_{n+1}]$ (indices modulo q) satisfy (2) of Definition 2.1. Let

$$t := \max_{n \in \{0, \dots, q-1\}} \|(x_n, y_n, z_n)\|_\infty.$$

At first suppose that $t \geq 2$. We start with proving that $t \leq m(a, b)$ and deduce that for all $n \in \{0, \dots, q-1\}$ we have $(x_n, y_n, z_n) = (t, -t, t)$. We first claim that $x_n \neq -t$ for all $n \in \{0, \dots, q-1\}$. Suppose x_n equals $-t$. Then, by the fact that $[i_n, \pi(x_n, y_n, z_n), j_n] \in \mathcal{D}$, we have

$$\langle (x_n, y_n, z_n), \mathbf{v}_\beta \rangle = -t + y_n(b\beta^{-1} + \beta^{-2}) + z_n\beta^{-1} \geq 0.$$

Since $|y_n| \leq t$, $|z_n| \leq t$ and $t \geq 2$ we necessarily have that y_n and z_n are positive and at least one of them is strictly greater than 1. Furthermore, we have

$$y_n(b\beta^{-1} + \beta^{-2}) + z_n\beta^{-1} \geq t \Rightarrow y_n(a\beta^{-1} + b\beta^{-2} + \beta^{-3}) + z_n(b\beta^{-1} + \beta^{-2}) \geq t\beta^{-1} + y_n a\beta^{-1} + z_n b\beta^{-1}.$$

Using Lemma 4.2 we can estimate

$$\begin{aligned} |\langle (x_{n+1}, y_{n+1}, z_{n+1}), \mathbf{v}_\beta \rangle| &= |\langle (y_n, z_n, z_{n+1}), \mathbf{v}_\beta \rangle| \\ &\geq |y_n(a\beta^{-1} + b\beta^{-2} + \beta^{-3}) + z_n(b\beta^{-1} + \beta^{-2})| - |z_{n+1}\beta^{-1}| \\ &\geq t\beta^{-1} + y_n a\beta^{-1} + z_n b\beta^{-1} - t\beta^{-1} = y_n a\beta^{-1} + z_n b\beta^{-1} > 1, \end{aligned}$$

which contradicts the assumption that $[\pi(x_{n+1}, y_{n+1}, z_{n+1}), j_{n+1}] \in \Gamma_{srs}$. Therefore, for all $n \in \{0, \dots, q-1\}$, $x_n \neq -t$.

By this consideration and Lemma 4.2, we may assume without loss of generality that $x_0 = t$. We have

$$(5.2) \quad \langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle = t \underbrace{(a\beta^{-1} + b\beta^{-2} + \beta^{-3})}_{=1} + y_0(b\beta^{-1} + \beta^{-2}) + z_0\beta^{-1}.$$

By the definition of t , it is clear that we have $-t \leq y_0, z_0 \leq t$. Suppose that $-t + 1 \leq y_0 \leq t$. We will derive a contradiction. There are two cases.

Case 1. $0 \leq y_0 \leq t$: in this case (5.2) reduces to

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq t(a-1)\beta^{-1} + tb\beta^{-2} + t\beta^{-3}.$$

Our assumption that $t \geq 2$ implies that, if $a > 1$, we have $t(a-1) \geq a$. Therefore

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq a\beta^{-1} + tb\beta^{-2} + t\beta^{-3} > 1.$$

If $a = 1$ then $b = 1$, which is the classical Tribonacci case, it is easy to verify that

$$t(\beta^{-2} + \beta^{-3}) \geq 2(\beta^{-2} + \beta^{-3}) > 1,$$

too. In both cases, this contradicts the fact that $[\pi(x_0, y_0, z_0), j_0] \in \Gamma_{srs}$.

Case 2. $-t + 1 \leq y_0 \leq -1$: here (5.2) gives

$$\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle \geq 1 + (t - 1) \left((a - b - 1)\beta^{-1} + (b - 1)\beta^{-2} + \beta^{-3} \right) + (z_0 + t - 1)\beta^{-1}.$$

If $a > b + 1$ this expression is again greater than 1 since $z_0 \geq -t$. Also for $a = b + 1$ and $z_0 \geq -t + 1$ as well as for $a = b$ and $z_0 \geq 0$ we have $\langle (x_0, y_0, z_0), \mathbf{v}_\beta \rangle > 1$. Thus we have two subcases left.

- Suppose $a = b + 1$ and $z_0 = -t$. Whenever the edge from $[i_0, \pi(x_0, y_0, z_0), j_0]$ to $[i_1, \pi((x_1, y_1, z_1)), j_1]$ is of Type 1, the significant difference is

$$ax_1 + by_1 + z_1 - x_0 \geq -a.$$

Using (4.4) we obtain

$$z_1 \geq -ax_1 - by_1 + x_0 - a = -ay_0 - bz_0 + x_0 - a \geq a + bt + t - a > t.$$

Similarly, if the edge is of Type 2, we deduce $z_1 < -t$. In both cases this contradicts the definition of t .

- The case $a = b$ and $-t \leq z_0 \leq -1$ is treated analogously.

Therefore the only possibility is $y_0 = -t$. By Lemma 4.2 this implies that $x_1 = \pm t$. Thus, by the beginning of this proof, $x_1 = t$. We can prove in a similar way that $y_1 = -t$. Now it follows from Lemma 4.2 that the edge is necessarily of Type 2 and, hence, $z_0 = t$. We infer that $(x_n, y_n, z_n) = (t, -t, t)$ for all $n \in \{0, \dots, q - 1\}$.

Since all the edges are of Type 2, the significant difference is $at - bt + t + t = t(a - b + 2) \leq a$ by (4.5). This yields $t \leq \frac{a}{a-b+2}$ and, since t is an integer, $t = \lfloor \frac{a}{a-b+2} \rfloor = m(a, b)$, as it was claimed.

Up to now we have proved that, for a cycle $([i_n, \pi(x_n, y_n, z_n), j_n])_{n \in \{0, \dots, q-1\}}$ in Γ_{srs} , if we set $t := \max_{n \in \{0, \dots, q-1\}} \|(x_n, y_n, z_n)\|_\infty$ and suppose $t \geq 2$, then $t \leq m(a, b)$ and for all n , $(x_n, y_n, z_n) = (t, -t, t)$.

To determine the exact set of vertices we use Lemma 4.4. More precisely, for an edge of Type 2 to exist from $[i, \pi(t, -t, t), j]$ to $[i', \pi(t, -t, t), j']$, the list in row (j, i) and column (i', j') in Table 7 must necessarily contain $t(a - b + 2)$. Thus, we search in Table 7 for values between $a - b + 2$ ($t = 1$) and a ($t = m(a, b)$). Note that we allow $t = 1$ - this case will be needed later. The cells of Table 8 contain all such pairs $((i, j), (i', j'))$. We do not have to take care of the pairs $((2, 3), (2, 1))$ and $((1, 3), (2, 1))$ since $a - b + 2 > a - b > a - b - 1$.

(1, 1)	(1, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(3, 1)	(1, 2)	(1, 1)	(2, 1)	(2, 1)	(2, 1)
(1, 1)	(2, 2)	(3, 1)	(2, 2)	(1, 1)	(3, 1)	(2, 1)	(3, 1)	(1, 1)	(3, 2)	(3, 1)	(3, 2)

TABLE 8

Each cell represents possible edges with the same initial vertex and terminal vertex. An edge that has its origin (destination, respectively) in a vertex that is not destination (origin, respectively) of another edge cannot be part of a strongly connected component of the graph. Therefore, we successively delete the cells whose first element does not appear as second element of another cell and vice versa. Finally, only the four cells highlighted in grey remain:

$$((1, 1), (1, 1)), ((1, 1), (2, 1)), ((2, 1), (1, 1)) \text{ and } ((2, 1), (2, 1)).$$

Consequently, if a strongly connected component contains a cycle with $t \geq 2$, then a vertex of this component is either $[1, \pi(t, -t, t), 1]$ or $[2, \pi(t, -t, t), 1]$. Note that for $t(a - b + 2) = a$ the vertex $[1, \pi(t, -t, t), 1]$ has no outgoing edge and therefore this point cannot be a vertex of the strongly connected component.

Therefore, we have shown that the vertices of the strongly connected components are contained in $C(t)$ for $2 \leq t \leq \frac{a}{a-b+2}$.

Shape	from	to	Type	significant difference
1	$[i, \pi(0, 1, 0), j]$	$[i', \pi(1, 0, -1), j']$	Type 1	$a - 1 \geq 0$
2	$[i, \pi(0, 1, -1), j]$	$[i', \pi(1, -1, 0), j']$	Type 1	$a - b \geq 0$
3	$[i, \pi(0, 1, -1), j]$	$[i', \pi(1, -1, 1), j']$	Type 1	$a - b + 1 \geq 1$
4	$[i, \pi(1, 0, -1), j]$	$[i', \pi(0, 1, 0), j']$	Type 2	$b + 1 \geq 2$
5	$[i, \pi(1, 0, -1), j]$	$[i', \pi(0, 1, -1), j']$	Type 2	b
6	$[i, \pi(1, -1, 0), j]$	$[i', \pi(1, 0, -1), j']$	Type 2	a
7	$[i, \pi(1, -1, 1), j]$	$[i', \pi(1, -1, 0), j']$	Type 2	$a - b + 1 \geq 1$
8	$[i, \pi(1, -1, 1), j]$	$[i', \pi(1, -1, 1), j']$	Type 2	$a - b + 2 \geq 2$

TABLE 9

We now treat the case of the possible strongly connected components whose vertices $[i, \pi(\mathbf{x}), j]$ satisfy $\|\mathbf{x}\|_\infty \leq 1$. There are 27 \mathbb{Z}^3 -vectors whose maximum norm is less or equal to 1. For these vertices to belong to the self-replicating boundary graph we must have $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < 1$. Thus, we can restrict to

$$\mathbf{x} \in \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, -1), (1, 0, -1), (1, -1, 0), (1, -1, 1), (0, 1, 1), (-1, 1, 1)\}.$$

The vectors $(0, 1, 1)$ and $(-1, 1, 1)$ can be excluded. Their possible successors in a strongly connected component would have the form $[i, \pi(1, 1, z'), j]$ with $z' \geq -1$. But then $\langle (1, 1, z'), \mathbf{v}_\beta \rangle \geq 1 + v_2 - v_3 > 1$, thus $[\pi(1, 1, z'), j]$ is not in the tiling set Γ_{srS} , a contradiction.

The point $\mathbf{x} = (0, 0, 0)$ does not give rise to a strongly connected component for the following reason. By Lemma 4.2, the only possible predecessor for a vertex of the form $[i', \mathbf{0}, j']$ is of the shape $[i, \mathbf{0}, j]$. By Lemma 4.4 the list in row (i, j) (row (j, i) , respectively) and column (i', j') in Table 7 must contain 0. Since by definition $i \neq j$ and $i' \neq j'$ we easily see that such edges may only occur for $(i, j) = (1, 3)$ and $(i', j') = (1, 2)$ and for $(i, j) = (2, 3)$ and $(i', j') = (1, 2)$. This leads to $[1, (0, 0, 0), 3] \rightarrow [1, (0, 0, 0), 2]$ and $[2, (0, 0, 0), 3] \rightarrow [1, (0, 0, 0), 2]$ as the only possible edges in the strongly connected component. But combining these two edges does not give rise to a component. Therefore, $\mathbf{x} = (0, 0, 0)$ cannot show up in a strongly connected component.

Since $\mathbf{x} = (0, 0, 0)$ does not induce vertices of a strongly connected component we can exclude $\mathbf{x} = (0, 0, 1)$ completely (again, by Lemma 4.2).

By Lemma 4.2, the remaining possibilities for \mathbf{x} provide eight different shapes of edges shown in Table 9.

Similarly as before we use Lemma 4.4 and Table 7 and write down all possible edges. Table 10 consists of eight blocks corresponding to the eight shapes of edges. According to Lemma 4.4 the cells of a block contain all pairs (i, j) and (i', j') with suitable significant difference found in Table 7. Note that $[\pi(0, 1, 0), 2], [\pi(0, 1, 0), 3] \notin \Gamma_{srS}$. Hence, we do not write pairs $((i, j), (i', j'))$ involving these elements.

Now we can easily determine the possible strongly connected component algorithmically by successively deleting pairs (edges) whose origin does not appear as destination of another edge, in the same way as we have done in Table 8. In particular, we delete a cell whenever its left pair does not appear as a right pair in another cell of a suitable block or its right pair does not appear as a left pair in another cell of a suitable block (according to the shape of edges). A detailed proceeding can be found in the annex. The remaining cells are highlighted in grey. The corresponding vertices read as follows.

$$\begin{aligned} & [1, \pi(0, 1, 0), 1], [1, \pi(0, 1, -1), 1], [1, \pi(0, 1, -1), 2], [3, \pi(0, 1, -1), 2], [1, \pi(1, 0, -1), 1], \\ & [2, \pi(1, 0, -1), 1], [3, \pi(1, 0, -1), 1], [2, \pi(1, -1, 0), 1], [1, \pi(1, -1, 1), 1], [2, \pi(1, -1, 1), 1]. \end{aligned}$$

Block 1										$[i, \pi(0, 1, 0), j] \longrightarrow [i', \pi(1, 0, -1), j']$												
(1, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 1)	(2, 1)	(1, 1)	(2, 2)	(1, 1)	(2, 3)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 1)	(2, 1)	(1, 1)	(2, 2)	(1, 1)	(2, 3)
(1, 1)	(3, 1)	(1, 1)	(3, 2)	(1, 1)	(3, 3)																	
Block 2										$[i, \pi(0, 1, -1), j] \longrightarrow [i', \pi(1, -1, 0), j']$												
(1, 1)	(1, 1)	(1, 2)	(1, 1)	(2, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 3)	(1, 2)	(2, 3)	(1, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 3)	(1, 2)	(2, 3)	(1, 2)	
(1, 1)	(1, 3)	(1, 1)	(2, 1)	(3, 2)	(2, 1)	(1, 1)	(2, 2)	(1, 3)	(2, 2)	(3, 3)	(2, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 3)	(2, 2)	(3, 3)	(2, 2)	
(1, 1)	(2, 3)	(1, 1)	(3, 1)	(1, 1)	(3, 2)	(1, 3)	(3, 2)	(1, 1)	(3, 3)			(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 3)	(2, 2)	(3, 3)	(2, 2)	
Block 3										$[i, \pi(0, 1, -1), j] \longrightarrow [i', \pi(1, -1, 1), j']$												
(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
(1, 1)	(2, 2)	(1, 3)	(2, 2)	(1, 1)	(3, 1)	(1, 2)	(3, 1)	(1, 1)	(3, 2)	(1, 3)	(3, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
Block 4										$[i, \pi(1, 0, -1), j] \longrightarrow [i', \pi(0, 1, 0), j']$												
(1, 1)	(1, 1)	(2, 1)	(1, 1)	(1, 1)	(2, 1)	(1, 3)	(2, 1)	(2, 1)	(2, 1)	(2, 3)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
(1, 1)	(3, 1)	(2, 1)	(3, 1)									(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
Block 5										$[i, \pi(1, 0, -1), j] \longrightarrow [i', \pi(0, 1, -1), j']$												
(1, 1)	(1, 1)	(2, 1)	(1, 1)	(3, 1)	(1, 2)	(1, 1)	(2, 1)	(1, 3)	(2, 1)	(2, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
(2, 3)	(2, 1)	(3, 1)	(2, 2)	(1, 1)	(3, 1)	(2, 1)	(3, 1)	(3, 1)	(3, 2)			(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
Block 6										$[i, \pi(1, -1, 0), j] \longrightarrow [i', \pi(1, 0, -1), j']$												
(2, 1)	(1, 1)	(3, 1)	(1, 2)	(2, 1)	(2, 1)	(3, 1)	(2, 2)	(2, 1)	(3, 1)	(3, 1)	(3, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
Block 7										$[i, \pi(1, -1, 1), j] \longrightarrow [i', \pi(1, -1, 0), j']$												
(1, 1)	(1, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(3, 1)	(1, 2)	(1, 1)	(2, 1)	(2, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
(1, 1)	(2, 2)	(3, 1)	(2, 2)	(1, 1)	(3, 1)	(2, 1)	(3, 1)	(1, 1)	(3, 2)	(3, 1)	(3, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
Block 8										$[i, \pi(1, -1, 1), j] \longrightarrow [i', \pi(1, -1, 1), j']$												
(1, 1)	(1, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(3, 1)	(1, 2)	(1, 1)	(2, 1)	(2, 1)	(2, 1)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	
(1, 1)	(2, 2)	(3, 1)	(2, 2)	(1, 1)	(3, 1)	(2, 1)	(3, 1)	(1, 1)	(3, 2)	(3, 1)	(3, 2)	(1, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 3)	(1, 2)	(1, 1)	(2, 1)	(1, 2)	(2, 1)	

TABLE 10

We note that these vertices match with the vertices of $C(1)$ for $a > b > 1$. Hence, in this case the lemma is proved.

In the case $b = 1$, no type of edge that start in $[1, \pi(1, -1, 1), 1]$ (shapes 7 and 8) exists. Thus, the vertex $[1, \pi(1, -1, 1), 1]$ cannot be contained in the strongly connected components. This shows the lemma in the case $a > b = 1$.

Finally, suppose $a = b$. We see that there is no edge of shape 4, because the significant difference $a + 1$ does not show up. But this would be the only incoming edge for the vertex $[1, \pi(0, 1, 0), 1]$. Hence we deduce that $[1, \pi(0, 1, 0), 1]$ cannot be a vertex of the strongly connected components in this case. The same applies to the vertex $[1, \pi(1, 0, -1), 1]$. Hence, the lemma holds also for all $a = b \geq 1$. \square

Proof of Theorem 3.1. From Lemma 5.1 we know that for all vertices $[i, \gamma, j]$ of \mathcal{S} we have $[\gamma, j] \in \Gamma_{srs}$. It is also easy to verify that all edges satisfy (2) of Theorem 4.1. Furthermore, each vertex lies on a path ending in a strongly connected component. Therefore, \mathcal{S} is a subgraph of $\mathcal{G}_{srs}^{(B)}$.

Now we are going to check that $\mathcal{G}_{srs}^{(B)}$ is a subgraph of \mathcal{S} .

In Lemma 5.2 we showed that the vertices of the strongly connected components of $\mathcal{G}_{srs}^{(B)}$ are contained in the sets $C(1), \dots, C(m(a, b))$. Observe that \mathcal{S} contains all these vertices. The fact that every vertex in $\mathcal{G}_{srs}^{(B)}$ lies on a path that ends up in a strongly connected component immediately implies that such a path passes a vertex of \mathcal{S} .

We claim that it is impossible to add additional edges (and vertices) to \mathcal{S} that satisfy (1) and (2) in Theorem 4.1. This obviously proves the theorem. For this purpose we go through all types of vertices of \mathcal{S} and investigate the possible incoming edges. The strategy is always the same: we use Lemma 4.2 to show that a predecessor $[i, \gamma, j]$ of a vertex $[i', \gamma', j']$ can obtain at most two different values for γ and then we use Lemma 4.4 to determine i and j . Here, we show this

explicitly for vertices of the form $[i', \pi(0, 0, 0), j']$. For the other vertices everything runs more or less analogously. A detailed proof can be found in the annex. By Lemma 4.2 the only possible incoming edges have an initial vertex of the shape $[i, \pi(0, 0, 0), j]$. Moreover, $i < j$ by Observe the definition of the self-replicating boundary graph. We have to investigate the following pairs.

- $(i', j') = (1, 2)$: we use Table 7 to find out the possible edges. In column (1, 2) there occur three lists that may include 0. The list in line (1, 1) is not relevant since $i = j = 1$ is not allowed. Hence, we only consider $(i, j) = (1, 3)$ or $(i, j) = (2, 3)$. For $a = b$ row (1, 3) contains strictly positive values only, hence, this edge cannot occur in this case. For $a \neq b$ we consult Γ_σ and see that there is only one edge from 3 to 2 labelled by $(1^b, 3, \varepsilon)$. Hence, we have only one possible edge $[1, \pi(0, 0, 0), 3] \xrightarrow{\pi(b, 0, 0)} [1, \pi(0, 0, 0), 2]$ (of Type 1). Analogously, $(i, j) = (2, 3)$ gives one edge from $[2, \pi(0, 0, 0), 3]$ labelled by $\pi(a, 0, 0)$ provided that $a = b$. In both cases the edges can already be found in Adjacency Table 1.
- $(i', j') = (1, 3)$ **or** $(2, 3)$: these vertices cannot have any incoming edge since in columns (1, 3) and (2, 3) of Table 7 the only list including 0 is the one in row (1, 1), which is not relevant here.

□

6. PROOF OF THEOREM 3.2 AND THEOREM 3.3

The proof of Theorem 3.2 runs analogously to the first part of Theorem 3.1.

Proof of Theorem 3.2. It is quite easy to see that \mathcal{L} is a subgraph of $\mathcal{G}_{lat}^{(B)}$. Indeed, all vertices lie on a finite path that ends in a vertex of $\mathcal{G}_{srs}^{(B)}$. These vertices satisfy (2.6). Using a similar argumentation as in the second part of the proof of Theorem 4.1, one can show that actually each vertex of \mathcal{L} satisfies (2.6). □

The lack of statements similar to Proposition 2.3 and Theorem 4.1 leads us to consider an alternative strategy to prove that \mathcal{L} coincides with $\mathcal{G}_{lat}^{(B)}$. The self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ gives us a list of all subtiles that intersect with the Rauzy fractal in the aperiodic tiling. We use this information to construct a tube around the central tile. To show that our graph \mathcal{L} is exactly the lattice boundary graph, we will prove that the neighbours occurring in \mathcal{L} cover the whole tube. As a consequence, the neighbour set cannot be bigger. In the last step we deduce the exact set of vertices and edges. However, the computation in the general case seems to be difficult. Hence, we will restrict here to the most simple case, when $m(a, b) = 1$.

According to Definition 2.4 and the prefix-suffix automaton Γ_σ , the Rauzy fractal \mathcal{T} is the solution of a graph directed function system: for every $i \in \mathcal{A}$,

$$(6.1) \quad \mathcal{T}(i) = \bigcup_{\sigma(j)=pis} \pi(\mathbf{1}(p)) + \mathbf{h}(\mathcal{T}(j)).$$

For convenience, we denote by $\mathcal{B}(i)$ the finite set

$$(6.2) \quad \mathcal{B}(i) := \{\pi(\mathbf{1}(p)) + \mathbf{h}(\mathcal{T}(j)) \mid \exists(p, i, s) \in \mathcal{P} : \sigma(j) = pis\}.$$

Now consider a vertex $[i, \gamma, j]$ of the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$ with $\gamma \neq \{\mathbf{0}\}$. By [22, Theorem 5.6], this is equivalent to the fact that $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma) \neq \emptyset$. Now we may ask for which $B \in \mathcal{B}(j)$ we have $\mathcal{T}(i) \cap (B + \gamma) \neq \emptyset$. In other words, we want a characterisation of the set

$$\mathcal{O}([i, \gamma, j]) := \{\gamma + B \mid B \in \mathcal{B}(j), \mathcal{T}(i) \cap (B + \gamma) \neq \emptyset\}.$$

Lemma 6.1. *Let $[i, \gamma, j]$ a vertex of the self-replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. Then the elements of $\mathcal{O}([i, \gamma, j])$ are given by outgoing edges of $[i, \gamma, j]$. In particular,*

$$\begin{aligned} \mathcal{O}([i, \gamma, j]) = & \{\eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) \mid [i, \gamma, j] \xrightarrow{\eta} [i', \gamma', j'] \text{ is an edge of Type 1}\} \cup \\ & \{\eta' + \mathbf{h}(\mathcal{T}(i')) \mid [i, \gamma, j] \xrightarrow{\eta'} [i', \gamma', j'] \text{ is an edge of Type 2}\} \end{aligned}$$

Proof. At first we show that each element of $\mathcal{O}([i, \gamma, j])$ is contained in the set on the right hand side of this equality. Let $C = \gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \in \mathcal{O}([i, \gamma, j])$. Thus $\mathcal{T}(i) \cap C \neq \emptyset$. The use of (6.1) yields

$$\bigcup_{B \in \mathcal{B}(i)} B \cap (\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset.$$

Now, there must be at least one $B = \pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(i')) \in \mathcal{B}(i)$ that satisfies the equation.

$$\pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(i')) \cap \gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \neq \emptyset.$$

Suppose that $\gamma = \pi(\mathbf{x})$. Hence,

$$(6.3) \quad \mathcal{T}(i') \cap (\mathbf{h}^{-1}(\pi(\mathbf{x}) + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))) + \mathcal{T}(j')) \neq \emptyset.$$

By (2.2) we have

$$\mathcal{T}(i') \cap (\pi(\mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1))) + \mathcal{T}(j')) \neq \emptyset.$$

Now note that $\langle \mathbf{x}, \mathbf{v}_\beta \rangle < v_j = \langle \mathbf{l}(j), \mathbf{v}_\beta \rangle$. Since

$$\beta \langle \mathbf{l}(j'), \mathbf{v}_\beta \rangle = \langle \mathbf{M}\mathbf{l}(j'), \mathbf{v}_\beta \rangle = \langle \mathbf{l}(\sigma(j')), \mathbf{v}_\beta \rangle \geq \langle \mathbf{l}(j) + \mathbf{l}(p_2), \mathbf{v}_\beta \rangle$$

we immediately see that $\langle \mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1)), \mathbf{v}_\beta \rangle < v_{j'}$. Analogously, we can show that $\langle \mathbf{M}^{-1}(\mathbf{x} + \mathbf{l}(p_2) - \mathbf{l}(p_1)), \mathbf{v}_\beta \rangle > -v_{i'}$.

However, we either have $[i', \mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), j'] \in \mathcal{D}$ and $[\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), j'] \in \Gamma_{srs}$ or $[j', -\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), i'] \in \mathcal{D}$ and $[-\mathbf{h}^{-1}(\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1))), i'] \in \Gamma_{srs}$. Remember that for a vertex $[i, \gamma, j]$ of the boundary graph, we necessarily have $\mathcal{T}(i) \cap (\mathcal{T}(j) + \gamma) \neq \emptyset$. Together with (6.3), this leads to the conclusion that one of the triples occurs as vertices in $\mathcal{G}_{srs}^{(B)}$ and we see that it has an incoming edge from $[i, \gamma, j]$. In the first case it is of Type 1 and labelled by $\pi(\mathbf{l}(p_1))$, in the second case it is of Type 2 and labelled by $\pi(\mathbf{l}(p_2)) + \gamma$. However, we see that

$$C \in \{\eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) \mid [i, \gamma, j] \xrightarrow{\eta} [i', \gamma', j'] \text{ is an edge of Type 1}\} \cup \\ \{\eta' + \mathbf{h}(\mathcal{T}(i')) \mid [i, \gamma, j] \xrightarrow{\eta'} [i', \gamma', j'] \text{ is an edge of Type 2}\}.$$

To prove the reverse inclusion, consider an edge $[i, \gamma, j] \xrightarrow{\eta} [i', \gamma', j']$. We have $\mathcal{T}(i') \cap (\gamma' + \mathcal{T}(j')) \neq \emptyset$. Suppose the edge is of Type 1. Then

$$\mathbf{h}(\mathcal{T}(i')) \cap (\gamma + \pi(\mathbf{l}(p_2)) - \pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset$$

by the definition of the self replicating boundary graph $\mathcal{G}_{srs}^{(B)}$. Furthermore, we have $\eta = \pi(\mathbf{l}(p_1))$, $\pi(\mathbf{l}(p_1)) + \mathbf{h}(\mathcal{T}(i')) \in \mathcal{O}(i)$ and $\pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) \in \mathcal{O}(j)$. Thus

$$\mathcal{T}(i) \cap (\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j'))) \neq \emptyset$$

and, hence,

$$\gamma + \pi(\mathbf{l}(p_2)) + \mathbf{h}(\mathcal{T}(j')) = \eta + \mathbf{h}(\gamma' + \mathcal{T}(j')) \in \mathcal{O}([i, \gamma, j]).$$

For edges of Type 2 the proof runs analogously. \square

Now, for each vertex $[i, \gamma, j]$ we consider the set $\mathcal{O}([i, \gamma, j])$. This set consists of the γ -translates of all subsets of $\mathcal{T}(j)$ induced by the decomposition (6.2) that intersect with $\mathcal{T}(i)$. The union of the elements of $\mathcal{O}([i, \gamma, j])$ for all vertices of $\mathcal{G}_{srs}^{(B)}$ with $\gamma \neq \mathbf{0}$ gives the mentioned tube. In Lemma 6.3 we show that in the lattice tiling the neighbours of the Rauzy fractal induces by the elements of $\{\pm\pi(0, 1, -1), \pm\pi(1, 0, -1), \pm\pi(1, -1, 0)\}$ cover all of this tube. Lemma 6.2 is a preparation to Lemma 6.3.

Lemma 6.2. *For all $a \geq b \geq 1$ we have*

$$\pi(0, 1, 0) + \mathcal{T}(2) \subset \mathcal{T}(1).$$

Proof. By (6.1) for $i = 2$ we have $\mathcal{T}(2) = \pi(a, 0, 0) + \mathbf{h}(\mathcal{T}(1))$. Hence, by (2.2) and the shape of \mathbf{M} ,

$$(6.4) \quad \pi(0, 1, 0) + \mathcal{T}(2) = \pi(\mathbf{M}(1, 0, 0)) + \mathbf{h}(\mathcal{T}(1)) = \mathbf{h}(\pi(1, 0, 0) + \mathcal{T}(1)).$$

Set

$$R := \bigcup_{k=0}^{a-1} (\pi(k, 0, 0) + \mathbf{h}(\mathcal{T}(1))) \cup \bigcup_{k=0}^{b-1} (\pi(k, 0, 0) + \mathbf{h}(\mathcal{T}(2))).$$

Now we use again (6.1) to obtain

$$\begin{aligned} \pi(0, 1, 0) + \mathcal{T}(2) &= \mathbf{h}(\pi(1, 0, 0) + \mathcal{T}(1)) \\ &= \mathbf{h}(\pi(1, 0, 0) + R) \cup \mathbf{h}(\pi(1, 0, 0) + \mathbf{h}(\mathcal{T}(3))) \\ &= \mathbf{h}(\pi(1, 0, 0) + R) \cup [\mathbf{h}(\pi(1, 0, 0) + \mathbf{h}(\pi(b, 0, 0))) + \mathbf{h}^2(\pi(a, 0, 0)) + \mathbf{h}^3(\mathcal{T}(1))]. \end{aligned}$$

Now observe that

$$(6.5) \quad \pi(1, 0, 0) + \mathbf{h}(\pi(b, 0, 0)) + \mathbf{h}^2(\pi(a, 0, 0)) = \pi((\mathbf{I}_3 + b\mathbf{M} + a\mathbf{M}^2)(1, 0, 0)) = \mathbf{h}^3(\pi(1, 0, 0)),$$

(where \mathbf{I}_3 denotes the 3×3 identity matrix) since $x^3 - ax^2 - bx - 1$ is the characteristic (and minimal) polynomial of \mathbf{M} . Hence,

$$\pi(0, 1, 0) + \mathcal{T}(2) = \mathbf{h}(\pi(1, 0, 0) + R) \cup \mathbf{h}^4(\pi(1, 0, 0) + \mathcal{T}(1)).$$

Iterating this procedure, we obtain

$$(6.6) \quad \pi(0, 1, 0) + \mathcal{T}(2) = \overline{\bigcup_{n=0}^{\infty} \mathbf{h}^{3n+1}(\pi(1, 0, 0) + R)}$$

since \mathbf{h} is a contraction. Now we claim that $\mathbf{h}(\pi(1, 0, 0) + R) \subset \mathcal{T}(1)$. By definition of R we have

$$(6.7) \quad \mathbf{h}(\pi(1, 0, 0) + R) = \bigcup_{k=1}^a (\mathbf{h}(\pi(k, 0, 0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup \bigcup_{k=1}^b (\mathbf{h}(\pi(k, 0, 0)) + \mathbf{h}^2(\mathcal{T}(2))).$$

On the other hand, by (6.1), we have

$$\begin{aligned} \mathcal{T}(1) &\supset \mathbf{h}(\mathcal{T}(1)) \cup \mathbf{h}(\mathcal{T}(2)) \cup \mathbf{h}(\mathcal{T}(3)) \\ &= \bigcup_{k=0}^{a-1} (\mathbf{h}(\pi(k, 0, 0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup \bigcup_{k=0}^{b-1} (\mathbf{h}(\pi(k, 0, 0)) + \mathbf{h}^2(\mathcal{T}(2))) \cup (\mathbf{h}(\pi(0, 0, 0)) + \mathbf{h}^2(\mathcal{T}(3))) \\ &\quad \cup (\mathbf{h}(\pi(a, 0, 0)) + \mathbf{h}^2(\mathcal{T}(1))) \cup (\mathbf{h}(\pi(b, 0, 0)) + \mathbf{h}^2(\mathcal{T}(2))) \end{aligned}$$

which contains the set (6.7) and thus yields the claim. Observing that $\mathcal{T}(1) \supset \mathbf{h}^{3i}(\mathcal{T}(1))$ and that by the claim $\mathbf{h}^{3i+1}(\pi(1, 0, 0) + R) \subset \mathbf{h}^{3i}(\mathcal{T}(1))$ for all $i \in \mathbb{N}$ we obtain the assertion from (6.6). \square

Lemma 6.3. *Let $m(a, b) = 1$. For each $O \subset \mathcal{O}([i, \gamma, j])$ of each vertex $[i, \gamma, j]$ of $\mathcal{G}_{srs}^{(B)}$ with $\gamma \neq \mathbf{0}$ there exists a vertex $[i', \gamma', j']$ of \mathcal{L} with $[\gamma', j'] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ such that one of the following conditions hold:*

- (1) $i = i'$ and $O \subset \gamma' + \mathcal{T}(j')$;
- (2) $i = j'$ and $O \subset -\gamma' + \mathcal{T}(i')$.

Proof. We prove the lemma by analysing the vertices of $\mathcal{G}_{srs}^{(B)}$ one by one. Since for every $O \in \mathcal{O}([i, \gamma, j])$ we have $O \subset \gamma + \mathcal{T}(j)$ the lemma holds for all vertices $[i, \gamma, j]$ with $\gamma \in \{\pi(0, 1, -1), \pi(1, 0, -1), \pi(1, -1, 0)\}$. Indeed, these vertices appear in \mathcal{L} too. Thus, we have 6 more vertices to investigate.

$[1, \pi(0, 0, 1), 1]$: for convenience, define the 3 sets

$$\begin{aligned} C &:= \{\pi(k, 0, 1) + \mathbf{h}(\mathcal{T}(1)) \mid b-1 \leq k \leq a-2\}, \\ D &:= \{\pi(a-1, 0, 1) + \mathbf{h}(\mathcal{T}(1))\}, \\ E &:= \{\pi(b-1, 0, 1) + \mathbf{h}(\mathcal{T}(2))\}. \end{aligned}$$

Using Lemma 6.1 we easily obtain that

$$\mathcal{O}([1, \pi(0, 0, 1), 1]) = C \cup D \cup E.$$

We claim that the vertices $[i', \pi(1, 0, -1), 1]$ for $i' \in \{1, 2, 3\}$ cover all elements of C , D and E according to (2). Indeed, the triples are vertices of \mathcal{L} , $[\pi(1, 0, -1), 1] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ and at the third position we find 1. Furthermore, by (6.1), we have that

$$\begin{aligned} -\pi(1, 0, -1) + \mathcal{B}(1) &\supset \{\pi(k, 0, 1) + \mathbf{h}(\mathcal{T}(1)) \mid b-1 \leq k \leq a-1\} = C \\ -\pi(1, 0, -1) + \mathcal{B}(2) &= \{\pi(a-1, 0, 1) + \mathbf{h}(\mathcal{T}(1))\} = D \\ -\pi(1, 0, -1) + \mathcal{B}(3) &= \{\pi(b-1, 0, 1) + \mathbf{h}(\mathcal{T}(2))\} = E, \end{aligned}$$

which proves the claim.

$[i, \pi(0, 0, 1), 2]$: The vertex with $i = 1$ always occurs while $i = 2$ only exists for $a = b$. However, we have

$$\mathcal{O}([1, \pi(0, 0, 1), 2]) = \mathcal{O}([2, \pi(0, 0, 1), 2]) = \{\pi(a, 0, 1) + \mathbf{h}(\mathcal{T}(1))\}.$$

Now (6.1) and Lemma 6.2 yield

$$\pi(a, 0, 1) + \mathbf{h}(\mathcal{T}(1)) = \pi(0, 0, 1) + \mathcal{T}(2) \subset \pi(0, -1, 1) + \mathcal{T}(1).$$

Since $[1, \pi(0, 1, -1), 1]$ as well as $[1, \pi(0, 1, -1), 2]$ occur in \mathcal{L} the case is accomplished.

$[1, \pi(0, 1, 0), 1]$: note that $[2, (1, -1, 0), 1]$ is a vertex of \mathcal{L} . Also,

$$\mathcal{O}([1, \pi(0, 1, 0), 1]) = \{\pi(a-1, 1, 0) + \mathbf{h}(\mathcal{T}(1))\} = \{\pi(-1, 1, 0) + \mathcal{T}(2)\},$$

where we used (6.1). This proves the lemma in this case.

$[1, \pi(1, -1, 1), 1]$: we have for $b \geq 2$

$$\begin{aligned} \mathcal{O}([1, \pi(1, -1, 1), 1]) &= \{\pi(k, -1, 1) + \mathbf{h}(\mathcal{T}(1)) \mid 1 \leq k \leq b-2\} \\ &\cup \{\pi(k, -1, 1) + \mathbf{h}(\mathcal{T}(2)) \mid 1 \leq k \leq b-1\}. \end{aligned}$$

(there is nothing to prove for $b = 1$). Using (6.1) we easily obtain that $\pi(0, -1, 1) + \mathcal{T}(1)$ covers all elements of $\mathcal{O}([1, \pi(1, -1, 1), 1])$. This finishes the case since $[1, \pi(0, 1, -1), 1]$ is a vertex of \mathcal{L} .

$[2, \pi(1, -1, 1), 1]$: we have

$$\mathcal{O}([2, \pi(1, -1, 1), 1]) = \{\pi(b-1, -1, 1) + \mathbf{h}(\mathcal{T}(1)), \pi(b, -1, 1) + \mathbf{h}(\mathcal{T}(2))\}.$$

Now observe that $\{\pi(b-1, 0, 0) + \mathbf{h}(\mathcal{T}(1))\} \in \mathcal{B}(1)$ and $\mathcal{B}(3) = \{\pi(b, 0, 0) + \mathbf{h}(\mathcal{T}(2))\}$.

Hence, the lemma is proved since $[3, \pi(0, 1, -1), 2]$ and $[1, \pi(0, 1, -1), 2]$ are vertices of \mathcal{L} . \square

We are finally able to prove Theorem 3.2. However, we will exactly go through the proof to show which conditions we need.

Proof of Theorem 3.3. We already know from Theorem 3.2 that \mathcal{L} is a subgraph of $\mathcal{G}_{lat}^{(B)}$.

Thus, we just have to prove that \mathcal{L} contains $\mathcal{G}_{lat}^{(B)}$. At first we show that each vertex $[\tilde{i}, \tilde{\gamma}, \tilde{j}]$ of $\mathcal{G}_{lat}^{(B)}$ with $[\tilde{\gamma}, \tilde{j}] \in \Gamma_{lat} \setminus \{0\} \times \mathcal{A}$ is also a vertex of \mathcal{L} . Let $\xi \in \tilde{\gamma} + \mathcal{T}(\tilde{j}) \cap \mathcal{T}(\tilde{i})$. The tiles are the closure of their interiors, hence, there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of interior points of $\tilde{\gamma} + \mathcal{T}(\tilde{j})$ that converges to ξ . For each $n \in \mathbb{N}$ we can find an $\varepsilon_n > 0$ such that the open ball $K(\xi_n, \varepsilon_n)$ is completely contained in the interior of $\tilde{\gamma} + \mathcal{T}(\tilde{j})$. Since σ has the tiling property, we conclude that none of the ξ_n is contained in $\mathcal{T}(\tilde{i})$.

Now consider the aperiodic tiling induced by σ . We use the covering property to deduce that each of the ξ_n is contained in some translate of the self-replicating tiling. By the local finiteness, there are only finitely many possibilities. Thus, suppose that $\gamma + \mathcal{T}(j)$ (with $[\gamma, j] \in \Gamma_{srs}$) contains ξ_n for infinitely many $n \in \mathbb{N}$. By the above considerations and, again, by the tiling property we conclude that $\gamma \neq \mathbf{0}$. Since $\gamma + \mathcal{T}(j)$ is compact and contains an infinite subsequence of ξ_n , it necessarily includes the limit point ξ , too. Thus, $\gamma + \mathcal{T}(j) \cap \mathcal{T}(\tilde{i}) \neq \emptyset$ which makes $[\tilde{i}, \gamma, j]$ a vertex of $\mathcal{G}_{srs}^{(B)}$. Furthermore, as $\gamma + \mathcal{T}(j)$ includes points of the sequence $(\xi_n)_{n \in \mathbb{N}}$, it necessarily intersects with the respective neighbourhoods $K(\xi_n, \varepsilon_n)$. This shows that $\text{int}(\gamma + \mathcal{T}(j)) \cap \text{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \neq \emptyset$.

Now divide the subtile $\mathcal{T}(j)$ with respect to (6.1). Then there must be at least one $B \in \mathcal{B}(j)$ such that $\gamma + B$ includes ξ_n for infinitely many $n \in \mathbb{N}$. Similarly as before we have $\text{int}(\gamma + B) \cap \text{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \neq \emptyset$ and $\xi \in \gamma + B$. The latter relation yields $\gamma + B \in \mathcal{O}([\tilde{i}, \gamma, j])$. By Lemma 6.3 there exists a vertex $[i', \gamma', j'] \in \mathcal{L}$ with $[\gamma, j'] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ such that $i' = \tilde{i}$ and $\gamma + B \subset \gamma' + \mathcal{T}(j')$ or $j' = \tilde{i}$ and $\gamma + B \subset -\gamma' + \mathcal{T}(i')$. We claim that, in fact, the first relation holds. Indeed, suppose the second relation would hold. Then

$$\text{int}(\tilde{\gamma} + \mathcal{T}(\tilde{j})) \cap \text{int}(-\gamma' + \mathcal{T}(i')) \neq \emptyset$$

and, by the tiling property of σ , $\tilde{\gamma} + \mathcal{T}(\tilde{j}) = -\gamma' + \mathcal{T}(i')$. Hence, $[\tilde{j}, -\tilde{\gamma}, \tilde{i}]$ would be a vertex of \mathcal{L} . But $[\tilde{i}, \tilde{\gamma}, \tilde{j}]$ is a vertex of that $\mathcal{G}_{lat}^{(B)}$ with $\tilde{\gamma} \neq \{\mathbf{0}\}$. Thus, by definition, $\langle \mathbf{x}, \mathbf{v}_\beta \rangle > 0$ where $\tilde{\gamma} = \pi(\mathbf{x})$. The same consideration apply for \mathcal{L} which shows that $[\tilde{j}, -\tilde{\gamma}, \tilde{i}]$ impossibly can be a vertex of \mathcal{L} . Therefore, the first relation must hold necessarily. Now, the same considerations yield that $[\tilde{i}, \tilde{\gamma}, \tilde{j}]$ is a vertex of \mathcal{L} .

From the first part of the proof we can deduce that, whenever $[i', \gamma', j']$ with $[\gamma', j'] \in \Gamma_{lat} \setminus \{\mathbf{0}\} \times \mathcal{A}$ is a vertex of $\mathcal{G}_{lat}^{(B)}$, it is also a vertex of \mathcal{L} . Note that these 7 vertices (or only 6 vertices if $a = b < 4$) are also vertices of $\mathcal{G}_{srs}^{(B)}$. By Definition 2.1, $\mathcal{G}_{srs}^{(B)}$ contains all infinite paths starting from one of these vertices. Since \mathcal{L} also contains all of these paths, we conclude that \mathcal{L} contains $\mathcal{G}_{lat}^{(B)}$. \square

7. COMMENTS

We want to say a few words on possible proofs of Conjecture 3.4. For a, b that satisfy $m(a, b) \leq k$ for a given constant k the same strategy seems to work. But it requires additional assertions of the style of Lemma 6.2. The following considerations may yield another strategy. The sets $\mathcal{O}([i, \gamma, j])$ induce a neighbourhood of the central tile. By Lemma 6.1 it corresponds to the paths of length 1 of $\mathcal{G}_{srs}^{(B)}$. One may obtain a smaller neighbourhood by considering a refinement of $\mathcal{O}([i, \gamma, j])$. This would lead us to investigate longer paths of $\mathcal{G}_{srs}^{(B)}$. This will involve very lengthy hand calculations. We should rather use a computational implementation.

ANNEX - DETAILS TO THE TECHNICAL PROOFS

Details for the proof of Lemma 5.2. The following procedure describes explicitly how to reduce the number of cells in Table 10. At first we consider the outgoing edges for points of the type $[i, \pi(0, 1, 0), j]$ (left pairs in Block 1). The only possibility is $(i, j) = (1, 1)$ since $(1, 1)$ is the only pair that appears on the left side. Thus, an edge whose destination is of the same shape, $[i', \pi(0, 1, 0), j']$, cannot belong to the strongly connected components if $(i', j') \neq (1, 1)$. Hence, in Block 4 we can delete all cells that do not have $(1, 1)$ at their second position (the last 6 cells). Now we consider the vertices of the shape $[i, \pi(1, -1, 0), j]$. In Block 6 we find the outgoing edges. Only $(2, 1)$ and $(3, 1)$ occur on the left side. Therefore, in Block 2 and Block 7 we can cross out all cells whose right entry does not match with one of these two pairs (14 cells in Block 2, 8 cells in Block 7). Next we study the vertices of the shape $[i, \pi(0, 1, -1), j]$. The incoming edges are given in Block 5. The possible outgoing edges are listed in Block 2 and Block 3. Note that for $(i, j) = (1, 3)$ there is no incoming edge. Hence, we can delete the three cells with $(1, 3)$ on the left in Block 3. Now, there are only three different pairs, $(1, 1)$, $(1, 2)$ and $(3, 2)$, on the left in Block 2 and Block 3. So, in Block 5 we remove all cells with right entry different from one of these pairs (7 cells). The outgoing edges of vertices of the type $[i, \pi(1, 0, -1), j]$ are given in Block 4 and Block 5. We find the pairs $(1, 1)$, $(2, 1)$, and $(3, 1)$ there. Hence, we can delete 6 cells in Block 1 and 3 cells in Block 6. Now, for the edges that start in vertices of the shape $[i', \pi(1, -1, 0), j']$ (Block 6) we see that only $(i, j) = (2, 1)$ remained. Hence, we can cross out one more cell in Block 2 and 2 more cells in Block 7, respectively. Finally, for points of the shape $[i, \pi(1, -1, 1), j]$ we find the possible outgoing edges in Block 7 and Block 8. We deduce that $(i, j) \in \{(1, 1), (2, 1), (3, 1)\}$. Now we can delete the cells that do not have matching pairs on the right in Block 3 (3 cells) and Block 8 (6 cells). Observing that we just erased all edges that start in $[3, \pi(1, -1, 1), 1]$, we can remove two more cells from each of these two Blocks. \square

Details for the proof of Theorem 3.1. We go through all remaining types of vertices of \mathcal{S} in order to show that $\mathcal{G}_{srs}^{(B)}$ cannot contain edges different from those that are given in \mathcal{S} .

Vertices of the form $[i', \pi(0, 0, 1), j']$: similar as above, $[i, \pi(0, 0, 0), j]$ with $i < j$ is the only type of predecessor. The significant difference equals 1 and, thus, we are looking for the entry 1 in Table 7. In particular,

$(i', j') = (1, 1)$: since $i < j$ we see by Table 7 that $(i, j) = (1, 2)$ is the only possibility and gives only one edge (which is already included in \mathcal{S}).

$(i', j') = (1, 2)$: in column (1, 2) of Table 7 the only row that contains 1 is the row (1, 3). The corresponding edge starts in $[1, \pi(0, 0, 0), 3]$ and is contained in \mathcal{S} .

$(i', j') = (2, 2)$: similarly as before, row (1, 3) is the only row that contains 1 in column (2, 2) of Table 7. The corresponding edge is contained in \mathcal{S} provided that $[2, \pi(0, 0, 1), 2]$ is contained in \mathcal{S} , *i.e.*, $b = 1$.

The vertex $[1, \pi(0, 1, 0), 1]$: by Lemma 4.2 the incoming edges of Type 1 start in vertices of the form $[i, \pi(0, 0, 1), j]$ with the significant difference $b > 0$. In the corresponding column of Table 7 we find that row (1, 2) includes b . The associated edge is already included in \mathcal{S} . Row (1, 1) includes b provided that $a \neq b$, and in Γ_σ there are $a - b$ possibilities to choose edges $(p_1, 1, s_1)$ and $(p_2, 1, s_2)$ from 1 to 1 such that $\mathbf{l}(p_2) - \mathbf{l}(p_1) = (b, 0, 0)$. All the edges appear in \mathcal{S} . All incoming edges of Type 2 start from vertices of the type $[i, \pi(1, 0, -1), j]$. Now observe that we already collected all possibilities in Block 4 in the Table in Lemma 5.2. There are two cells whose right entry equals (1, 1). Their left entries give the possible pairs (i, j) . The first one is (1, 1) and yields $a - b - 1$ edges (hence, edges only if $a \geq b + 1$), the other one is (2, 1) and yields one edge provided that $a \neq b$. The edges are included in \mathcal{S} .

Vertices of the form $[i', \pi(0, 1, -1), j']$: the incoming edges of Type 1 have initial vertices of the shape $[i, \pi(0, 0, 1), j]$ with significant difference $b - 1 \geq 0$. As $[\pi(0, 0, 1), 3] \notin \Gamma_{srs}$ we conclude that $j \neq 3$.

$(i', j') = (1, 1)$: \mathcal{S} includes $a - b - 1$ edges that start in $[1, \pi(0, 0, 1), 1]$. There is also another edge that starts in $[1, \pi(0, 0, 1), 2]$ if $b > 1$, and in $[2, \pi(0, 0, 1), 2]$ if $b = 1$. By Table 7 and Γ_σ there is no other possibility.

$(i', j') = (1, 2)$ **or** $(3, 2)$: by Table 7 the only possibility is $(i, j) = (1, 1)$ since $j = 3$ is not allowed.

We already investigated the incoming edges of Type 2 in the proof of Lemma 5.2. They are of Shape 5, start in vertices of the form $[i, \pi(1, 0, -1), j]$ and Block 5 gives the possible pairs. All 4 cells that we find there correspond to edges that are included in \mathcal{S} .

The vertex $[2, \pi(1, -1, 0), 1]$ and vertices of the form $[i', \pi(1, 0, -1), j']$: In the blocks 1, 2, 6 and 7 of the table in Lemma 5.2 we can check that actually all possible incoming edges are already included in \mathcal{S} .

Vertices of the form $[i', \pi(t, -t, t), j']$: The incoming edges of Type 1 have initial vertices of the shape $[i, \pi(1 - t, t, -t), j]$ with significant difference $t(a - b + 2) - 1 = \delta_t - 1 > a - b$, those of Type 2 start in vertices of the form $[i, \pi(t, -t, t), j]$ with significant difference $t(a - b + 2) = \delta_t > a - b + 1 > 0$. We already studied the latter ones detailed in Lemma 5.2. We see that there cannot be other incoming edges than those that are included in \mathcal{S} . For examining the possible edges of Type 1 we investigate the three vertices of the present form.

$(i', j') = (1, 1)$ (**for** $t = 1, \dots, m(a, b)$): Note that column (1, 1) of Table 7 shows two lists that include strictly positive entries: the rows (1, 1) and (1, 2). The first one gives $a - \delta_t + 1$ edges starting from $[1, \pi(1 - t, t, -t), 1]$. The other one gives an edge starting from $[1, \pi(1 - t, t, -t), 2]$.

$(i', j') = (2, 1)$ (**for** $t = 1, \dots, m(a, b)$): Again, we consult Table 7 and find two lists with suitable entries, (1, 1) and (1, 2). They give the edges that are included in \mathcal{S} .

$(i', j') = (3, 1)$ (**for** $t = 1, \dots, m(a, b) - 1$): Note that $t \leq m(a, b) - 1$ induces that $\delta_t < a$. Thus, in Table 7 the only row of interest is (1, 1). It gives the edge starting from $[1, \pi(1 - t, t, -t), 1]$.

- Vertices of the form $[i', \pi(2-t, t-1, -t), j']$ (for $t = 2, \dots, m(a, b)$):** Incoming edges of Type 1 can only start at vertices of the form $[i, \pi(t-2, 2-t, t-1), j]$ with significant difference $a - (t-1)(a-b+2) = a - \delta_{t-1} > 0$, those of Type 2 start at vertices of the form $[i, \pi(3-t, t-2, 1-t), j]$ with significant difference $a - (t-1)(a-b+2) + 1 = a - \delta_{t-1} + 1 > 0$.
- $(i', j') = (1, 1)$: The two rows which have positive entries in column (1, 1) of Table 7 are (1, 1) and (1, 2). They induce the $a - \delta_{t-1} + 1$ edges of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 1]$ and the single edge of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 2]$ that are included in \mathcal{S} . On the other hand, the edges of Type 2 start in $[1, \pi(3-t, t-2, 1-t), 1]$ and $[2, \pi(3-t, t-2, 1-t), 1]$ and are included in \mathcal{S} , too.
- $(i', j') = (2, 1)$: In Table 7 we find in column (2, 1) that the rows (1, 1) and (1, 2) yield the edges of Type 1 that are included in \mathcal{S} . Note that by the definition of $m(a, b)$ we have $a - (t-1)(a-b+2) > a - b + 2 > a - b$ and, hence, neither $(i, j) = (3, 1)$ nor $(i, j) = (3, 2)$ come into question. For edges of Type 2 we only have the possibilities $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$ and obtain edges that already are contained in \mathcal{S} . For the same reason as before, (i, j) cannot be (1, 3) or (2, 3).
- $(i', j') = (3, 1)$: Similar as before, we see that there is always one incoming edge of Type 1 starting at $[1, \pi(t-2, 2-t, t-1), 1]$ and of Type 2 starting at $[1, \pi(3-t, t-2, 1-t), 1]$. Since $a - \delta_{t-1} < a - (t-1)(a-b+2) + 1 \leq a - 1 < a$ we cannot have $(i, j) = (1, 2)$ (Type 1) or $(i, j) = (2, 1)$ (Type 2), respectively.
- Vertices of the form $[i', \pi(t-1, 1-t, t), j']$ (for $t = 2, \dots, m(a, b)$):** The incoming edges originate in $[i, \pi(2-t, t-1, 1-t), j]$ (Type 1 with significant difference $(t-1)(a-b+2) = \delta_{t-1}$) and $[i, \pi(t-1, 1-t, t-1), j]$ (Type 2 with significant difference $(t-1)(a-b+2) + 1 = \delta_{t-1} + 1 > 1$). Note that $\delta_{t-1} \leq b - 2$.
- $(i', j') = (1, 1)$: Analogously as before we easily find that the only possible incoming edges of Type 1 are $a - \delta_{t-1}$ edges starting at $[1, \pi(2-t, t-1, 1-t), 1]$ and one edge starting at $[1, \pi(2-t, t-1, 1-t), 2]$. The incoming edges of Type 2 start at $[1, \pi(t-1, 1-t, t-1), 1]$ ($a - \delta_{t-1} - 1$ edges) and $[2, \pi(t-1, 1-t, t-1), 1]$ (one edge).
- $(i', j') = (1, 2)$: In Table 7 we find that the lists in row (1, 1) and (1, 3) include suitable values. Indeed, $[1, \pi(2-t, t-1, 1-t), 1]$ is the origin of $b - \delta_{t-1}$ edges of Type 1. By (5.1) we see that $[\pi(2-t, t-1, 1-t), 3] \notin \Gamma_{srs}$ and, hence, $j = 3$ is no option. On the other hand, for the the edges of Type 2 we have one edge starting at $[3, \pi(t-1, 1-t, t-1), 1]$ besides the $b - \delta_{t-1} - 1$ incoming edges that have their origin in $[1, \pi(t-1, 1-t, t-1), 1]$.
- Vertices of the form $[i', \pi(1-t, t, -t), j']$ (for $t = 2, \dots, m(a, b)$):** The incoming edges of Type 1 have initial vertices of the form $[i, \pi(t-1, 1-t, t), j]$ with significant difference $a - t(a-b+2) + 1 = a - \delta_t + 1 > 0$, those of Type 2 originate in vertices of the shape $[i, \pi(2-t, t-1, -t), j]$ with significant difference $a - t(a-b+2) + 2 = a - \delta_t + 1 > 2 > 0$.
- $(i', j') = (1, 1)$: We see that for an edge of Type 1 we must have $(i, j) \in \{(1, 1), (1, 2)\}$ and, respectively, $(i, j) \in \{(1, 1), (2, 1)\}$ for an edge of Type 2. All possible edges are included in \mathcal{S} .
- $(i', j') = (1, 2)$: $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$ are the only options for edges of Type 1 and $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$ give the only possibilities for edges of Type 2. Since all these edges are contained in \mathcal{S} we finally showed the claim. \square

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