ON THE FUNDAMENTAL GROUP OF THE SIERPIŃSKI-GASKET

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Dedicated to Professor Peter Kirschenhofer on the occasion of his 50th birthday

ABSTRACT. We give a description of the fundamental group $\pi(\triangle)$ of the Sierpiński-gasket $\triangle$. It turns out that this group is isomorphic to a certain subgroup of an inverse limit $\varprojlim G_n$ formed by the fundamental groups G_n of natural approximations of $\triangle$. This subgroup, and with it $\pi(\triangle)$, can be described in terms of sequences of words contained in an inverse limit of semi-groups.

1. INTRODUCTION

The present paper is devoted to the description of the fundamental group of the Sierpiński-gasket $\triangle$ (see Figure 1). It turns out that this fundamental group can be viewed as a subset of an inverse limit of the fundamental groups of certain natural approximations of $\triangle$. Before we give more details we would like to state some definitions and earlier results that are related to our topic.

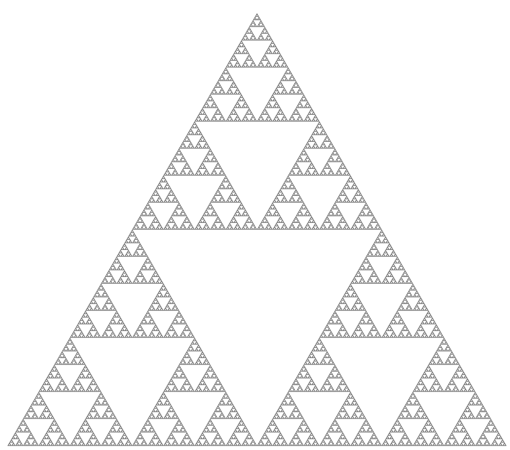

FIGURE 1. The Sierpiński-gasket

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One of the possibilities to define the Sierpiński-gasket is to use a so-called *iterated function system.* Let

$$f_1(x) := \frac{x}{2}, \quad f_2(x) := \frac{x}{2} + \frac{1}{2}, \quad f_3(x) := \frac{x}{2} + \frac{1+\sqrt{-3}}{4}.$$

Then it is well-known that $\triangle \subset \mathbb{C}$ is the unique non-empty compact subset of $\mathbb{C}$ satisfying the set equation

$$\triangle = \bigcup_{j=1}^{3} f_j(\triangle)$$

(see for instance Hutchinson [21]). Since f_1, f_2, and f_3 are similarities, $\triangle$ is a *self-similar* set. Topological properties of self-similar sets have been studied extensively in the literature. For instance Hata [18, 19] proves that a connected self-similar set is a locally connected continuum. Moreover, he establishes criteria for the connectivity of a self-similar set, deals with their cut points and proves a criterion for a self-similar set to be homeomorphic to an arc. Cut points play a role also in Winkler [27]. Related questions are addressed by Bandt and Keller [2], where the authors get information on the topological properties of self-similar sets by studying their dynamics. More recently, topological properties of self-similar sets with non-empty interior attracted interest. We mention the survey paper by Akiyama and Thuswaldner [1], where many results are stated. Some results on the structure of the fundamental group of self-similar sets are shown in Luo and Thuswaldner [22].

In describing the fundamental group $\pi(\triangle)$, the main difficulty consists in the fact that $\triangle$ is not semilocally simply connected. This makes it impossible to apply the classical methods like van Kampen's theorem and the theory of covering spaces in order to compute the fundamental group of $\triangle$.

Spaces that are not semilocally simply connected have been studied for a long time. We want to review some of the known results on such spaces. The standard example of a non-semilocally simply connected space is the so-called Hawaiian Earring (see Figure 2) which is defined by

$$H := \bigcup_{n \geq 1} \left\{ z \in \mathbb{C} \ : \ \left| z - \frac{1}{n} \right| = \frac{1}{n} \right\}.$$

It is not semilocally simply connected in the origin. Properties of the fundamental

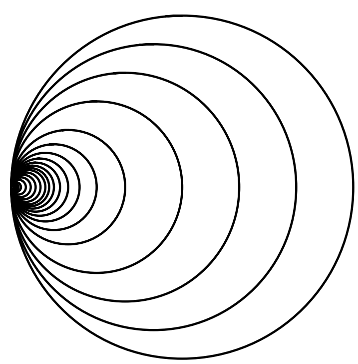

FIGURE 2. The Hawaiian Earring

group of H were studied implicitly by Higman [20] introducing the notion of an *unrestricted free product* of groups. Morgan and Morrisson [24] determine $\pi(H)$ as a subgroup of an inverse limit of finite free products of cyclic groups. Their proof was simplified by de Smit [11] who also showed that $\pi(H)$ is uncountable and not free. Zastrow [29] gives a description of $\pi(H)$ in terms of a subset of a projective limit of groups that is related to our approach (see in particular [29, Definition 2.3]).

Also in the more general context of one dimensional spaces results on fundamental groups have been proved. We mention [17] where it is shown that a one-dimensional locally connected continuum has a trivial fundamental group if and only if it is a dendrite. Moreover, Curtis and Fort [8] showed that higher homotopy groups of one-dimensional separable metric spaces are always trivial. More recently, in a big project consisting of three papers, Cannon and Conner [3, 4, 5] thoroughly study fundamental groups (and so-called big fundamental groups) of one-dimensional spaces. In particular, [3] is devoted to the fundamental group of the Hawaiian Earring. The authors give a combinatorial description of this group in terms of "big" words. In [5] they prove some important properties of the fundamental groups of one-dimensional spaces. For instance, generalizing a result by Curtis and Fort [9] they show that for a one-dimensional space X the following assertions are equivalent: $\pi(X)$ is free, $\pi(X)$ is countable, X has a universal cover, X is locally simply connected. Conner and Lamoreaux [7] generalize some of the results of [5] to larger classes of subsets of the plane.

In the present paper we embed the fundamental group of $\triangle$ into an inverse limit of groups which is easily seen to be equal to the Čech homotopy group $\check{\pi}(\triangle)$ of $\triangle$. The fact that the fundamental group of a one-dimensional space is always isomorphic to a subgroup of its Čech homotopy group is proved by Eda and Kawamura [13] and independently by Cannon and Conner [5]. For the Menger sponge this was already shown by use of a more explicit construction by Curtis and Fort [10, Section 3]. Related results for subsets of closed surfaces can be found in Fischer and Zastrow [16]. In Eda [12] criteria for the isomorphy of the fundamental group of two non-locally semisimply connected spaces are studied. More recently, Conner and Eda [6] proved that certain spaces can be recovered from their fundamental groups, the Sierpiński-gasket is among these spaces. Finally, we mention that homology groups of non-locally semisimply connected spaces are studied by Eda and Kawamura [14, 15].

The starting point of the present paper is a remark contained in [5, Section 2]. In an example the authors describe the implications of their results for the fundamental groups of Sierpiński and Menger curves. Among other things, they showed that these spaces have uncountable fundamental groups which are not free and that they do not have an universal cover. On the other hand, the authors mention that these groups have no known combinatorial (word) structure. In the present paper we want to describe the fundamental group of the Sierpiński-gasket $\triangle$ by some word structure. Our description differs from the combinatorial word description of the Hawaiian Earring group by Cannon and Conner [3] in several respects. The main difference is that in [3] letters correspond to loops in H based in a single base point whereas in our description of $\pi(\triangle)$ each letter is related to a local cut point (later called dyadic point) of $\triangle$. For the definition of a local cut point we refer to Whyburn and Duda [26, Appendix 2]. As a consequence we have restrictions on the admissible finite words in our representation. Moreover, we do not obtain a representation of $\pi(\triangle)$ as a subgroup of an unrestricted free product of groups in the sense of Higman [20] since certain finiteness conditions on the occurrence of letters are not fulfilled (cf. Remark 1.2).

In what follows we want to give a short overview of the content of the present paper. It is an evident idea to consider for a loop f in $\triangle$ the sequence of homotopy classes $[f]_n$ of f in the approximating spaces $\triangle_n$ that arise when the usual construction process of recursively removing the open middle triangle is stopped at level n. Applying the result of Eda and Kawamura [13] mentioned above we can show that the sequence $([f]_n)_{n\geq 0}$ characterizes f exactly up to homotopy. The natural ambient space for the sequences $([f]_n)_{n\geq 0}$ is the inverse limit $\varprojlim G_n$ of the fundamental groups G_n of $\triangle_n$. We will show that $\varprojlim G_n$ is canonically isomorphic to $\check{\pi}(\triangle)$ (see Proposition 2.8). Thus in view of the above mentioned result of Eda and Kawamura there is an injective mapping

$$\varphi : \pi(\triangle) \hookrightarrow \varprojlim G_n.$$

With an easy example (see Example 2.11) it becomes clear that $\varprojlim G_n$ contains elements which do not represent homotopy classes for loops in $\triangle$. So the objective arises to describe the subgroup of $\varprojlim G_n$ that corresponds to the fundamental group of $\triangle$.

Our approach to this task pursues the following strategy: Instead of investigating the problem directly in $\varprojlim G_n$ we consider an intermediate semigroup structure $\varprojlim S_n$ in which the set $S(\triangle)$ of all (based) loops in $\triangle$ is described up to reparametrization (see Figure 3).

$$\begin{array}{ccc} S(\triangle) & \xrightarrow{\sigma} & \varprojlim S_n \\ \downarrow {\scriptstyle [\cdot]} & & {\scriptstyle \mathrm{Red}} \downarrow \\ \pi(\triangle) & \overset{\varphi}{\hookrightarrow} & \varprojlim G_n \cong \check{\pi}(\triangle) \end{array}$$

FIGURE 3.

To this end at every approximation level n we represent a loop f by a (finite) word $\sigma_n(f) \in S_n$ consisting of the sequence of transition points of order n (later called dyadic points) between the subtriangles of $\triangle_n$ that the loop passes. We will define the bonding mappings $\gamma_n : S_n \to S_{n-1}$ ($n \geq 1$) in a way that we just omit the transition points of order n (see (2.3)) and $\gamma_{nk} : S_n \to S_k$ ($n > k$) denotes the composition $\gamma_{k+1} \circ \ldots \circ \gamma_n$. An appropriate reduction process on $\sigma_n(f)$ leads then to a canonical representative $\mathrm{Red}_n(\sigma_n(f))$ of the homotopy class $[f]_n$ which as a byproduct gives rise to an adequate representation of the elements in $\check{\pi}(\triangle)$. We mention here that in Zastrow [28] another combinatorial representations of loops based on edges is used.

We finally succeed in characterizing the elements of the fundamental group of $\triangle$ by a, after all, surprisingly simple stabilizing condition in the inverse semigroup limit $\varprojlim S_n$. Our main theorem reads as follows.

Theorem 1.1. *An element $(\omega_n)_{n\geq 0}$ of $\varprojlim G_n$ is in $\varphi(\pi(\triangle))$ if and only if for all $k \geq 0$ the sequence $(\gamma_{nk}(\omega_n))_{n\geq k}$ is eventually constant.*

Remark 1.2. (a) Essentially this condition means that exactly those $(\omega_n)_{n\geq 0} \in \varprojlim G_n$ correspond to elements of the fundamental group of $\triangle$ for which, for any order k, the number of alterations between distinct transition points of order k in ω_n is bounded in n. Note that this does not imply that the number of occurrences of a single transition point in ω_n is bounded in n.

(b) Let $\omega = (\omega_n)_{n\geq 0} \in \varprojlim G_n$ be an element of $\varphi(\pi(\triangle))$. In view of Theorem 1.1 there exists a “stabilized sequence” $\bar{\omega} = (\bar{\omega}_n)_{n\geq 0}$ with $\bar{\omega}_n = \gamma_{\ell n}(\omega_\ell)$ which is well defined for $\ell > \ell_n$ large enough. We will show that

(i) $(\bar{\omega}_n)_{n\geq 0} \in \varprojlim S_n$ and

(ii) $\mathrm{Red}(\bar{\omega}_n)_{n\geq 0} = (\omega_n)_{n\geq 0}$.

Thus the sequence $(\bar{\omega}_n)_{n\geq 0}$ can be regarded as the canonical representation of ω in $\varprojlim S_n$. The group operation in $\pi(\triangle)$ in terms of stabilized sequences then reads as follows: for $\omega, \omega' \in \varphi(\pi(\triangle))$ we have

$$\bar{\omega} * \bar{\omega'} = \overline{\mathrm{Red}(\bar{\omega} \cdot \bar{\omega'})},$$

i.e., the product of two stabilized sequences is formed by concatenation and reduction at every level, followed by stabilization.

The crucial step towards Theorem 1.1 is the fact that though σ is not surjective, restricting the domain of the reduction map $\mathrm{Red} : \varprojlim S_n \to \varprojlim G_n$ to the range of σ does not affect its image, i.e., $\mathrm{ran}(\mathrm{Red} \circ \sigma) = \mathrm{ran}(\mathrm{Red})$ where ran(g) denotes the range of a map g (cf. Proposition 3.4).

Moreover, we employ considerable effort to completely describe the kernel and the range of σ to enlighten the relevance of $\varprojlim S_n$ independently of its expedience with respect to the description of the fundamental group of $\triangle$: The elements in the range of σ are characterized by a completeness condition and they precisely describe the set of all loops in $\triangle$ up to re-parametrization.

The organization of the two forthcoming chapters is as follows: In Section 2.1 we introduce a digital representation for the points of the Sierpiński-gasket $\triangle$ by retracing the usual construction process of recursively removing the open middle triangle. Thereby we obtain two sequences of approximating spaces to $\triangle$, and the points in $\triangle$ naturally split into the two classes of dyadic and generic points. In Section 2.2 it is explicated how a loop in $\triangle$ can be represented by a finite word over the alphabet of dyadic points of order $\leq n$ at every approximation level n. In Section 2.3 we introduce the inverse limit of semigroups $\varprojlim S_n$ and show that the groupoid $S(\triangle)$ of all loops in $\triangle$ can be mapped by a homomorphism into $\varprojlim S_n$ by means of the sequence of representations of a loop attained in Section 2.2. In Section 2.4 we introduce the set of reduced words G_n which turns out to be isomorphic to the fundamental group of $\triangle_n$. The $(G_n)_{n\geq 0}$ give rise to an inverse limit of groups $\varprojlim G_n$ and an appropriate reduction map on elements of $\varprojlim S_n$ is defined such that the diagram in Figure 3 commutes. Employing a result of Eda and Kawamura [13] we see that φ is injective and thus the fundamental group of $\triangle$ is a subgroup of $\varprojlim G_n$. Example 2.11 demonstrates that φ is not surjective. This provided the initial motivation for considering $\varprojlim S_n$.

In Section 3.1 we develop the machinery to study the range and the kernel of σ which is accomplished in Propositions 3.3–3.5 in full detail. In Section 3.2 we finally prove the characterization of the elements in $\varprojlim G_n$ representing a homotopy class in $\pi(\triangle)$ given in Theorem 1.1.

2. Preliminaries

2.1. **Digital representations of the Sierpiński-gasket $\triangle$.** For our purposes we need a digital representation of the points of the Sierpiński-gasket $\triangle$. To this end we follow the construction process of $\triangle$ that recursively removes the open middle triangle at each stage. We start with a triangle (including its inside) $\triangle_0$ in the plane. Just to have a concrete metric at hand we assume that $\triangle_0$ is equilateral with side length 1. The vertices of $\triangle_0$ are denoted by 0, 1 and 2. By joining the midpoints of the sides $\triangle_0$ is subdivided in four smaller triangles $\langle 0\rangle$, $\langle 1\rangle$, $\langle 2\rangle$ and the middle triangle, where $\langle i\rangle$ is the subtriangle that contains the vertex i. Removing the interior of the middle triangle from $\triangle_0$ we obtain the first approximation $\triangle_1$, i.e.,

$$\triangle_1 = \langle 0\rangle \cup \langle 1\rangle \cup \langle 2\rangle.$$

With the remaining triangles $\langle i\rangle$, $i = 0, 1, 2$, we proceed in the same way: $\langle i\rangle$ is divided into the four subtriangles $\langle i0\rangle$, $\langle i1\rangle$, $\langle i2\rangle$, and the middle triangle the interior of which is cut out in the next step. Thus we get the second approximation

$$\triangle_2 = \bigcup_{i,j\in\{0,1,2\}} \langle ij\rangle,$$

and so on and so forth. We obtain a decreasing sequence $\triangle_0 \supset \triangle_1 \supset \triangle_2 \ldots$ of compact spaces and hence the intersection $\triangle = \bigcap_{n\in\mathbb{N}} \triangle_n$, the Sierpiński-gasket, is a compact space as well. $\triangle$ consists of two types of points which we call dyadic and generic:

Dyadic points: these are points P which lie in two different subtriangles at some stage (and consequently in all the following stages) in the construction process described before. The smallest level at which P appears as a vertex of two different subtriangles is called the order of P. For instance $\{P\} = \langle 01\rangle \cap \langle 02\rangle = \langle 012\rangle \cap \langle 021\rangle = \ldots$ defines a point P of order 2. We represent P by $(0, 1/2)$ or $(0, 2/1)$ (see Figue 4). In general a dyadic point of order n has a finite representation of the form

$$P = (a_1, a_2, \ldots, a_{n-1}, a/b) = (a_1, a_2, \ldots, a_{n-1}, b/a)$$

with $a_i, a, b \in \{0, 1, 2\}$ and $a \neq b$, and this means $\{P\} = \langle a_1a_2 \ldots a_{n-1}a\rangle \cap \langle a_1a_2 \ldots a_{n-1}b\rangle$. We consider the vertices $0, 1, 2$ of $\triangle_0$ as dyadic points of order 0. Let in the following D_n denote the set of all dyadic points of order $\leq n$. In D_n there is a natural relation $\sim_n$ describing the neighborhood of dyadic points at level n: for $P, Q \in D_n$ we have $P \sim_n Q$ if and only if $P \neq Q$ and there is a subtriangle $\langle a_1 \ldots a_n\rangle$ of $\triangle_n$ to which P and Q belong. At every stage n a dyadic point $P \in D_n$, $P \neq 0, 1, 2$ has exactly four neighbors, and the points $0, 1$ and 2 have exactly two neighbors each.

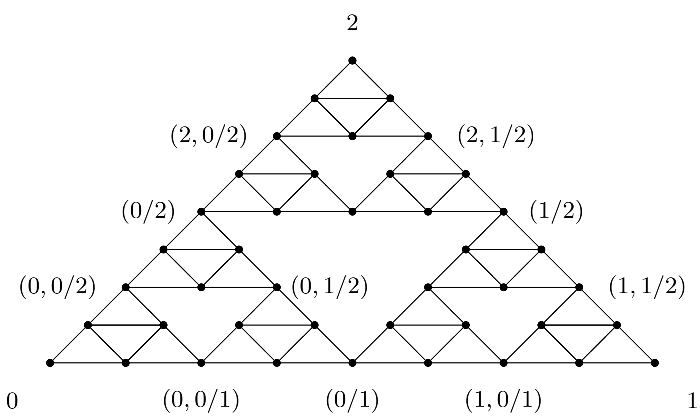

FIGURE 4

Generic points: these are points P of $\triangle$ such that at every stage n there is a unique subtriangle of $\triangle_n$ to which the point P belongs. If $P \in \langle a_1a_2 \ldots a_n\rangle$, $n \in \mathbb{N}$, then P has the infinite representation $P = (a_1, a_2, \ldots)$ with $a_i \in \{0, 1, 2\}$, where the sequence $(a_n)_{n\in\mathbb{N}}$ is not ultimately constant.

Formally $\triangle$ can be obtained as the quotient space of the compact space X of one-sided infinite sequences over the three letter alphabet $\{0, 1, 2\}$, i.e., $X = \{0, 1, 2\}^{\mathbb{N}}$

with the discrete topology on the factors, where a pair of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is identified if there is an n_0 such that $a_n = b_n$ for $n < n_0$ and $a_n = b_{n_0} \neq a_{n_0} = b_n$ for $n > n_0$. In the approach described before this means that $P = (a_1, a_2, \dots, a_{n_0-1}, a_{n_0}/b_{n_0})$ is a dyadic point of order n_0.

The spaces $\triangle_n$, $n \geq 0$, provide an encasing approximation to the Sierpiński-gasket. In the following we will also consider an approximation from inside. Let $\triangle^n$ denote the boundary of $\triangle_n$ considered as a subspace of the plane. Then $\triangle = \overline{\bigcup_{n\in\mathbb{N}} \triangle^n}$ where the bar means the closure operator in the plane: $\bigcup_{n\in\mathbb{N}} \triangle^n$ contains exactly those points $P = (a_n)$ such that eventually the digits a_n are out of a two-element subset of $\{0, 1, 2\}$, in particular this set contains all dyadic points. On the other hand every generic point of $\triangle$ is the limit of a sequence of dyadic points.

Concerning homotopy the spaces $\triangle_n$ and $\triangle^{n-1}$, $n \geq 1$, provide the same level of approximation to the Sierpiński-gasket $\triangle$. There exists a deformation p_n that retracts $\triangle_n$ to $\triangle^{n-1}$: For every subtriangle $T = \langle a_1 a_2 \dots a_{n-1}\rangle$ of $\triangle_{n-1}$ the map p_n projects the points of $\triangle_n \cap T$ from the center of T to the boundary of T. Hence the fundamental groups $\pi(\triangle_n)$ and $\pi(\triangle^{n-1})$ are isomorphic (cf. [25, Theorem 1.22 and Theorem 3.10]).

2.2. **Representation of loops in $\triangle$.** To describe the fundamental group $\pi(\triangle)$ we have to consider continuous loops $f : [0,1] \to \triangle$. Since $\triangle$ is path connected throughout we may assume $f(0) = f(1) = 0$. Our next aim is to represent loops based at 0 in $\triangle_n$ and $\triangle$ by a finite word over the alphabet D_n for every $n \geq 0$.

Let us fix n and assume that $f : [0,1] \to \triangle_n$ is a continuous loop in $\triangle_n$ with $f(0) = f(1) = 0$. The pre-images $\{f^{-1}(P) | P \in D_n\}$ form a finite family of disjoint compact subsets of the interval $[0,1]$. Therefore this family is separated, i.e., there is $m \in \mathbb{N}$ such that for all $i = 1, 2, \dots, m$ the set $f^{-1}(P) \cap [\frac{i-1}{m}, \frac{i}{m}]$ is non-empty for at most one P. We list these points P as i increases and in the arising sequence we cancel out consecutive repetitions. Thus we obtain a finite word $P_1 P_2 \dots P_k =: \sigma_n(f)$ over D_n which is independent of the chosen m. Obviously $\sigma_n(f)$ has the following properties:

$$P_1 = P_k = 0, \tag{2.1}$$

$$P_i \sim_n P_{i+1} \text{ for all } i = 1, \dots k-1. \tag{2.2}$$

In the following we will also consider the loop that emerges from $\sigma_n(f)$ by connecting the listed points straight-lined in the order they appear and call it the piecewise linear loop corresponding to $\sigma_n(f)$. In order to disburden the notation we will not distinguish between the string $\sigma_n(f)$ and the associated loop as long as no confusion can arise.

Now let $f : [0,1] \to \triangle$ be a loop in $\triangle$ based at 0. Since $\triangle \subset \triangle_n$, the image $\sigma_n(f)$ is well defined for each $n \in \mathbb{N}$ and represents f at approximation level n.

Proposition 2.1. *In $\triangle_n$ the loop f and the piecewise linear loop $\sigma_n(f)$ are homotopic.*

Proof. Let $\sigma_n(f) = P_1 \dots P_k$. For every $i = 1, \dots, k$ there is a maximal interval $[s_i, t_i]$ such that $f(s_i) = f(t_i) = P_i$, $f([s_i, t_i]) \cap D_n = \{P_i\}$ and $0 = s_1 \leq t_1 < s_2 \leq t_2 < \dots < s_k \leq t_k = 1$. This means that $f([s_i, t_i])$ is contained in the interior – as a subset of $\triangle_n$ – of the union of the (at most) two subtriangles of $\triangle_n$ that intersect in P_i. Since this set is simply connected $f \upharpoonright [s_i, t_i]$ is homotopic to the constant loop at P_i.

Moreover, the conditions on s_i and t_i imply that $f([t_i, s_{i+1}])$ is a subset of the subtriangle of $\triangle_n$ that contains P_i and P_{i+1} and hence $f \upharpoonright [t_i, s_{i+1}]$ is homotopic to the straight line between P_i and P_{i+1}.

Putting the pieces together we obtain the assertion. $\square$

In order to describe the fundamental group of $\triangle$, Proposition 2.1 suggests to represent a loop f, as a first step, by the sequence $(\sigma_n(f))_{n\geq 0}$. In the next section we will elaborate an appropriate ambient space for the sequences $(\sigma_n(f))_{n\geq 0}$.

2.3. **The inverse system $(S_n, \gamma_n)_{n\geq 0}$ of semigroups.** The semigroups S_n, $n \geq 0$, are defined in the following way: The elements of S_n are finite words $\omega_n = P_1 \dots P_k$ over the alphabet D_n such that (2.1) and (2.2) are satisfied. These words ω_n are called admissible and they are supposed to represent paths in $\triangle_n$. (2.1) means that we consider only cyclic paths with base point 0, and (2.2) reflects that with respect to homotopy constant parts of paths do not matter and that in a continuous path a dyadic point can only be followed by a neighboring dyadic point.

The semigroup operation $\cdot$ on S_n is defined by concatenation of words and cancellation of one of the adjacent letters 0 at the interface:

$$P_1 \dots P_k \cdot Q_1 \dots Q_l = P_1 \dots P_k Q_2 \dots Q_l.$$

The bonding mapping

$$\gamma_n : S_n \to S_{n-1}, \quad n \geq 1, \tag{2.3}$$

eliminates from an element of S_n all points of order n, and then cancels consecutive repetitions of points of order $< n$ arising in this process. Obviously the result is an admissible word in S_{n-1} and γ_n is a semigroup epimorphism. Thus we may consider the inverse semigroup-limit

$$\varprojlim S_n = \{(\omega_n)_{n\geq 0} \mid \gamma_k(\omega_k) = \omega_{k-1} \text{ for all } k \geq 1\}$$

corresponding to the sequence $(S_n, \gamma_n)_{n\geq 0}$.

Let $(S(\triangle), \cdot)$ denote the groupoid of continuous loops $f : [0,1] \to \triangle$ (based at 0), where multiplication $\cdot$ is just the usual concatenation of loops. As a general principle we denote the operations in the groupoid $S(\triangle)$ and in the semigroups S_n and $\varprojlim S_n$ by $\cdot$ (or omit the operation symbol), whereas for the group operations, for instance in the fundamental group $\pi(\triangle)$, we use the notation $*$.

Next we will provide a digital description of loops at the semigroup level.

Proposition 2.2. *The map*

$$\sigma : \begin{cases} S(\triangle) & \to & \varprojlim S_n \\ f & \mapsto & (\sigma_n(f))_{n\geq 0} \end{cases}$$

is a homomorphism from the groupoid $(S(\triangle), \cdot)$ into the semigroup $(\varprojlim S_n, \cdot)$.

Proof. Firstly we show that σ is well defined: Let f be an element of $S(\triangle)$. Then the word $\sigma_n(f)$ contains the dyadic points of D_n which are passed by the loop f in the order they appear in f without consecutive repetitions. When we apply γ_n to $\sigma_n(f)$, obviously we end up with the same word in S_{n-1} we obtain when we list the dyadic points f passes at level $n-1$, i.e., $\gamma_n(\sigma_n(f)) = \sigma_{n-1}(f)$, and thus $\sigma(f) \in \varprojlim S_n$.

σ is a homomorphism since concatenation of loops in $S(\triangle)$ correlates exactly to the concatenation of words in the components S_n, $n \geq 0$. To put it more formally, for $f, g \in S(\triangle)$ we have:

$$\sigma(f \cdot g) = (\sigma_n(f \cdot g))_{n\geq 0} = (\sigma_n(f) \cdot \sigma_n(g))_{n\geq 0} = \\ (\sigma_n(f))_{n\geq 0} \cdot (\sigma_n(g))_{n\geq 0} = \sigma(f) \cdot \sigma(g).$$

$\square$

2.4. **The inverse system $(G_n, \delta_n)_{n\geq 0}$ of groups.** In order to describe the homotopy of loops in $\triangle$ we have to consider an appropriate reduction process for the semigroup words in $\varprojlim S_n$. In the following for $f \in S(\triangle)$ let $[f]$ denote the homotopy class of f in $\triangle$, and let $[f]_n$ denote the homotopy class of f in $\triangle_n$, i.e., in the latter case f is considered as a map with range $\triangle_n$.

In a first step we will describe the elements of the fundamental group of $\triangle_n$. Very briefly we recall here the standard approach to the fundamental group of a simplicial complex (cf. [25, chapter 7]): One considers edge paths in $\triangle_n$ which start and end in the same vertex, say in 0. In principle an edge path is the same as an admissible word over D_n, i.e., an element of S_n, except that also constant edges are allowed. Two edge paths are defined to be equivalent if one can be obtained from the other by a finite number of elementary moves. In our setting an elementary move is a substitution on subwords consisting of consecutive letters of the form

$$PQP \quad\longleftrightarrow\quad P \qquad \text{or} \qquad PQR \quad\longleftrightarrow\quad PR \tag{2.4}$$

where P, Q, R are the distinct vertices of a simplex in the simplicial complex which in our case means that P, Q, R form a subtriangle of $\triangle_n$. As the arrows indicate these transformations may be performed in both directions. The equivalence classes of edge paths then constitute the elements of the fundamental group with concatenation as the group operation (cf. [25, Theorem 7.36]).

In our attempt we proceed slightly different: We call an element $\omega_n \in S_n$ reduced if ω_n cannot be shortened by an elementary move as described in (2.4). A reduced word in S_n can be identified with a sequence of subtriangles of $\triangle_n$ such that any three consecutive subtriangles are pairwise different. Let G_n denote the set of all reduced words of S_n and $\mathrm{Red}_n : S_n \to G_n$ the mapping that performs elementary moves until the word is reduced.

Proposition 2.3. Red_n *is well defined and for* $\omega_n \in S_n$ *the loop corresponding to* $\mathrm{Red}_n(\omega_n)$ *forms a canonical representative of the homotopy class of the loop corresponding to* ω_n *in* $\triangle_n$.

Proof. Obviously, by performing an elementary move on an element of S_n we stay in the same homotopy class for the corresponding loops. All we have to show is that two different reduced words correspond to non-homotopic loops. Here we use the fact that $\triangle_n$ and $\triangle^{n-1}$ have isomorphic homotopy groups ($\triangle^{n-1}$ is a deformation retract of $\triangle_n$).

Since $\triangle^{n-1}$ is a connected 1-complex its homotopy group is a free group, freely generated by the edges not contained in a fixed spanning tree T (cf. [25, Corollary 7.35]). Starting with two different reduced words $\omega_n \neq \bar{\omega}_n$ in G_n by retracting the loops corresponding to ω_n and $\bar{\omega}_n$ to $\triangle^{n-1}$, we end up with two different words $\omega_{n-1} \neq \bar{\omega}_{n-1}$ over the alphabet D_{n-1} such that any three consecutive letters of these words are pairwise different elements of D_{n-1} (a reduced word in G_n correlates to a sequence of subtriangles in $\triangle_n$; every subtriangle in $\triangle_n$ contains exactly one vertex in D_{n-1}; the sequence of these vertices is exactly what we obtain by the retraction).

Suppose the two emerging loops corresponding to ω_{n-1} and $\bar{\omega}_{n-1}$ are homotopic in $\triangle^{n-1}$, then due to the fact that the homotopy group of $\triangle^{n-1}$ is a free group the two words must contain the same edges not contained in the tree T in the corresponding order. Moreover, there is a unique path in T connecting these edges. Since ω_{n-1} and $\bar{\omega}_{n-1}$ do not contain subwords of the form PQP, ω_{n-1} and $\bar{\omega}_{n-1}$ must be identical in the parts connecting the edges not in T, and hence they must coincide on the whole, which is a contradiction. $\square$

Now it is obvious how to define the group operation for $\omega_n, \bar{\omega}_n \in G_n$:

$$\omega_n * \bar{\omega}_n = \mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n),$$

where $\omega_n \cdot \bar{\omega}_n$ is the product in S_n. Together with the results in [25, chapter 7] we obtain:

Proposition 2.4. *The fundamental group* $(\pi(\triangle_n), *)$ *is isomorphic to* $(G_n, *)$ *by means of the isomorphism* $\varphi_n : [f]_n \mapsto \mathrm{Red}_n(\sigma_n(f))$ *where* f *is a continuous loop in* $\triangle_n$*. Furthermore, the reduction map* $\mathrm{Red}_n : S_n \to G_n$*, associating to every admissible word its reduced form, is a semigroup epimorphism, i.e.,* $(G_n, *)$ *is isomorphic to* $(S_n/\ker(\mathrm{Red}_n), \cdot)$*.*

Now we elaborate a bonding between the groups G_n.

Lemma 2.5. *For* $n \geq 1$ *the map*

$$\delta_n : \begin{cases} G_n & \to & G_{n-1} \\ \omega_n & \mapsto & \mathrm{Red}_{n-1}(\gamma_n(\omega_n)) \end{cases}$$

is a group epimorphism.

Proof. Let $\omega_n, \bar{\omega}_n \in G_n$. We have

$$\delta_n(\omega_n * \bar{\omega}_n) = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n))).$$

On the other hand we get

$$\delta_n(\omega_n) * \delta_n(\bar{\omega}_n) = \mathrm{Red}_{n-1}(\mathrm{Red}_{n-1}(\gamma_n(\omega_n)) \cdot \mathrm{Red}_{n-1}(\gamma_n(\bar{\omega}_n))) =$$
$$\mathrm{Red}_{n-1}(\gamma_n(\omega_n) \cdot \gamma_n(\bar{\omega}_n)) = \mathrm{Red}_{n-1}(\gamma_n(\omega_n \cdot \bar{\omega}_n)).$$

Due to Proposition 2.3 it is thus sufficient to show that the loops $\gamma_n(\mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n))$ and $\gamma_n(\omega_n \cdot \bar{\omega}_n)$ are homotopic in $\triangle_{n-1}$. It is obvious by the definition of γ_n that $[\alpha_n]_{n-1} = [\gamma_n(\alpha_n)]_{n-1}$ for every $\alpha_n \in S_n$. Further we have $[\alpha_n]_n = [\mathrm{Red}_n(\alpha_n)]_n$ and hence also $[\alpha_n]_{n-1} = [\mathrm{Red}_n(\alpha_n)]_{n-1}$. Altogether we obtain

$$[\gamma_n(\omega_n\bar{\omega}_n)]_{n-1} = [\omega_n\bar{\omega}_n]_{n-1} = [\mathrm{Red}_n(\omega_n\bar{\omega}_n)]_{n-1} = [\gamma_n(\mathrm{Red}_n(\omega_n\bar{\omega}_n))]_{n-1}$$

and we are done.

δ_n is surjective: Suppose $\omega_{n-1} = P_1P_2 \dots P_k$ in G_{n-1} is given. Put $\omega_n = P_1Q_1P_2Q_2 \dots Q_{k-1}P_k$, where Q_i is the (unique) element of D_n with $P_i \sim_n Q_i \sim_n P_{i+1}$. One can check easily that ω_n is reduced and $\delta_n(\omega_n) = \omega_{n-1}$. □

As a consequence of the last lemma we can consider the inverse group-limit

$$\varprojlim G_n = \{(\omega_n)_{n\geq 0} \mid \delta_k(\omega_k) = \omega_{k-1} \text{ for all } k \geq 1\}.$$

Next we show that the reduction maps $\mathrm{Red}_n : S_n \to G_n$ can be lifted to a map on the inverse limits.

Lemma 2.6. *For every* $n \geq 1$ *the following diagram commutes:*

$$\begin{array}{ccc} S_n & \xrightarrow{\gamma_n} & S_{n-1} \\ \downarrow {\scriptstyle \mathrm{Red}_n} & & {\scriptstyle \mathrm{Red}_{n-1}} \downarrow \\ G_n & \xrightarrow{\delta_n} & G_{n-1} \end{array}$$

Proof. Let ω_n be in S_n. We have to show that $\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\gamma_n(\omega_n))$. Since $\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n)))$ it suffices to prove that $\gamma_n(\omega_n)$ and $\gamma_n(\mathrm{Red}_n(\omega_n))$ are homotopic in $\triangle_{n-1}$. However, this was already accomplished in the proof of Lemma 2.5. □

Proposition 2.7. *The map*

$$\mathrm{Red}: \begin{cases} \varprojlim S_n & \to & \varprojlim G_n \\ (\omega_n)_{n\geq 0} & \mapsto & (\mathrm{Red}_n(\omega_n))_{n\geq 0} \end{cases}$$

is a well defined semigroup homomorphism.

Proof. If $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ then $\gamma_n(\omega_n) = \omega_{n-1}$ for every $n \geq 1$. Thus Lemma 2.6 yields $\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\omega_{n-1})$. This shows that Red is well defined. The fact that Red is a homomorphism follows because Red_n is a homomorphism by Proposition 2.4. □

Now we figure out that the fundamental group $(\pi(\triangle), *)$ can be embedded into the group-limit $(\varprojlim G_n, *)$. To this matter we need a lemma on the Čech homoptopy group $\check{\pi}(\triangle)$ of $\triangle$ (see e.g. [23, p. 130][1] or [13, Appendix A] for a definition of $\check{\pi}$).

Proposition 2.8. *The Čech homoptopy group $\check{\pi}(\triangle)$ is isomorphic to $\varprojlim G_n$.*

Proof. Since $\triangle = \bigcap_{n\geq 0} \triangle_n$ and $\triangle_0 \supset \triangle_1 \supset \triangle_2 \ldots$ is a nested sequence of compact polyhedra we have that

$$\check{\pi}(\triangle) = \varprojlim \pi(\triangle_n) \tag{2.5}$$

where for each $n \in \mathbb{N}$ the bonding mapping $j_n : \pi(\triangle_n) \to \pi(\triangle_{n-1})$ is induced by the inclusion $\triangle_n \hookrightarrow \triangle_{n-1}$ (see [23, Chapter II, §3]).

According to Proposition 2.4 we have that $\pi(\triangle_n) \cong G_n$. Let $\varphi_n : \pi(\triangle_n) \to G_n$ be the canonical isomorphism between these groups. It is now easy to see that the diagram

$$\begin{array}{ccc} \pi(\triangle_n) & \xrightarrow{j_n} & \pi(\triangle_{n-1}) \\ \cong \downarrow \varphi_n & & \cong \downarrow \varphi_{n-1} \\ G_n & \xrightarrow{\delta_n} & G_{n-1} \end{array}$$

is commutative. Indeed, for each $n \geq 1$ and each continuous loop f in $\triangle_n \subset \triangle_{n-1}$ we have $[\sigma_n(f)]_n = [f]_n$ by Proposition 2.1. In particular, $[\sigma_n(f)]_{n-1} = [f]_{n-1}$ and $[\sigma_{n-1}(f)]_{n-1} = [f]_{n-1}$ hold. Also we observed in the proof of Lemma 2.5 that $[\gamma_n(\omega_n)]_{n-1} = [\omega_n]_{n-1}$ holds for $\omega_n \in S_n$. Hence,

$$[\gamma_n(\sigma_n(f))]_{n-1} = [\sigma_n(f)]_{n-1} = [f]_{n-1} = [\sigma_{n-1}(f)]_{n-1}.$$

Combining this with Lemma 2.6 we get

$$\begin{aligned} \delta_n(\varphi_n([f]_n)) &= \delta_n(\mathrm{Red}_n(\sigma_n(f))) = \mathrm{Red}_{n-1}(\gamma_n(\sigma_n(f)) \\ &= \mathrm{Red}_{n-1}(\sigma_{n-1}(f)) = \varphi_{n-1}([f]_{n-1}) \\ &= \varphi_{n-1}(j_n([f]_n)) \end{aligned}$$

which proves the commutativity of the above diagram. Together with 2.5 the diagram implies the assertion of the lemma. □

We are now in a position to prove the following result.

Proposition 2.9. *The map*

$$\varphi: \begin{cases} \pi(\triangle) & \to & \varprojlim G_n \\ [f] & \mapsto & \mathrm{Red}(\sigma(f)) \end{cases}$$

is a well defined group monomorphism.

[1]Note that the Čech homotopy group is called *shape group* in this text.

Proof. Because $\triangle$ is a one-dimensional continuum, [13, Corollary 1.2] implies that the canonical homomorphism from $\pi(\triangle)$ to $\check{\pi}(\triangle)$ is a monomorphism. Since φ is the composition of this monomorphism with the isomorphism between $\check{\pi}(\triangle)$ and $\varprojlim G_n$ established in Proposition 2.8 we get the result. $\square$

The next theorem gives an interim survey of what we have established up to this point.

Theorem 2.10. *The fundamental group $(\pi(\triangle), *)$ of the Sierpiński-gasket is isomorphic to a subgroup of $(\varprojlim G_n, *)$. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} S(\triangle) & \xrightarrow{\sigma} & \varprojlim S_n \\ \downarrow {\scriptstyle [\,.\,]} & & {\scriptstyle \mathrm{Red}} \downarrow \\ \pi(\triangle) & \overset{\varphi}{\hookrightarrow} & \varprojlim G_n \end{array}$$

However, the next example shows that φ is not surjective:

Example 2.11. Let C_0 be the (piecewise linear) loop that starting at 0 passes around the boundary of $\triangle_0$ in positive direction (i.e. passing from 0 to 1, then 2 and back to 0). By C_0^{-1} we mean the same cycle passed in the opposite direction. C_1 denotes the loop around the subtriangle $\langle 0\rangle$ in $\triangle_1$ (i.e. passing through 0, (0/1), (0/2) and 0), C_2 the loop around $\langle 00\rangle$ in $\triangle_2$, and so on. Now we consider the following sequence of words:

$$\begin{aligned}
\omega_0 &= \omega_1 = 0 \\
\omega_2 &= \mathrm{Red}_2(\sigma_2(C_0C_1C_0^{-1})) \\
\omega_3 &= \mathrm{Red}_3(\sigma_3(C_0C_1C_0^{-1}C_2)) \\
\omega_4 &= \mathrm{Red}_4(\sigma_4(C_0C_1C_0^{-1}C_2C_0C_3C_0^{-1})) \\
\omega_5 &= \mathrm{Red}_5(\sigma_5(C_0C_1C_0^{-1}C_2C_0C_3C_0^{-1}C_4)) \\
&\dots
\end{aligned}$$

It can be checked easily that $(\omega_n)_{n\geq 0}$ is an element of $\varprojlim G_n$. For instance, if we apply δ_4 to ω_4, the loop C_3 disappears since it is null-homotopic in $\triangle_3$, and consequently also the C_0 and C_0^{-1} neighboring C_3 cancel out and we arrive at ω_3.

Suppose there exists f in $S(\triangle)$ such that $\varphi([f]) = (\omega_n)_{n\geq 0}$. Then due to the construction of $\omega_n = [f]_n$ the loop f has to traverse the circle C_0 infinitely many times, which is not possible.

Maybe it is instructive to see here that $(\omega_n)_{n\geq 0}$ is even not in $\mathrm{Red}(\varprojlim S_n)$. Suppose there is $(\alpha_n)_{n\geq 0}$ in $\varprojlim S_n$ with $\mathrm{Red}((\alpha_n)_{n\geq 0}) = (\omega_n)_{n\geq 0}$. If we consider only the dyadic points of order 1 that appear in ω_{2n}, we see that the sequence (0/1) (1/2) (0/2) (1/2) (0/1) repeats n times. This means that at least this sequence of $5n$ points of order 1 also appears in α_{2n} (maybe some more which cancel out by performing Red_{2n}). However, when projecting down from S_{2n} to S_1 in $\varprojlim S_n$ no cancelation in between these $5n$ points can occur. As a consequence α_1 would contain infinitely many points which is a contradiction.

We aim at describing the fundamental group of the Sierpiński-gasket. Retrospectively, Theorem 2.10 provides the motivation for investigating the semigroup limit $\varprojlim S_n$: $\pi(\triangle) \cong \varphi(\pi(\triangle)) = \mathrm{Red}(\sigma(S(\triangle)))$. Therefore we have to study the range of σ in $\varprojlim S_n$ and the range of Red in $\varprojlim G_n$. This will be accomplished in the next section.

3. A characterization of the elements in $\varphi(\pi(\triangle))$

3.1. **The range and the kernel of $\boldsymbol{\sigma}$.** We associate to a fixed element $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2}\dots P_{nk_n})_{n\geq 0}$ in $\varprojlim S_n$ a graph $G = (V, E)$ with vertices V and directed edges E. We think of the graph G as organized in rows: in the nth row, $n \geq 0$, we have for every letter appearing in the word ω_n a corresponding vertex, i.e. $V = \{(n, j) \mid n \geq 0, 1 \leq j \leq k_n\}$. Edges connect certain vertices from row n to vertices in row $n+1$, namely, $((n,i),(n+1,j)) \in E$ if and only if $P_{ni} = P_{n+1,j}$ and in the course of γ_{n+1} that maps ω_{n+1} to ω_n the point $P_{n+1,j}$ is projected to P_{ni}. Consequently any vertex (n,i) in row n has at least one successor up to a finite number of successors (not bounded from above for growing n) in row $n+1$, and (n,i) has exactly one predecessor in row $n-1$ if and only if the order of P_{ni} is $< n$.

Example 3.1. We consider the following element in $\varprojlim S_n$ one can think of as a "pseudo-path" that passes from 0 on the baseline of $\triangle^0$ arbitrarily near to 1 without touching 1 and then goes the same way back to 0. A phenomenon arising in this example will turn out to be important in the further investigation:

$$\omega_0 = 0,\ \omega_1 = 0(0/1)0,\ \omega_2 = 0(0,0/1)(0/1)(1,0/1)(0/1)(0,0/1)0,\ \dots .$$

Figure 5 shows the graph associated to $(\omega_n)_{n\geq 0}$ where we denote the vertices by the corresponding dyadic points P_{ni} instead of the index (n,i) we usually use.

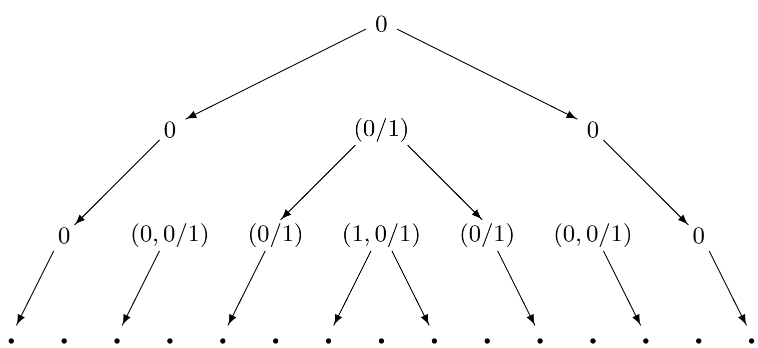

Figure 5

By a branch B we mean a directed path in G which cannot be extended. As description for B we use the sequence of vertices contained in B, i.e. $B = (n, i_n)_{n\geq n_0}$ where $P = P_{n,i_n}$ for all $n \geq n_0$, is a point of order n_0. We say that the branch B corresponds to the dyadic point P.

The set $\mathcal{B}$ of all branches in G carries a natural total order $\leq$: Let $B_1 = (n, i_n)_{n\geq n_1}$, $B_2 = (n, j_n)_{n\geq n_2}$ be two branches then we define $B_1 < B_2$ if and only if there exists $n \geq \max\{n_1, n_2\}$ such that $i_n < j_n$. Consequently we then have $i_m < j_m$ for all $m \geq n$, and $i_m \leq j_m$ for all m with $\max\{n_1, n_2\} \leq m < n$ which reflects the property that branches do not cross in G if we display the vertices in every row n in the order they appear in ω_n. It is straightforward to check that $\leq$ is a total order on $\mathcal{B}$. For instance, $B_1 \leq B_2$ and $B_2 \leq B_1$ implies $B_1 = B_2$ since branches are maximal with respect to extension.

The order $\leq$ on $\mathcal{B}$ is dense: Let $B_1 < B_2$ be defined as before with $i_n < j_n$. Then $j_{n+1} - i_{n+1} \geq 2$ since the points corresponding to B_1 and B_2 are of order $\leq n$ and thus $P_{n+1,i_{n+1}} \not\sim_{n+1} P_{n+1,j_{n+1}}$. Hence any branch B starting at vertex $(n+1, i_{n+1}+1)$ satisfies $B_1 < B < B_2$.

In the following we will consider Dedekind cuts in $(\mathcal{B}, \leq)$: A cut $(\mathcal{B}_1, \mathcal{B}_2)$ is a partition of $\mathcal{B}$ into two (nonempty) subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $B \in \mathcal{B}_1$, $\bar{B} < B$ implies $\bar{B} \in \mathcal{B}_1$, and $B \in \mathcal{B}_2$, $\bar{B} > B$ implies $\bar{B} \in \mathcal{B}_2$.

Rational and irrational cuts: The cut $(\mathcal{B}_1, \mathcal{B}_2)$ is called *rational* if either $\mathcal{B}_1$ has a largest element or $\mathcal{B}_2$ has a least element. In the remaining case $(\mathcal{B}_1, \mathcal{B}_2)$ is called *irrational.*

Every cut $(\mathcal{B}_1, \mathcal{B}_2)$ converges to a uniquely defined element of $\triangle$ in the following sense: For all $n \geq 0$ put

$$\begin{aligned} l_n &= \max\{i \mid \exists\, B \in \mathcal{B}_1 : B \text{ contains } (n,i)\} \\ r_n &= \min\{j \mid \exists\, B \in \mathcal{B}_2 : B \text{ contains } (n,j)\} \end{aligned}$$

Obviously we have $1 \leq l_n \leq r_n \leq k_n$ for all $n \geq 0$.

Lemma 3.2. *For the cut $(\mathcal{B}_1, \mathcal{B}_2)$ we have $\lim\limits_{n\to\infty} P_{n,l_n} = \lim\limits_{n\to\infty} P_{n,r_n}$.*

Proof. By construction of l_n and r_n we have either $l_n = r_n$ and thus $P_{n,l_n} = P_{n,r_n}$ or $r_n = l_n + 1$ and thus $P_{n,l_n} \sim_n P_{n,r_n}$. Hence it is sufficient to prove the existence of $\lim\limits_{n\to\infty} P_{n,l_n}$.

We prove now for all $n \geq 0$ that $P_{n+1,l_{n+1}}$ lies in the same subtriangle T_n of $\triangle_n$ as P_{n,l_n}: We suppose $P_{n,l_n} \sim_n P_{n,r_n}$, the other case $P_{n,l_n} = P_{n,r_n}$ is proved similarly. Let $B_1 = (\ldots, (n, l_n), (n+1, i), \ldots)$ be a branch in $\mathcal{B}_1$ such that i is a large as possible. Further, let $B_2 = (\ldots, (n, r_n), (n+1, j), \ldots)$ be a branch in $\mathcal{B}_2$ such that j is a small as possible. Note that $P_{n+1,i} = P_{n,l_n}$, $P_{n+1,j} = P_{n,r_n}$ and $l_{n+1} \geq i$. Evidently, all points $P_{n+1,k}$ with $i < k < j$ are of order $n+1$ and lie in the same subtriangle T_n of $\triangle_n$ as P_{n,l_n} and P_{n,r_n}, and it is clear by construction that $P_{n+1,l_{n+1}}$ is one of the points $P_{n+1,k}$ or coincides with P_{n,l_n}.

Thus we obtain a sequence of subtriangles $(T_n)_{n\geq 0}$ with $T_n \supset T_{n+1}$, $\operatorname{diam}(T_n) = 2^{-n}$, $P_{n,l_n} \in T_n$, and hence $\lim\limits_{n\to\infty} P_{n,l_n}$ exists. □

The limit of the cut $(\mathcal{B}_1, \mathcal{B}_2)$ is defined to be the point $\lim\limits_{n\to\infty} P_{n,l_n} = \lim\limits_{n\to\infty} P_{n,r_n}$ in $\triangle$. As the proof of Lemma 3.2 shows, a rational cut has a dyadic limit point, namely the point corresponding to the largest branch in $\mathcal{B}_1$ or the smallest branch in $\mathcal{B}_2$, respectively. An irrational cut may converge to a dyadic or to a generic point.

Complete elements: We call $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ *complete* if every irrational cut in the set of branches $\mathcal{B}$ associated to $(\omega_n)_{n\geq 0}$ converges to a generic point.

Coming back to Example 3.1 we see that $(\omega_n)_{n\geq 0}$ defined there is not complete: Let $\mathcal{B}_1$ consist of all branches which turn left when following them downwards, $\mathcal{B}_2$ all that turn right. Then obviously this cut is irrational and converges to the dyadic point 1.

Next we prove that completeness is a necessary condition for $(\omega_n)_{n\geq 0}$ to be an element of $\sigma(S(\triangle))$.

Proposition 3.3. *For all $f \in S(\triangle)$ the representation $\sigma(f)$ in $\varprojlim S_n$ is complete.*

Proof. Put $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2}\ldots P_{n,k_n})_{n\geq 0} = (\sigma_n(f))_{n\geq 0}$ and let $B = (n, i_n)_{n\geq n_0}$ be a branch in the graph G which is associated to $(\omega_n)_{n\geq 0}$.

We will assign to B an interval $[s_B, t_B] \subseteq [0,1]$: Firstly, as we already explicated in the beginning of the proof of Proposition 2.1, for every $n \geq n_0$ we can associate to P_{n,i_n} the interval $[s_n, t_n]$ such that $f([s_n, t_n]) \cap D_n = \{P_{n,i_n}\}$. The definition of the edges in the graph G yields $[s_{n+1}, t_{n+1}] \subseteq [s_n, t_n]$, and so we obtain a nonempty interval $[s_B, t_B] = \bigcap\limits_{n\geq 0} [s_n, t_n]$ such that f is constant on $[s_B, t_B]$ with the dyadic point corresponding to B as the constant value.

We list some properties of this relationship between branches and intervals. The order on the branches is preserved by this construction, i.e., if $B_1 = (n, i_n^{(1)})_{n\geq n_1}, B_2 = (n, i_n^{(2)})_{n\geq n_2}$ are two branches then $B_1 < B_2$ implies $t_{B_1} < s_{B_2}$: $B_1 < B_2$ means that there is an n such that $i_n^{(1)} < i_n^{(2)}$ and thus for the intervals $[s_{nk}, t_{nk}]$ associated to $P_{n,i_n^{(k)}}$, $k = 1, 2$, we have $t_{n1} < s_{n2}$. Hence $t_{B_1} = \inf_{n\geq n_1} t_{n1} < \sup_{n\geq n_2} s_{n2} = s_{B_2}$.

As a consequence different branches lead to disjoint intervals. Further, it is evident that for every $u \in [0, 1]$ such that $f(u)$ is a dyadic point there exists a unique branch B with $u \in [s_B, t_B]$.

To sum up, the family $\{[s_B, t_B] \mid B \in \mathcal{B}\}$ forms a partition of $f^{-1}(\bigcup_{n\geq 0} D_n)$ which inherits the order on the set of all branches $\mathcal{B}$ in the sense explained above.

Now we are in position to prove that every irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$ converges to a generic point in $\triangle$: The irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ corresponds to an irrational cut in $\{[s_B, t_B] \mid B \in \mathcal{B}\}$. Put $s = \sup_{B\in\mathcal{B}_1} s_B$ and $t = \inf_{B\in\mathcal{B}_2} s_B$. Since the cut is irrational it is irrelevant whether we take s_B or t_B when forming the inf and the sup, and moreover we have $s > s_{B_1}$ and $t < t_{B_2}$ for all $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$.

Obviously $s \leq t$ and we claim that f is constant in the interval $[s, t]$ with a generic point as constant value: Suppose there exists $u \in [s, t]$ such that $f(u)$ is a dyadic point. Then there is a branch $\bar{B}$ with $u \in [s_{\bar{B}}, t_{\bar{B}}]$. However, due to the definition of $s = \sup_{B\in\mathcal{B}_1} s_B$ all intervals corresponding to branches of $\mathcal{B}_1$ are strictly below s and thus cannot contain u. The same applies to all branches of $\mathcal{B}_2$ since their intervals lie above t. Hence $\bar{B}$ is not in $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}$ which is a contradiction. So f does not attain a dyadic point as value on the interval $[s, t]$. Suppose f is not constant on $[s, t]$. Then $f([s, t])$ is a connected subset of $\triangle$ containing at least two points and therefore also contains a dyadic point.

Finally we show that the cut $(\mathcal{B}_1, \mathcal{B}_2)$ converges to the generic point $f(s)$. Put $l_n = \max\{i \mid \exists B \in \mathcal{B}_1 : B \text{ contains } (n, i)\}$. Thus for every $n \geq 0$ there exists a branch $B_n = (m, i_m^{(n)})_{m\geq m_n} \in \mathcal{B}_1$ such that $(n, l_n) = (n, i_n^{(n)})$ and thus $P_{n,l_n} = P_{n,i_n^{(n)}}$. As a consequence $f(s_{B_n}) = P_{n,l_n}$ where as usual $[s_{B_n}, t_{B_n}]$ is the interval corresponding to B_n.

Since $\mathcal{B}_1$ has no largest element for every $B = (n, i_n)_{n\geq n_0} \in \mathcal{B}_1$ there exists $\bar{B} = (n, j_n)_{n\geq \bar{n}_0} \in \mathcal{B}_1$ with $\bar{B} > B$, i.e. there is an $n \in \mathbb{N}$ such that $i_n < j_n \leq l_n = i_n^{(n)}$. This means that for all $B \in \mathcal{B}_1$ there is an $n \in \mathbb{N}$ such that $s_B < s_{B_n}$. So we infer $\lim_{n\to\infty} s_{B_n} = s$, and using the continuity of f we obtain

$$\lim_{n\to\infty} P_{n,l_n} = \lim_{n\to\infty} f(s_{B_n}) = f(s)$$

and we are done. □

We have already seen that non-complete elements in $\varprojlim S_n$ exist (see Example 3.1). Proposition 3.3 thus shows that $\sigma : S(\triangle) \to \varprojlim S_n$ is not surjective.

The next proposition aims at finding f in $S(\triangle)$ such that $\sigma(f)$ approximates a given $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ best possible.

Proposition 3.4. *For every $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ there exists $f \in S(\triangle)$ such that $\mathrm{Red}(\sigma(f)) = \mathrm{Red}((\omega_n)_{n\geq 0})$, i.e., $\mathrm{ran}(\mathrm{Red} \circ \sigma) = \mathrm{ran}(\mathrm{Red})$. Moreover, if $(\omega_n)_{n\geq 0}$ is complete then even $\sigma(f) = (\omega_n)_{n\geq 0}$ holds for some $f \in S(\triangle)$.*

Proof. Let $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2} \dots P_{n,k_n})_{n\geq 0}$ be a fixed element of $\varprojlim S_n$. We will define a sequence of functions $(f_n)_{n\geq 0}$ by induction on n such that f_n is piecewise linear with range in $\triangle^n$ and $\sigma_k(f_n) = \omega_k$ for all $k \leq n$.

We start with $n = 0$, $\omega_0 = P_{01}P_{02}\ldots P_{0,k_0}$, and divide $[0,1]$ into $2k_0 - 1$ subintervals of equal length by the points

$$0 = u_{01} < v_{01} < u_{02} < v_{02} < \ldots < u_{0,k_0} < v_{0,k_0} = 1.$$

Define $f_0(t) = P_{0i}$ for $t \in [u_{0i}, v_{0i}]$, $1 \le i \le k_0$, and f_0 to be the linear connection of P_{0i} and $P_{0,i+1}$ in the interval $[v_{0i}, u_{0,i+1}]$, $1 \le i < k_0$. Obviously $\sigma_0(f_0) = \omega_0$.

Suppose f_n is already defined: $f_n(t) = P_{ni}$ for $t \in [u_{ni}, v_{ni}]$, $1 \le i \le k_n$, f_n is the linear connection of P_{ni} and $P_{n,i+1}$ in the interval $[v_{ni}, u_{n,i+1}]$, $1 \le i < k_n$, and thus $\sigma_k(f_n) = \omega_k$ for all $k \le n$. We explain in detail how to define $f_{n+1}(t)$ for $t \in [u_{n1}, v_{n1}]$ and $t \in [v_{n1}, u_{n2}]$. In the equality $\gamma_{n+1}(\omega_{n+1}) = \omega_n$ we analyze the action of γ_{n+1} on the individual letters of ω_{n+1}: Figure 6 is part of the graph G

$$\begin{array}{cccccccccc}
 & P_{n1} & & & & & & P_{n2} & & \cdots \\
\swarrow & & \searrow & & & & \swarrow & & \searrow & \cdots \\
P_{n+1,1} & \ldots & P_{n+1,i_1} & P_{n+1,i_1+1} & \ldots & P_{n+1,i_2} & P_{n+1,i_2+1} & \ldots & P_{n+1,i_3} & \cdots
\end{array}$$

FIGURE 6

we associated to $(\omega_n)_{n\ge 0}$ in the beginning of this section and should be interpreted as follows: $P_{n+1,1}$ respectively P_{n+1,i_1} is the first respectively last letter in ω_{n+1} that is projected to P_{n1} by γ_{n+1}; P_{n+1,i_1+1} up to P_{n+1,i_2} are all of order $n+1$ and disappear by applying γ_{n+1}, and so on.

Now we define $f_{n+1}(t)$ for $t \in [u_{n1}, v_{n1}]$ analogously to f_0 in $[0,1]$: divide $[u_{n1}, v_{n1}]$ into $2i_1 - 1$ subintervals of equal length and define f_{n+1} in these subintervals alternately to be constant with value $P_{n+1,i}$, $1 \le i \le i_1$, and to connect $P_{n+1,i}$ with $P_{n+1,i+1}$ linearly, $1 \le i \le i_1 - 1$.

Next, the interval $[v_{n1}, u_{n2}]$ is divided into $2(i_2 - i_1) + 1$ subintervals. Here f_{n+1} alternately connects $P_{n+1,i}$ with $P_{n+1,i+1}$ linearly, $i_1 \le i \le i_2$, and is constant with value $P_{n+1,i}$, $i_1 + 1 \le i \le i_2$.

In the same manner we proceed with the rest of the intervals and obtain f_{n+1} satisfying our requirements.

We compare f_n with f_{n+1} (see Figure 7). For $1 \le i \le k_n$:

$$t \in [u_{ni}, v_{ni}] \, : \begin{cases} f_n(t) & \ldots \ \text{constant } P_{ni} \\ f_{n+1}(t) & \ldots \ \text{stays in the two subtriangles } T_1 \text{ and} \\ & \qquad T_2 \text{ of } \triangle_n \text{ that intersect in } P_{ni}, \end{cases}$$

and for $1 \le i \le k_n - 1$:

$$t \in [v_{ni}, u_{n,i+1}] \, : \begin{cases} f_n(t) & \ldots \ \text{connects } P_{ni} \text{ and } P_{n,i+1} \text{ linearly} \\ f_{n+1}(t) & \ldots \ \text{stays in the subtriangle } T_2 \text{ of } \triangle_n \text{ to} \\ & \qquad \text{which } P_{ni} \text{ and } P_{n,i+1} \text{ belong.} \end{cases}$$

Summing up we obtain $\|f_n - f_{n+1}\|_\infty \le 2^{-n}$ where $\|.\|_\infty$ denotes the maximum norm for $t \in [0,1]$. Consequently f_n converges for $n \to \infty$ uniformly to a continuous $f : [0,1] \to \triangle$.

By construction we have $f_m(u_{ni}) = P_{ni}$, $1 \le i \le k_n$, for all $m \ge n$ and thus also $f(u_{ni}) = P_{ni}$, $1 \le i \le k_n$. This means that $\sigma_n(f)$ contains at least all letters appearing in the word ω_n in the proper order, but it may happen that $\sigma_n(f)$ in between the P_{ni} contains further dyadic points of order $\le n$ and some of the P_{ni} appear in multiplied form. To illustrate this we consider the interval $[u_{ni}, u_{n,i+1}]$:

f_{n+1} and all f_m with $m \ge n+1$ stay for $t \in (u_{ni}, u_{n,i+1})$ in the interior of the union of the two subtriangles $\mathrm{int}(T_1 \cup T_2)$ of $\triangle_n$ (interior as a subset of $\triangle_n$). This implies that $f = \lim\limits_{m\to\infty} f_m$ stays in the union of the (closed) subtriangles

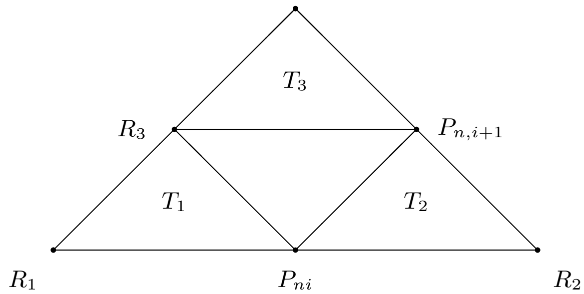

FIGURE 7

$T_1 \cup T_2$. Hence $\sigma_n(f \restriction [u_{ni}, u_{n,i+1}]) = P_{ni}Q_1Q_2 \dots Q_l P_{n,i+1}$, $l \geq 0$, where $Q_i \in \{R_1, R_2, R_3, P_{ni}, P_{n,i+1}\}$. However, since $f([u_{ni}, u_{n,i+1}]) \cap (T_3 \setminus \{R_3, P_{n,i+1}\}) = \emptyset$, the two letters R_3 and $P_{n,i+1}$ can never occur in immediate succession in $P_{ni}Q_1Q_2 \dots Q_l P_{n,i+1}$. This implies that $\mathrm{Red}_n(\sigma_n(f \restriction [u_{ni}, u_{n,i+1}])) = P_{ni}P_{n,i+1}$ and hence on the whole $\mathrm{Red}_n(\sigma_n(f)) = \mathrm{Red}_n(\omega_n)$.

Of course, configurations for P_{ni} and $P_{n,i+1}$ different to the one displayed in Figure 7 are possible. However, as can be checked easily the consequences concerning the respective subtriangles T_1, T_2 and T_3 are always the same.

The first part of the proposition is proved. Now we show that $\sigma_n(f) = \omega_n$ for all $n \geq 0$ if $(\omega_n)_{n \geq 0}$ is complete.

We have two sets of branches: The set $\mathcal{B}_f$ corresponding to $\sigma(f)$ and $\mathcal{B}_\omega$ corresponding to $(\omega_n)_{n\geq 0}$. As pointed out above the vertices of the graph G_ω associated to $(\omega_n)_{n\geq 0}$ form a subset of the vertices of the graph G_f associated to $\sigma(f)$. In order to distinguish between these two graphs we use the following notation: Put $\sigma_n(f) = (Q_{n1} \dots Q_{n,\bar{k}_n})$, $n \geq 0$, and let $V_f = \{(n,j)^{(f)} \mid n \geq 0, 1 \leq j \leq \bar{k}_n\}$ be the vertices in G_f.

Next it will be outlined that in a canonical way to every branch $B = (n, i_n)_{n \geq n_0}$ in $\mathcal{B}_\omega$ a branch in $\mathcal{B}_f$ is associated. Two cases may occur:

(1) The interval $[u, v] = \bigcap\limits_{n \geq n_0} [u_{n,i_n}, v_{n,i_n}]$ corresponding to B is a singleton. Recall that when constructing f_n we assigned to every P_{ni} the interval $[u_{ni}, v_{ni}]$ on which f_n has constant value P_{ni}. Thus the property $u = v$ is equivalent to the feature that in G_ω for an infinite number of n the vertex (n, i_n) has more than one successor: if there is more than one successor of (n, i_n) then $[u_{n+1,i_{n+1}}, v_{n+1,i_{n+1}}]$ has length less than 1/3 of $[u_{n,i_n}, v_{n,i_n}]$. Let P be the point corresponding to the branch B then in this case $f(u) = P$ and in every neighborhood of u, f has infinitely many different dyadic values. Anyway, turning to the graph G_f we see that there is a unique branch $\bar{B} = (n, j_n)^{(f)}_{n \geq n_0}$ in $\mathcal{B}_f$ such that $Q_{n,j_n} = P$ corresponds to the interval $[s_{n,j_n}, t_{n,j_n}]$ in the sense utilized in the proof of Proposition 2.1 with $u \in [s_{n,j_n}, t_{n,j_n}]$ for all $n \geq n_0$.

(2) The interval $[u, v]$ corresponding to B satisfies $u < v$. This means that there exists an index n_1 such that for all $n \geq n_1$ the interval $[u_{n,i_n}, v_{n,i_n}] = [u, v]$. In this case f_n has constant value P on $[u, v]$ for all $n \geq n_1$ and hence f satisfies this, as well. Again, there exists a unique branch $\bar{B} = (n, j_n)^{(f)}_{n \geq n_0}$ in $\mathcal{B}_f$ such that $Q_{n,j_n} = P$ corresponds to the interval $[s_{n,j_n}, t_{n,j_n}]$ with $[u, v] \subseteq [s_{n,j_n}, t_{n,j_n}]$.

In the following we will identify $B \in \mathcal{B}_\omega$ with the respective $\bar{B} \in \mathcal{B}_f$ from (1) or (2) and thus we may consider $\mathcal{B}_\omega$ as a subset of $\mathcal{B}_f$.

We have already proved in Proposition 3.3 that $\mathcal{B}_f$ is complete. Now we show that $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$, i.e. for all $B_1, B_2 \in \mathcal{B}_f$ with $B_1 < B_2$ there exists $B \in \mathcal{B}_\omega$ such that $B_1 < B < B_2$: First of all, it is sufficient to prove this for $B_1, B_2 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$:

– if $B_1, B_2 \in \mathcal{B}_\omega$ then there exists an according B since $\leq$ is a dense order on $\mathcal{B}_\omega$,
– if $B_1 \in \mathcal{B}_\omega$, $B_2 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$, then, since $\mathcal{B}_f$ is dense, there exists $B_3 \in \mathcal{B}_f$ with $B_1 < B_3 < B_2$; if $B_3 \in \mathcal{B}_\omega$ we are done and if $B_3 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$ then the problem is reduced to $B_3 < B_2$, the case we will deal with.

Let B_i correspond to the interval $[s_i, t_i]$, $f(s_i) = Q_i$, $i = 1, 2$. As $B_1 < B_2$ we have $t_1 < s_2$. Since $\mathcal{B}_f$ is dense there exist $B_3 \in \mathcal{B}_f$ with $B_1 < B_3 < B_2$ and since f cannot be constant on $[t_1, s_2]$ we can choose B_3 such that the point Q_3 corresponding to B_3 satisfies $Q_1 \neq Q_3 \neq Q_2$. Consequently there is $s_3 \in (t_1, s_2)$ with $f(s_3) = Q_3$. We fix some $k \geq 0$ such that the distance $d(Q_3, Q_i)$ is larger than 2^{-k+2}, $i = 1, 2$. Since $(f_m)_{m\geq 0}$ converges uniformly to f we have $\|f - f_m\|_\infty < 2^{-k}$ for all $m \geq m_k$ with appropriate m_k. So for $m \geq m_k$ we have

$$d(Q_1, f_m(t_1)) < 2^{-k}, \quad d(Q_3, f_m(s_3)) < 2^{-k}.$$

Hence $f_m(t)$ must pass from the 2^{-k}-neighborhood of Q_1 for $t = t_1$ to the 2^{-k}-neighborhood of Q_3 for $t = s_3$ and since f_m is alternately constant/linear f_m assumes a dyadic point P (of order $\leq m$) as constant value for some interval in (t_1, s_3). Since $\sigma_m(f_m) = \omega_m$ there is a branch $B \in \mathcal{B}_\omega$ containing this P which by construction satisfies $B_1 < B < B_3 < B_2$.

Finally we show $\sigma(f) \neq (\omega_n)_{n\geq 0}$ (which is equivalent to $\mathcal{B}_f \setminus \mathcal{B}_\omega \neq \emptyset$) implies that $(\omega_n)_{n\geq 0}$ is not complete: Let $\bar{B} = (n, i_n)^{(f)}_{n\geq n_0} \in \mathcal{B}_f \setminus \mathcal{B}_\omega$ such that for all $n \geq n_1$ the vertices $(n, i_n)^{(f)}$ in $\bar{B}$ have smallest possible i_n. For instance this is possible if $(n_1 - 1, i_{n_1 - 1})^{(f)}$ is a vertex in $G_f \setminus G_\omega$. We consider the following cut in $\mathcal{B}_\omega$:

$$\mathcal{B}_1 = \{B \in \mathcal{B}_\omega \mid B < \bar{B}\}, \quad \mathcal{B}_2 = \{B \in \mathcal{B}_\omega \mid B > \bar{B}\}.$$

First we show that $(\mathcal{B}_1, \mathcal{B}_2)$ is irrational: for $B_1 \in \mathcal{B}_1$ we have $B_1 < \bar{B}$ and since $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$ there is $B \in \mathcal{B}_\omega$ such that $B_1 < B < \bar{B}$ showing that $\mathcal{B}_1$ has no largest element. Analogously one learns that $\mathcal{B}_2$ has no least element.

Now we prove that $(\mathcal{B}_1, \mathcal{B}_2)$ converges to the point $\bar{Q}$ corresponding to $\bar{B}$. Let $(\mathcal{B}_1^f, \mathcal{B}_2^f)$ be the cut in $\mathcal{B}_f$ with smallest element $\bar{B}$ in $\mathcal{B}_2^f$ and

$$\begin{aligned} l_n^f &= \max\{j \mid \exists B_1 \in \mathcal{B}_1^f : B_1 \text{ contains } (n, j)^f\}, \\ l_n &= \max\{j \mid \exists B_1 \in \mathcal{B}_1 : B_1 \text{ contains } (n, j)^f\}. \end{aligned}$$

Due to our choice of $\bar{B}$ we have for all $n \geq n_1$ that $l_n^f = i_n - 1$ and $Q_{n, l_n^f} \sim_n \bar{Q}$. Further let $B_n^f \in \mathcal{B}_f$ be the largest branch containing $(n, l_n^f)^{(f)}$ (starting from Q_{n, l_n^f} taking always the rightmost vertex as successor). As a consequence all branches B with $B_n^f < B < \bar{B}$ correspond to a dyadic point in the subtriangle T_n of $\triangle_n$ that contains $\bar{Q}$ and Q_{n, l_n^f}. Since $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$ there exists $B_n \in \mathcal{B}_\omega$ such that $B_n^f < B_n < \bar{B}$. Hence the points P_n corresponding to B_n must lie in the subtriangle T_n and if P_n is of order r_n then also Q_{k, l_k} lies in T_n for all $k \geq r_n$. So we have proved

$$\lim_{n\to\infty} Q_{n, l_n^f} = \lim_{k\to\infty} Q_{k, l_k} = \bar{Q}.$$

Summing up this means that the irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}_\omega$ converges to the dyadic point $\bar{Q}$ and hence $(\omega_n)_{n\geq 0}$ is not complete, in contrast to our assumption. Thus $\mathcal{B}_\omega = \mathcal{B}_f$, i.e., $\sigma(f) = (\omega_n)_{n\geq 0}$, and we are done. $\square$

We now have precise information on the range of σ. In order to get an idea what the sub-semigroup $\sigma(S(\triangle)) \cong S(\triangle)/\ker(\sigma)$ of $\varprojlim S_n$ describes we have to investigate the kernel of σ.

A first observation in this direction is that $\ker(\sigma)$ is a sub-relation of the homotopy relation of elements $f, g \in S(\triangle)$: $\sigma(f) = \sigma(g)$ implies

$$\varphi([f]) = \mathrm{Red}(\sigma(f)) = \mathrm{Red}(\sigma(g)) = \varphi([g]),$$

and since φ is injective we obtain $[f] = [g]$.

It is palpable that $\ker(\sigma)$ will be related with the re-parameterization of loops. Therefore, following Curtis and Fort [9] we say that two loops $f, g \in S(\triangle)$ are Fréchet equivalent, $f \approx g$ for short, if and only if there exist functions $\alpha, \beta : [0,1] \to [0,1]$ which are monotonously increasing and surjective (and hence continuous) such that $f \circ \alpha = g \circ \beta$. In Curtis and Fort [9, Appendix] it is shown that this is an equivalence relation.

Proposition 3.5. *If $f \approx g$ then $\sigma(f) = \sigma(g)$.*

Proof. First we show that $\sigma_n(f) = \sigma_n(f \circ \alpha)$ for all $n \geq 0$ where $f \circ \alpha = g \circ \beta$ with properties as defined above. We recall that $\sigma_n(f)$ is the sequence of points in D_n that arises when we raster the separated set $f^{-1}(D_n)$ with appropriate small intervals and list the corresponding points. For a letter P appearing in $\sigma_n(f)$ let again $[s, t]$ be the maximal interval such that $f(s) = f(t) = P$ and $f([s, t]) \cap D_n = \{P\}$. Since α is surjective P appears also in $\sigma_n(f \circ \alpha)$ and the monotonicity of α preserves the order of points in $\sigma_n(f)$, in particular $[\min \alpha^{-1}(\{s\}), \max \alpha^{-1}(\{t\})]$ is the interval corresponding to letter P with respect to the loop $f \circ \alpha$.

The rest is obvious: $\sigma_n(f) = \sigma_n(f \circ \alpha) = \sigma_n(g \circ \beta) = \sigma_n(g)$. $\square$

The converse of Proposition 3.5 is established in the following.

Proposition 3.6. *If $\sigma(f) = \sigma(g)$ then $f \approx g$.*

Proof. For $n \geq 0$ let $\omega_n = \sigma_n(f) = \sigma_n(g) = P_{n1} P_{n2} \ldots P_{n,k_n}$. As usual we assign to $(\omega_n)_{n \geq 0}$ the graph G with vertices (n, i), $n \geq 0$, $1 \leq i \leq k_n$, and an edge connecting (n, i) to $(n+1, j)$ if the letter $P_{n+1,j}$ in ω_{n+1} is projected to P_{ni} when performing $\gamma_{n+1}(\omega_{n+1}) = \omega_n$.

In the first step we will show that the parametrization $f_n : [0,1] \to \triangle$ of the piecewise linear loop corresponding to $\sigma_n(f)$ from Proposition 2.1, yields a sequence $(f_n(t))_{n \geq 0}$ which converges uniformly to $f(t)$ for $t \in [0,1]$.

Let n be fixed. As in Proposition 2.1 we associate to every (n, i) the maximal interval $[s_{ni}, t_{ni}]$ such that $f(s_{ni}) = f(t_{ni}) = P_{ni}$, $D_n \cap f([s_{ni}, t_{ni}]) = \{P_{ni}\}$ and $0 = s_{n1} \leq t_{n1} < s_{n2} \leq t_{n2} < \ldots < s_{n,k_n} \leq t_{n,k_n} = 1$. We parameterize the piecewise linear loop corresponding to $\sigma_n(f)$ by f_n such that f_n is constant with value P_{ni} in the interval $[s_{ni}, t_{ni}]$, $1 \leq i \leq k_n$, and connects P_{ni} and $P_{n,i+1}$ linearly in the interval $[t_{ni}, s_{n,i+1}]$, $1 \leq i \leq k_n - 1$. For $t \in [s_{ni}, t_{ni}]$ the loop $f(t)$ is contained in one of the (at most) two subtriangles of $\triangle_n$ to which P_{ni} belongs, and for $t \in [t_{ni}, s_{n,i+1}]$ the loop $f(t)$ is contained in the subtriangle T of $\triangle_n$ to which P_{ni} and $P_{n,i+1}$ belong. Thus we infer that the maximum norm $\|f_n - f\|_\infty \leq \mathrm{diam}(T) = 2^{-n}$ and (f_n) converges uniformly to f.

What was done for f can be realized mutatis mutandis with g where the piecewise linear approximations will be denoted by g_n, and $[u_{ni}, v_{ni}]$ is the generic notation for the interval corresponding to the vertex (n, i) with respect to g.

In the following we will need another correlation, namely we associate to the vertex (n, i) also the interval

$$[a_{ni}, b_{ni}] = [(s_{ni} + u_{ni})/2, (t_{ni} + v_{ni})/2].$$

With this concept we now consider $\alpha_n, \beta_n : [0,1] \to [0,1]$ such that

$$\alpha_n(a_{ni}) = s_{ni}, \quad \alpha_n(b_{ni}) = t_{ni},$$
$$\beta_n(a_{ni}) = u_{ni}, \quad \beta_n(b_{ni}) = v_{ni},$$

and α_n, β_n are piecewise linear between these points. Evidently, we then have

$$f_n \circ \alpha_n = g_n \circ \beta_n.$$

We recall what was accomplished in Proposition 3.4: Starting from an arbitrary $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ a sequence f_n of loops was constructed converging uniformly to some $f \in S(\triangle)$. Moreover, it was shown that $\sigma(f) = (\omega_n)_{n\geq 0}$ provided $(\omega_n)_{n\geq 0}$ is complete. Now we perform the same starting with $(\omega_n)_{n\geq 0} = \sigma(f) = \sigma(g)$ which is complete by Proposition 3.3. Instead of using subintervals of equal length as in the proof of Proposition 3.4, we here employ the given family $[a_{ni}, b_{ni}]$, $n \geq 0$, $1 \leq i \leq k_n$ which creates appropriate subdivisions. However, this difference does not influence the validity of the rest of the proof at all. What we obtain is the sequence $h_n = f_n \circ \alpha_n = g_n \circ \beta_n$ converging uniformly to some $h \in S(\triangle)$ with $\sigma(h) = \sigma(f) = \sigma(g)$. Moreover, we will show that the interval $[x_{ni}, y_{ni}]$ associated to the vertex (n,i) with respect to h in the usual way, i.e., $[x_{ni}, y_{ni}]$ is the maximal interval with the properties $h(x_{ni}) = h(y_{ni}) = P_{ni}$, $h([x_{ni}, y_{ni}]) \cap D_n = \{P_{ni}\}$, must coincide with $[a_{ni}, b_{ni}]$. Indeed, since $\alpha_m([a_{ni}, b_{ni}]) = [s_{ni}, t_{ni}]$ and thus $h_m([a_{ni}, b_{ni}]) = f_m([s_{ni}, t_{ni}])$ for all $m \geq n$, and since the sequence (f_m) converges uniformly to f and $f([s_{ni}, t_{ni}]) \cap D_n = \{P_{ni}\}$, we conclude that $h([a_{ni}, b_{ni}]) \cap D_n = \{P_{ni}\}$. This shows $[a_{ni}, b_{ni}] \subseteq [x_{ni}, y_{ni}]$. Now suppose $x_{ni} < a_{ni}$. We have

$$P_{ni} = h(x_{ni}) = \lim_{m\to\infty} f_m(\alpha_m(x_{ni})) \tag{3.1}$$

and $\alpha_m(x_{ni}) \in (t_{n,i-1}, s_{ni})$ for $m \geq n$ since $x_{ni} > b_{n,i-1}$. From $f((t_{n,i-1}, s_{ni})) \cap D_n = \emptyset$ and (3.1) we infer that there exists a subsequence (m_k) with $\lim_{k\to\infty} \alpha_{m_k}(x_{ni}) = s_{ni}$. Next, in any proper interval $[t, s_{ni}]$ the path f assumes dyadic points of arbitrary high order near to P_{ni}. Therefore in the graph G corresponding to $\sigma(f) = \sigma(g)$ there exists a sequence (m_k, i_k) with $s_{m_k i_k} < s_{ni}$ and $\lim_{k\to\infty} s_{m_k i_k} = s_{ni}$, and with the same argument for g we obtain $u_{m_k i_k} < u_{ni}$ and $\lim_{k\to\infty} u_{m_k i_k} = u_{ni}$. This implies $a_{m_k i_k} < a_{ni}$ and $\lim_{k\to\infty} a_{m_k i_k} = a_{ni}$ and hence there exists $\tilde{k}$ such that $x_{ni} < a_{m_{\tilde{k}} i_{\tilde{k}}}$. Now, for all $k \geq \tilde{k}$ we have $\alpha_{m_k}(x_{ni}) \leq \alpha_{m_k}(a_{m_{\tilde{k}} i_{\tilde{k}}}) = s_{m_{\tilde{k}} i_{\tilde{k}}} < s_{ni}$. We conclude that

$$s_{ni} = \lim_{k\to\infty} \alpha_{m_k}(x_{ni}) \leq s_{m_{\tilde{k}} i_{\tilde{k}}} < s_{ni},$$

a contradiction, and hence $x_{ni} = a_{ni}$. Similarly it is shown that $y_{ni} > b_{ni}$ is impossible and hence $[x_{ni}, y_{ni}] = [a_{ni}, b_{ni}]$.

Let again $\mathcal{B}$ denote the set of branches in G. To every branch $B = (n, i_n)_{n\geq n_0}$ we assign the interval $[s_B, t_B] = \bigcap\limits_{n\geq n_0} [s_{n,i_n}, t_{n,i_n}]$ and the intervals $[u_B, v_B]$, $[a_B, b_B]$ accordingly.

In the next step we will elaborate that the sequences $(\alpha_n(x))_{n\geq 0}$ and $(\beta_n(x))_{n\geq 0}$ converge pointwise for a good deal of x. First we consider $x \in [0,1]$ such that there exists $B = (n, i_n)_{n\geq n_0} \in \mathcal{B}$ with $x \in [a_B, b_B] = \bigcap\limits_{n\geq n_0} [a_{n,i_n}, b_{n,i_n}]$. (In the following we will refer to this case by (I).) This implies $x \in [a_{n,i_n}, b_{n,i_n}] = [(s_{n,i_n} + u_{n,i_n})/2, (t_{n,i_n} + v_{n,i_n})/2]$ for all $n \geq n_0$. Recall that

$$\lim_{n\to\infty} s_{n,i_n} = s_B, \ \lim_{n\to\infty} t_{n,i_n} = t_B, \ \lim_{n\to\infty} u_{n,i_n} = u_B, \ \lim_{n\to\infty} v_{n,i_n} = v_B,$$

and that

$$\alpha_n(x) = s_{n,i_n} + \frac{t_{n,i_n} - s_{n,i_n}}{b_{n,i_n} - a_{n,i_n}}(x - a_{n,i_n})$$

if $b_{n,i_n} > a_{n,i_n}$, and $\alpha_n(x) = s_{n,i_n} = t_{n,i_n}$ otherwise. In general we have $\alpha_n(x) \in [s_{n,i_n}, t_{n,i_n}]$. Therefore, if $t_B = s_B$ we infer $\lim_{n\to\infty} \alpha_n(x) = s_B$, and if $t_B > s_B$ we obtain $\lim_{n\to\infty} \alpha_n(x) = s_B + \frac{t_B - s_B}{b_B - a_B}(x - a_B)$. In any case the limit exists and we define $\alpha(x) = \lim_{n\to\infty} \alpha_n(x)$. Analogously we can proceed with $\beta_n(x)$ and define $\beta(x) = \lim_{n\to\infty} \beta_n(x)$.

Now we deal with the case that $x \notin [a_B, b_B]$ for all $B \in \mathcal{B}$ (case (II)). Then x defines a cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$ by putting $\mathcal{B}_1 = \{B \in \mathcal{B} \mid x > b_B\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B} \mid x < a_B\}$. We recapitulate what was shown in the proof of Proposition 3.3: The cut $(\mathcal{B}_1, \mathcal{B}_2)$ is irrational and if we define $a = \sup_{B\in\mathcal{B}_1} a_B = \sup_{B\in\mathcal{B}_1} b_B$ and $b = \inf_{B\in\mathcal{B}_2} a_B = \inf_{B\in\mathcal{B}_2} b_B$ then $x \in [a, b]$ and h is constant in the interval $[a, b]$ with a generic point Q which is the limit of the cut $(\mathcal{B}_1, \mathcal{B}_2)$ as constant value. With s, t and u, v defined accordingly, $a = (s + u)/2$, $b = (t + v)/2$, we further obtain $f([s, t]) = g([u, v]) = \{Q\}$. For $\tilde{x} \in [a, b]$ we define

$$\alpha(\tilde{x}) = \begin{cases} s = t & \text{if } a = b, \\ s + \frac{t-s}{b-a}(\tilde{x} - a) & \text{otherwise,} \end{cases}$$

$$\beta(\tilde{x}) = \begin{cases} u = v & \text{if } a = b, \\ u + \frac{v-u}{b-a}(\tilde{x} - a) & \text{otherwise.} \end{cases}$$

In order to justify this definition some warning is indicated here. One can easily construct an example of a loop f such that $\lim_{n\to\infty} \alpha_n(x)$ does not exist for some x. However, one always has $s \le \liminf \alpha_n(x) \le \limsup \alpha_n(x) \le t$ and since f is constant in $[s, t]$ this causes no problem.

Now we have to show that α and β comply with the intention they were constructed with.

$(f \circ \alpha)(x) = (g \circ \beta)(x)$ for all $x \in [0, 1]$: In case (I) $x \in [a_B, b_B]$ for some branch $B \in \mathcal{B}$ and we have

$$\|f(\alpha(x)) - f_n(\alpha_n(x))\| \le \|f(\alpha(x)) - f(\alpha_n(x))\| + \|f(\alpha_n(x)) - f_n(\alpha_n(x))\|.$$

The first part on the right hand side can be made arbitrarily small since f is continuous and $\alpha_n(x)$ converges to $\alpha(x)$ and the second part does so since f_n converges to f uniformly. The same applies to g and β. So we arrive at

$$f(\alpha(x)) = \lim_{n\to\infty} f_n(\alpha_n(x)) = \lim_{n\to\infty} g_n(\beta_n(x)) = g(\beta(x)).$$

In case (II) $x \notin [a_B, b_B]$ for any branch B we have with notations as before $\alpha(x) \in [s, t]$ and $\beta(x) \in [u, v]$ and hence $f(\alpha(x)) = Q = g(\beta(x))$. Just as a further remark we mention here that $h = f \circ \alpha$.

α and β are monotonously increasing functions: Let $x_1 < x_2$. Depending on whether case (I) or (II) apply to x_1 and x_2 four cases occur. We only work out one of the mixed cases in detail, the others can be treated similarly. So let $x_1 \in [a_B, b_B]$ for some branch B and let $x_2 \in [a, b]$ where $[a, b]$ is the interval corresponding to an irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ with respect to h. The relation $x_1 < x_2$ just means that $B \in \mathcal{B}_1$ and so we deduce

$$\alpha(x_1) \le t_B < \sup_{B_1\in\mathcal{B}_1} t_{B_1} = s = \alpha(a) \le \alpha(x_2).$$

The proof for the monotonicity of β works analogously.

α and β are surjective and thus continuous: From case (I) we see that

$$\operatorname{ran}(\alpha) \supseteq \bigcup_{B\in\mathcal{B}} [s_B, t_B] = f^{-1}\Big(\bigcup_{n\ge 0} D_n\Big) = D_f,$$

and for all components $[s,t]$ of the complement of D_f which correspond to an irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$, in (II) we tailored α such that the interval $[a,b]$ corresponding to $(\mathcal{B}_1, \mathcal{B}_2)$ with respect to h satisfies $\alpha([a,b]) = [s,t]$. Hence α is surjective, and with the respective proof for g, β is surjective, as well. $\square$

We summarize the last results in a separate statement.

Theorem 3.7. (i) *For f and g in $S(\triangle)$ we have $\sigma(f) = \sigma(g)$ if and only if f and g have a common re-parametrization, i.e. there exist $\alpha, \beta : [0,1] \to [0,1]$ monotonously increasing and surjective such that $f \circ \alpha = g \circ \beta$.*

(ii) *An element $(\omega_n)_{n\geq 0}$ in $\varprojlim S_n$ is a representation for a loop f in $S(\triangle)$, i.e. $(\omega_n)_{n\geq 0} = \sigma(f)$, if and only if $(\omega_n)_{n\geq 0}$ is complete.*

In other words, the complete elements of $\varprojlim S_n$ represent the elements of $S(\triangle)$ modulo re-parametrization.

3.2. **A description of the elements in the fundamental group $\pi(\triangle)$.** We have proved in Theorem 2.10 that $\varphi([f]) = \mathrm{Red}(\sigma(f))$ for all continuous loops f in $\triangle$. Since φ is an injection the fundamental group $\pi(\triangle)$ can be considered as a subgroup of $\varprojlim G_n$ and in this subsection we will characterize the elements of this subgroup.

In the following denote by γ_{nk} the projection $\gamma_{k+1} \circ \gamma_{k+2} \circ \ldots \circ \gamma_n : S_n \to S_k$, and analogously δ_{nk} denotes the composition of the corresponding δ_i's.

Before we prove the main result we need some preliminaries. Let $P_1 P_2 \ldots P_m$, $Q_1 Q_2 \ldots Q_k$ be two words over some alphabet. We define $P_1 P_2 \ldots P_m \preceq Q_1 Q_2 \ldots Q_k$ if and only if there exists $\alpha : \{1, \ldots, m\} \to \{1, \ldots, k\}$, α injective and order preserving, such that $P_i = Q_{\alpha(i)}$ for all $i \in \{1, \ldots, m\}$. This means that the first word is a subsequence of the second which differs from the notion subword we have used before (cf. elementary moves (2.4)).

Lemma 3.8. *Let $\omega_n, \bar{\omega}_n \in S_n$. Then*

(i) $\mathrm{Red}_n(\omega_n) \preceq \omega_n$,

(ii) *$\omega_n \preceq \bar{\omega}_n$ implies $\gamma_{nk}(\omega_n) \preceq \gamma_{nk}(\bar{\omega}_n)$ for all $k \leq n$,*

(iii) *if $(\omega_k)_{k\geq 0} \in \varprojlim G_n$ then $\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1})$ for all $k \leq n$.*

Proof. (i) is evident since Red_n eliminates just some letters from the word.

(ii) It is enough to prove that $\gamma_n(\omega_n) \preceq \gamma_n(\bar{\omega}_n)$. The bonding map γ_n first eliminates all points of order n from ω_n and $\bar{\omega}_n$, resulting in words ω_n' and $\bar{\omega}_n'$, respectively, and then cancels in each of these words all arising consecutive repetitions of letters of order $< n$, before arriving at $\gamma_n(\omega_n)$ and $\gamma_n(\bar{\omega}_n)$. Clearly, $\omega_n' \preceq \bar{\omega}_n'$, as testified by some order preserving injection α. Choose α in a way that $\alpha(i)$ is minimal for each i. Then α, restricted to the indices of the remaining letters, testifies $\gamma_n(\omega_n) \preceq \gamma_n(\bar{\omega}_n)$.

(iii) We have $\gamma_{nk}(\omega_n) = \gamma_{nk}(\delta_{n+1}(\omega_{n+1})) = \gamma_{nk}(\mathrm{Red}_n(\gamma_{n+1}(\omega_{n+1}))) \preceq \gamma_{nk}(\gamma_{n+1}(\omega_{n+1})) = \gamma_{n+1,k}(\omega_{n+1})$, where we used (i) and (ii) as $\preceq$ came in. $\square$

We are now in the position to give a proof of our main result.

Proof of Theorem 1.1. We fix the element $(\omega_n)_{n\geq 0}$ in $\varprojlim G_n$. First we prove the necessity of the condition. Let $(\bar{\omega}_n)_{n\geq 0}$ be in $\varprojlim S_n$ such that $\mathrm{Red}((\bar{\omega}_n)_{n\geq 0}) = (\omega_n)_{n\geq 0}$. Then for all $k \geq 0$ and all $n \geq k$ we have $\bar{\omega}_k = \gamma_{nk}(\bar{\omega}_n) \succeq \gamma_{nk}(\mathrm{Red}_n(\bar{\omega}_n)) = \gamma_{nk}(\omega_n)$ where we used (i) and (ii) of Lemma 3.8. By (iii) of Lemma 3.8 we get

$$\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1}) \preceq \ldots \preceq \bar{\omega}_k,$$

hence $(\gamma_{nk}(\omega_n))_{n\geq k}$ is eventually constant.

Now we prove the sufficiency of the condition. Put $\bar{\omega}_k = \gamma_{nk}(\omega_n)$ which is well defined for $n \geq n_k$, $k \geq 0$. We show that

(i) $(\bar{\omega}_k)_{k\geq 0} \in \varprojlim S_n$ and

(ii) $\mathrm{Red}(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$.

For $k \geq 1$ and $n \geq \max\{n_k, n_{k-1}\}$ we obtain $\gamma_k(\bar{\omega}_k) = \gamma_k(\gamma_{nk}(\omega_n)) = \gamma_{n,k-1}(\omega_n) = \bar{\omega}_{k-1}$. This shows (i).

Before we come to (ii) we prove $\delta_{nk} = \mathrm{Red}_k \circ \gamma_{nk}$: In Lemma 2.6 we showed $\mathrm{Red}_{i-1} \circ \gamma_i \circ \mathrm{Red}_i = \mathrm{Red}_{i-1} \circ \gamma_i$ for all $i \geq 1$. Obeying $\delta_i = \mathrm{Red}_{i-1} \circ \gamma_i$, iterated application of this identity leads immediately to the claimed relation.

Finally, for $k \geq 0$ and $n \geq n_k$ we infer $\mathrm{Red}_k(\bar{\omega}_k) = \mathrm{Red}_k(\gamma_{nk}(\omega_n)) = \delta_{nk}(\omega_n) = \omega_k$, which proves (ii). Due to Proposition 3.4 we can find $f \in S(\triangle)$ such that $\mathrm{Red}(\sigma(f)) = \mathrm{Red}(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$ and thus

$$(\omega_n)_{n\geq 0} = \mathrm{Red}(\sigma(f)) = \varphi([f]).$$

This completes the proof. $\square$

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