ON THE FUNDAMENTAL GROUP OF SELF-AFFINE PLANE TILES

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Abstract. Let $A \in \mathbb{Z}^{2\times 2}$ be an expanding matrix, $\mathcal{D} \subset \mathbb{Z}^2$ a set with $|\det(A)|$ elements and define $\mathcal{T}$ via the set equation $A\mathcal{T} = \mathcal{T} + \mathcal{D}$. If the two-dimensional Lebesgue measure of $\mathcal{T}$ is positive we call $\mathcal{T}$ a self-affine plane tile. In the present paper we are concerned with topological properties of $\mathcal{T}$. We show that the fundamental group $\pi_1(\mathcal{T})$ of $\mathcal{T}$ is either trivial or uncountable and provide criteria for the triviality as well as the uncountability of $\pi_1(\mathcal{T})$. Furthermore, we give a short proof of the fact that the closure of each component of $\mathrm{int}(\mathcal{T})$ is a locally connected continuum (we prove this result even in the more general case of plane IFS attractors fulfilling the open set condition). If $\pi_1(\mathcal{T}) = 0$ we even show that the closure of each component of $\mathrm{int}(\mathcal{T})$ is homeomorphic to a closed disk.

We apply our results to several examples of tiles which are studied in the literature.

1. Introduction

This paper is devoted to the study of topological properties of *integral self-affine tiles with standard digit set* in the plane. Before we discuss our aims in more detail we recall some definitions.

It is well known that for a family of contractions $f_1, \dots, f_k$ on $\mathbb{R}^d$, there is a unique non-empty compact set $T = T(f_1, \dots, f_k)$ with $T = \bigcup_i f_i(T)$ (cf. [12]). Here T is called the attractor of the *iterated function system* (IFS for short) $f_1, \dots, f_k$. If there exists a non-empty bounded open set $V \subset \mathbb{R}^d$ such that $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$ and $\bigcup_i f_i(V) \subset V$, then we say that $f_1, \dots, f_k$ (or T) satisfy the *open set condition* (cf. [8, p. 118]).

Special instances of IFS are so-called self-affine tiles. Let A be a real $d \times d$ matrix with all eigenvalues greater than 1. Suppose that $|\det(A)| > 1$ is an integer and let $\mathcal{D} \subset \mathbb{R}^d$ with $|\mathcal{D}| = |\det(A)|$. Then there exists a unique non-empty compact set $\mathcal{T} := \mathcal{T}(A, \mathcal{D})$ such that

$$A\mathcal{T} = \bigcup_{v \in \mathcal{D}} (\mathcal{T} + v).$$

If $\mathcal{T}$ has positive d-dimensional Lebesgue measure we call it a *self-affine tile.* In the present paper we are concerned with so-called *integral self-affine tiles with standard digit set.* These are tiles $\mathcal{T}(A, \mathcal{D})$ with integer matrix A whose digit set $\mathcal{D} \subset \mathbb{Z}^d$ is a complete set of coset representatives of $\mathbb{Z}^d/A\mathbb{Z}^d$. Moreover, we shall assume that $\mathcal{T}(A, \mathcal{D})$ tiles $\mathbb{R}^d$ by the lattice $\mathbb{Z}^d$, *i.e.*,

$$\mathcal{T} + \mathbb{Z}^d = \mathbb{R}^d$$

where

$$\mathrm{int}((\mathcal{T} + \gamma_1)) \cap (\mathcal{T} + \gamma_2) = \emptyset \quad \text{for} \quad \gamma_1 \neq \gamma_2 \quad (\gamma_1, \gamma_2 \in \mathbb{Z}^d). \tag{1.1}$$

Following Bandt and Wang [4] we shall call such a tile a $\mathbb{Z}^d$*-tile* for short. There are standard methods for checking whether $\mathcal{T}$ forms a tiling or not. We refer for instance to Vince [33, Theorem 4.2] where a list of tiling criteria is given.

The topological structure of attractors of IFS has been studied extensively in the literature. Let T be an IFS attractor. Hata [10] observed among many other results that if T is connected then it is a locally connected continuum. More recently, many results for plane IFS attractors with open set condition have been shown. For example Luo et al. [20] prove that connectivity of $\mathrm{int}(T)$ implies that T is homeomorphic to a closed disk, Ngai and Tang [24, 25] study the structure of the components of $\mathrm{int}(T)$ and Luo et al. [19] prove that connectivity of T implies connectivity of ∂T.

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For the case of $\mathbb{Z}^2$-tiles Bandt and Wang [4] provide an algorithmic criterion for the homeomorphy of a given $\mathbb{Z}^2$-tile to a disk. The structure of the set of "neighbors" of a $\mathbb{Z}^2$-tile was investigated by Scheicher and Thuswaldner [29] as well as Strichartz and Wang [31]. Recently Ngai and Tang [26] study "vertices" of $\mathbb{Z}^2$-tiles. For a survey on topological results of $\mathbb{Z}^2$-tiles with a long list of references we refer to Akiyama and Thuswaldner [2].

The aim of the present paper is to continue the study of the topological structure of $\mathbb{Z}^2$-tiles. In particular, we will prove the following results.

- In Section 4 we prove that the fundamental group of a $\mathbb{Z}^2$-tile $\mathcal{T}$ is either trivial or uncountable (Theorem 4.4). If it is trivial, then $\mathcal{T}$ is either homeomorphic to a closed disk or it contains a cut point.
- Section 5 is devoted to the study of the components U_i $(i \in \mathbb{N})$ of the interior of a $\mathbb{Z}^2$-tile $\mathcal{T}$. We give a short proof of the fact that $\overline{U_i}$ is a locally connected continuum. In fact, we are able to prove this result even for arbitrary plane IFS satisfying the open set condition (Proposition 5.2; this result has been shown already in [24, Theorem 1.2] with a much longer proof). Furthermore, we prove that $\pi_1(\mathcal{T}) = 0$ implies that all the sets $\overline{U_i}$ are homeomorphic to a closed disk (Theorem 5.3, for plane IFS with open set condition see Theorem 5.4).
- Sections 6 and 7 are devoted to criteria for checking whether a given $\mathbb{Z}^2$-tile $\mathcal{T}$ has trivial (Theorem 7.2) or uncountable fundamental group (Theorems 6.3 and 6.4). All these criteria can be checked by inspecting a certain graph related to the neighbors of $\mathcal{T}$.
- In the last section we apply our results to some examples.

Our results suggest that there are three levels of topological difficulty for $\mathbb{Z}^2$-tiles.

- $\mathcal{T}$ is homeomorphic to a closed disk.
- $\mathcal{T}$ has trivial fundamental group but is not homeomorphic to a closed disk. In this case, $\mathcal{T}$ has at least one cut point and the closure of each of its interior components is homeomorphic to a closed disk.
- $\mathcal{T}$ has uncountable fundamental group. This is the most involved case. Here in general we only know that the closures of the components of $\mathrm{int}(\mathcal{T})$ are locally connected continua.

The paper is organized as follows. Sections 2 and 3 contain preliminary results. In Sections 4, 5, 6 and 7 we state and prove our main results and Section 8 contains the detailed discussion of several examples.

2. Neighbor graph and adding machine

To each $\mathbb{Z}^2$-tile $\mathcal{T}(A, \mathcal{D})$ we may associate a kind of number system that permits to represent elements of $\mathbb{Z}^2$ in terms of powers of A with "digits" from the set $\mathcal{D}$. This enables us to view each element of $\mathbb{Z}^2$ as a *digit string* (*i.e.*, an element of $\mathcal{D}^{\mathbb{N}}$). If we add a fixed element $s \in \mathbb{Z}^2$ to an arbitrary element $z \in \mathbb{Z}^2$, it is natural to expect that the digit string corresponding to $z + s$ is somehow related to the digit strings of z and s. Indeed, this relation can be made precise with help of an "adding machine" represented by a certain graph $G^T(R)$. Since it turns out that properties of this adding machine play an important role in our results we want to recall its construction in full detail (cf. for instance [28, 29]).

We start with attaching a digit string to each element of $\mathbb{Z}^2$. To this matter define the mapping

$$\Phi(z) := A^{-1}(z - \delta),$$

where δ is the unique element of $\mathcal{D}$ with $\delta \equiv z (\mathrm{mod}\, A)$. If $\Phi^l(z)$ denotes the l-th iterate of Φ then for each $z \in \mathbb{Z}^2$ we call $(\Phi^j(z))_{j \geq 0}$ the orbit of z generated by Φ. Let $\|\cdot\|$ denote any norm in $\mathbb{R}^2$. Since A^{-1} is contractive it is easy to see (cf. [13, 14]) that there exists a constant C with the following property. For each $z \in \mathbb{Z}^2$ there exists an integer j_0 such that for each $j \geq j_0$ we have $\|\Phi^j(z)\| \leq C$. Since there are only finitely many elements of $\mathbb{Z}^2$ with bounded norm we conclude that the sequence

$$\Phi^0(z), \Phi^1(z), \Phi^2(z), \ldots \tag{2.1}$$

is ultimately periodic for each $z \in \mathbb{Z}^2$.

Let us call an element $p \in \mathbb{Z}^2$ *periodic*, if for some positive integer ω we have $\Phi^\omega(p) = p$, and denote the set of all periodic elements by $\mathcal{P}$. Thus (2.1) ends in a cycle of periodic points for each z and $\mathcal{P}$ is the *attractor* of the dynamical system $(\mathbb{Z}^2, \Phi)$ (this dynamical system has been studied for instance in [13, 14, 32]). Since the sequence $(\|\Phi^j(z)\|)_{j\geq 0}$ is ultimately less than or equal to C we conclude that $\mathcal{P}$ has only finitely many elements.

It is immediate from the definition of Φ and $\mathcal{P}$ that each $z \in \mathbb{Z}^2$ admits a unique representation of the shape

$$z = \sum_{\ell=0}^{L} A^\ell a_\ell + A^{L+1} p \tag{2.2}$$

with $p \in \mathcal{P}$, $a_\ell \in \mathcal{D}$ and L as small as possible. We call this representation the *A-adic* representation of z. Since p is a periodic point, there exists a positive integer ω, such that $p = \sum_{k=0}^{N\omega-1} A^k b_{k \bmod \omega} + A^{N\omega} p$, with $b_0, \dots, b_{\omega-1} \in \mathcal{D}$ and $N \in \mathbb{N}$ arbitrary. So we can rewrite (2.2) in the form

$$z = \sum_{\ell=0}^{L} A^\ell a_\ell + A^{L+1} \sum_{k=0}^{N\omega-1} A^k b_{k \bmod \omega} + A^{L+1+N\omega} p \tag{2.3}$$

for any $N \in \mathbb{N}$. In what follows we will denote the infinite repetition of a string $b_{\omega-1} \dots b_0$ by $(b_{\omega-1} \dots b_0)^\infty$ and identify the A-adic representation (2.3) of z with the infinite digit string

$$z = ((b_{\omega-1} \dots b_0)^\infty a_L \dots a_0)_A. \tag{2.4}$$

If $p = 0$ we write $z = (a_L \dots a_0)_A$ instead of $z = (0^\infty a_L \dots a_0)_A$. In this case we call the A-adic representation a finite representation of length $L+1$. This will be written as $\mathcal{L}(z) = L+1$. If the A-adic representation of z is not finite, we say that it has infinite length and write $\mathcal{L}(z) = \infty$. We will use the notation $\mathcal{L}$ for elements of $\mathbb{Z}^2$ as well as for strings of A-adic representations.

In a next step we want to give the description of the adding machine for A-adic representations. We define the directed labelled graph $G(\mathbb{Z}^2)$ as follows.

- Each $s \in \mathbb{Z}^2$ is a vertex of $G(\mathbb{Z}^2)$.
- Let s, s' be vertices of $G(\mathbb{Z}^2)$. There exists an edge $s \xrightarrow{d|d'} s'$ from s to s' labelled by $d|d'$ if and only if $As + d' = s' + d$ holds for a pair $(d, d') \in \mathcal{D} \times \mathcal{D}$.

For $R \subseteq \mathbb{Z}^2$ we denote by $G(R)$ the restriction of $G(\mathbb{Z}^2)$ to the set of vertices R. In a label $s \xrightarrow{d|d'} s'$ the number d' is uniquely determined by s, s' and d. Thus we sometimes omit d' and say that d is the label of this edge.

One important subgraph of $G(\mathbb{Z}^2)$ is the *neighbor graph* $G(S)$, where S is the set of "neighbors" of a $\mathbb{Z}^2$-tile $\mathcal{T}$, namely

$$S := \{s \in \mathbb{Z}^2 \,|\, \mathcal{T} \cap (\mathcal{T} + s) \neq \emptyset\}. \tag{2.5}$$

The graph $G(S)$ can be used in order to obtain a representation of $\partial\mathcal{T}$ as a so-called graph directed self-affine set (cf. for instance [29, 31]).

For $R \subseteq \mathbb{Z}^2$ denote by $G^T(R)$ the transposed graph of $G(R)$, which is obtained by changing the direction of every edge of $G(R)$, *i.e.*, there is an edge $s \xrightarrow{d|d'} s'$ from s to s' labelled by $d|d'$ if and only if $s + d = As' + d'$ holds for a pair $(d, d') \in \mathcal{D} \times \mathcal{D}$.

The graph $G^T(R)$ is *right resolving* in the sense that a pair $(s, d) \in \mathbb{Z}^2 \times \mathcal{D}$ determines at most one edge $s \xrightarrow{d|d'} s'$ in $G^T(R)$. Indeed, suppose that there exist edges $s \xrightarrow{d|d'} s'$ and $s \xrightarrow{d|d''} s''$ with $(s', d') \neq (s'', d'')$. Then

$$s + d = As' + d' = As'' + d''.$$

Reducing the second equation modulo $A\mathbb{Z}^2$ we see that $d' \equiv d'' (\text{mod } A\mathbb{Z}^2)$. Since $d', d'' \in \mathcal{D}$ and $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$ this implies that $d' = d''$ and thus $s' = s''$, a contradiction. By the same reasoning we see that the only edges of $G^T(R)$ starting at 0 are of the shape

$$0 \xrightarrow{d|d} 0, \tag{2.6}$$

i.e., there is no edge leading away from 0.

If $G^T(R)$ accepts the full shift, *i.e.*, if for each pair $(s,d)\in\mathbb{Z}^2\times\mathcal{D}$ there exists an edge $s\xrightarrow{d|d'}s'$, we say that R has property (C) and call $G^T(R)$ an *adding machine*. It has been shown in [29] that the set S in (2.5) has property (C).

We can regard the graph $G^T(R)$ as a *transducer automaton* (cf. [7] for a definition). In each label $d|d'$ we call d the input digit and d' the output digit. $G^T(R)$ can read each string of input digits contained in $\mathcal{D}^{\mathbb{N}}$ and produces a unique string of output digits.

Suppose that $G^T(R)$ is an adding machine and let an A-adic representation (2.4) of z be given. We feed $G^T(R)$ with (2.4) from right to left as input digits starting at the state $s\in R$. Then the output string is the unique A-adic representation of the vector $z+s$. This follows easily from the definition of Φ and $G^T(R)$ (see also Scheicher and Thuswaldner [28, Section 2] where this is discussed in more detail).

We need the following simple results.

Lemma 2.1. *Let $\mathcal{T}(A,\mathcal{D})$ be a $\mathbb{Z}^2$-tile. If $a\in\mathbb{Z}^2$ satisfies $\mathcal{L}(a)\le L$ then*

$$A^{-L}(\mathcal{T}+a)\subset\mathcal{T}.$$

If $a\in\mathbb{Z}^2$ satisfies $\mathcal{L}(a)>L$ we have

$$\mathrm{int}(A^{-L}(\mathcal{T}+a))\cap\mathcal{T}=\emptyset.$$

Proof. The first assertion is clear since $\mathcal{T}=\left\{\sum_{j\ge1}A^{-j}d_j \mid d_j\in\mathcal{D}\right\}$. Suppose that $a=(\dots b_1b_0)_A$ is the A-adic representation of a. Then $A^{-L}(\mathcal{T}+a)\subset\mathcal{T}+z$ for $z=(\dots b_{L+1}b_L)_A$. Since $\mathcal{L}(a)>L$ we have $z\ne0$. Together with (1.1) this implies the second assertion. □

Lemma 2.2. *Let $\mathcal{T}(A,\mathcal{D})$ be a $\mathbb{Z}^2$-tile and let $R\subset\mathbb{Z}^2$ with property (C) be given. Let $r_1,r_2\in R$ and let $a:=(\dots b_2b_1)_A$ be the A-adic representation of $a\in\mathbb{Z}^2$. Use $(\dots b_2b_1)$ as input string for $G^T(R)$ starting at r_1 and r_2 in order to produce the output strings o_1 and o_2, respectively. Then*

$$\mathcal{L}(o_1)<\mathcal{L}(o_2)$$

implies that

$$\mathcal{L}(r_1+a)<\mathcal{L}(r_2+a)$$

holds. Furthermore, setting $L:=\mathcal{L}(o_1)$, we get

$$\begin{aligned}A^{-L}(\mathcal{T}+r_1+a)&\subset\mathcal{T},\\ \mathrm{int}(A^{-L}(\mathcal{T}+r_2+a))\cap\mathcal{T}&=\emptyset.\end{aligned}$$

Proof. The first assertion is true because if we use $G^T(R)$ with the string $(\dots b_2b_1)$ starting at r_i then the output string o_i is the A-adic representation of r_i+a $(1\le i\le2)$. The second assertion follows from Lemma 2.1 □

Let $R_1,R_2\subset R$ where R has property (C). We say that a string of input digits $w=(a_{L-1}\dots a_0)$ leads from a set R_1 to a set R_2 in $G^T(R)$ if R_2 is minimal with the property that each walk in $G^T(R)$ with labelling w starting at a state $r\in R_1$ ends in a state $r'\in R_2$.

Lemma 2.3. *Let $\mathcal{T}(A,\mathcal{D})$ be a $\mathbb{Z}^2$-tile and assume that $R\subset\mathbb{Z}^2$ has property (C). For each $R_0\subset R$ there is a string of input digits w leading from R_0 to $\{0\}$.*

Proof. By [9, Proposition 3.2] each $s\in\mathbb{Z}^2$ and *a fortiori* each $s\in R$ has a representation of the shape

$$s=\sum_{i=0}^{L-1}A^i(a_i'-a_i)\qquad\text{with } a_i,a_i'\in\mathcal{D}. \tag{2.7}$$

There is an edge $r\xrightarrow{a|a'}r'$ in $G^T(R)$ if and only if $Ar'+a'=r+a$. Iterating this L times we conclude that there exists a walk

$$s\xrightarrow{a_0|a_0'}s_1\xrightarrow{a_1|a_1'}s_2\cdots s_{L-1}\xrightarrow{a_{L-1}|a_{L-1}'}0 \tag{2.8}$$

if and only if s has a representation of the shape (2.7).

Suppose that R_0 contains n non-zero elements. Select $s \in R_0$. Then s has a representation of the shape (2.7). Thus by the above considerations there exists a walk (2.8) leading from s to 0. Because by (2.6) no walk leads away from 0 this implies that w leads from R_0 to a set R_1 containing at most $n-1$ non-zero elements. Repeating this argument proves the lemma. □

Lemma 2.4. *Let $\mathcal{T}(A,\mathcal{D})$ be a $\mathbb{Z}^2$-tile and assume that $R \subset \mathbb{Z}^2$ has property (C). Then for each non-empty subset R_0 of R with $\#R_0 \geq 2$ there is a string of input digits w leading from R_0 to a set*

$$R_1 := \{b_1, \dots, b_m\}$$

with $b_i := d_i - d$ $(1 \leq i \leq m)$ for $d, d_1, \dots d_m \in \mathcal{D}$ containing at least one non-zero element.

Proof. Using Lemma 2.3 we know that there exists a string of input digits w leading from R_0 to $\{0\}$. Since the empty walk is a prefix of w there exists a maximal prefix v of w leading from R_0 to a set

$$R_1 = \{b_1, \dots, b_m\}$$

which contains at least one non-zero element. It remains to show that the b_i have the desired properties. By the maximality property of v there are $d, d_1, \dots, d_m \in \mathcal{D}$ such that the edges

$$b_i \xrightarrow{d|d_i} 0 \qquad (1 \leq i \leq m)$$

exist. By the definition of the edges of $G^T(R)$ this implies that the elements b_i $(1 \leq i \leq m)$ have the desired representations. □

We will need so-called *L-vertices* of a $\mathbb{Z}^2$-tile $\mathcal{T}$. Let S be the set of neighbors of $\mathcal{T}$ defined in (2.5). Then an L-vertex is an element of $\mathbb{R}^2$ where $\mathcal{T}$ coincides with L translates of the shape $\mathcal{T} + s$ $(s \in S \setminus \{0\})$. More precisely, for distinct $s_1, \dots, s_L \in S \setminus \{0\}$ we set

$$V_L(s_1, \dots, s_L) := \left\{ x \in \mathbb{R}^2 \;\middle|\; x \in \mathcal{T} \cap \bigcap_{j=1}^{L} (\mathcal{T} + s_j) \right\}.$$

The set of L-vertices of $\mathcal{T}$ is then defined by

$$V_L = \bigcup_{\{s_1,\dots,s_L\} \subset S \setminus \{0\}} V_L(s_1, \dots, s_L)$$

where the union is extended over all subsets of $S \setminus \{0\}$ containing L elements. A 2-vertex is sometimes simply called vertex. Furthermore, we define the set of *vertices between digital translates of $\mathcal{T}$* by

$V_{\mathcal{D}'} := \{x \in \mathbb{R}^2 \mid \text{there exist pairwise disjoint } d, d_1, d_2 \in \mathcal{D}' \text{ with } x \in (\mathcal{T}+d) \cap (\mathcal{T}+d_1) \cap (T+d_2)\}$

for each $\mathcal{D}' \subset \mathcal{D}$.

The next proposition gives a way how to characterize V_L with help of the graph $G(S \setminus \{0\})$.

Proposition 2.5. *Let $L \geq 1$ and let $s_{01}, \dots, s_{0L} \in S \setminus \{0\}$ be pairwise different. Then the following two assertions are equivalent.*

(i)
$$x = \sum_{j \geq 1} A^{-j} d_j \in V_L(s_{01}, \dots, s_{0L}).$$

(ii) *There exist the L infinite walks*

$$s_{0i} \xrightarrow{d_1|d_1'} s_{1i} \xrightarrow{d_2|d_2'} s_{2i} \xrightarrow{d_3|d_3'} \cdots \qquad (1 \leq i \leq L)$$

in $G(S \setminus \{0\})$.

Proof. For the proof we refer to Strichartz and Wang [31, Appendix] or to Akiyama and Thuswaldner [1, Section 8]. □

In an obvious way this result can also be used to characterize $V_{\mathcal{D}'}$. Indeed, it is easy to see that $V_{\mathcal{D}'}$ is a finite union of translations of sets of the shape $V_2(s_1, s_2)$.

3. Preliminaries from plane topology

In the following sections we will frequently use definitions and results from plane topology (see for instance Kuratowski [15, 16] and Hatcher [11]). In order to make the paper more readable we want to list some of them in this section.

We start with the definition of *locally connected continuum.* First, recall that a topological space is *locally connected* at a point x if every open neighborhood of x contains a connected neighborhood of x. A topological space that is locally connected at each point is called a *locally connected space.*

Definition 1. A Hausdorff-space X is called a *continuum* if it is compact and connected. If it is also locally connected it is termed *locally connected continuum.*

According to the theorem of Hahn-Mazurkiewicz-Sierpiński (cf. [16, §50, II, Theorem 2]) a continuum is locally connected if and only if it is the continuous image of an interval.

A space whose every pair of points can be joined by a continuum is said to be a semi-continuum.

Definition 2. A set is said to be a *cut* of a space if its complement is not a semi-continuum. If a cut consists of a single point it is called a *cut point.*

According to [16, §50, II, Theorem 4] a cut point x of a locally connected metric continuum X is a separation of X, *i.e.*, $X \setminus \{x\}$ is disconnected.

Proposition 3.1 (Thorhorst; see [16, §61, II, Theorem 4]). *Let $C \subset \mathbb{S}^2$ be a locally connected continuum and let R be a component of $\mathbb{S}^2 \setminus C$. Then ∂R is a locally connected continuum.*

If C contains no cut point, then $\overline{R}$ is homeomorphic to a closed disk.

Proposition 3.2 (Schönflies; see [16, §61, II, Theorem 10]). *Let $C \subset \mathbb{S}^2$ be a locally connected continuum and suppose that $\mathbb{S}^2 \setminus C$ has infinitely many components $\{R_i\}_{i\in\mathbb{N}}$. Then* $\mathrm{diam}(R_i) \to 0$ *for $i \to \infty$.*

Proposition 3.3 (see [16, §61, IV, Theorem 11]). *Let $C \subset \mathbb{S}^2$ be a locally connected continuum whose complement is connected. Then C is an absolute retract. If, moreover, $\#C \geq 2$ and if C contains no cut point then C is homeomorphic to a closed disk.*

Proposition 3.4 (see [16, §62, VI, Theorem 1]). *Let $C \subset \mathbb{S}^2 \setminus \{p, q\}$ be a locally connected set which cuts between p and q. Then C contains a simple closed curve which cuts between p and q.*

Proposition 3.5 (see [16, §62, IV, Theorem 7]). *Let $C_0, C_1, C_2 \subset \mathbb{S}^2$ be three connected sets and let $x_0, x_1 \in \mathbb{S}^2 \setminus (C_0 \cup C_1 \cup C_2)$.*

If none of the sets $C_k \cup C_{k+1}$ cuts $\mathbb{S}^2$ between x_0 and x_1 and if $C_0 \cap C_1 \cap C_2 \neq \emptyset$ then also $C_0 \cup C_1 \cup C_2$ does not cut between x_0 and x_1 (here $k = 0, 1, 2$ and the indices are reduced modulo 3).

Recall that $M \subset \mathbb{S}^2$ is called *locally simply connected* if for every $x \in M$ there exists an arcwise connected and simply connected neighborhood V of x.

Proposition 3.6 (cf. Conner and Lamoreaux [6, Theorem 3.1]). *Let K be a connected, locally arcwise connected subset of the Euclidean plane. Then the following assertions are equivalent.*

- (i) $\pi_1(K)$ *is not free.*
- (ii) $\pi_1(K)$ *is uncountable.*
- (iii) K *is not locally simply connected.*
- (iv) K *has no universal cover.*

Proposition 3.7 (cf. Pommerenke [27, p. 279]). *Let g be a univalent function defined on the open unit disk D and let $G = g(D)$. The following assertions are equivalent:*

- (i) ∂G *is locally connected.*
- (ii) *$g(z)$ has a continuous extension to $\overline{D}$.*

We end this section with some easy remarks on arcs. An arc in $\mathbb{S}^2$ is an injective continuous mapping $h : [0, 1] \to \mathbb{S}^2$. Sometimes, we also say that the image $h([0, 1])$ is an arc. Let $B \subset \mathbb{S}^2$ be a closed set intersecting an arc h, then $h([0, 1]) \cap B$ is also closed. Thus

$$h^{-1}(B) \subset [0, 1]$$

is compact and therefore has a minimal element t. We call the point $h(t)$ the *first intersection of the arc h with B*. The last intersection of h with B is defined analogously.

Lemma 3.8. *Let $h_1, h_2 : [0,1] \to \mathbb{S}^2$ be two arcs in $\mathbb{S}^2$ intersecting each other. Let $h_1(t_1)$ be the first intersection of h_1 with h_2. Then there is a unique t_2 such that $h_1(t_1) = h_2(t_2)$. Furthermore,*

$$h_1([0,t_1]) \cup h_2([t_2,1])$$

is an arc.

Proof. First note that the uniqueness of t_2 is a consequence of the injectivity of h_2. Let $t_1' := \frac{t_1}{2}$ and define the mapping $h : [0,1] \to \mathbb{S}^2$ by

$$h(t) := \begin{cases} h_1(2t) & 0 \le t \le t_1', \\ h_2\left(\frac{t_2-1}{t_1'-1}t + \frac{t_1'-t_2}{t_1'-1}\right) & t_1' \le t \le 1. \end{cases}$$

Then $h([0,1]) = h_1([0,t_1]) \cup h_2([t_2,1])$. As continuity is clear, we just need to show that h is injective. Suppose that there exist $s_1 \ne s_2$ such that $h(s_1) = h(s_2)$. If $s_1, s_2 \le t_1'$ or $s_1, s_2 \ge t_1'$ this is a contradiction to the injectivity of h_1 and h_2, respectively. Thus we may assume that $s_1 < t_1' < s_2$. By the definition of h this implies that

$$h_1(s_1) = h_2\left(\frac{t_2-1}{t_1'-1}s_2 + \frac{t_1'-t_2}{t_1'-1}\right).$$

But since $s_1 < t_1$ this contradicts the minimality of t_1 and we are done. □

4. Possible cardinalities of the fundamental group of a tile

In this section we will prove that the fundamental group of a $\mathbb{Z}^2$-tile is either trivial or uncountable. This result together with a sketched proof is contained in the survey paper [2]. Here we give the result with its full proof.

Proposition 4.1. *Let $K \subset \mathbb{S}^2$ be a locally arcwise connected set. If $\mathbb{S}^2 \setminus K$ is disconnected then K contains a non-trivial loop.*

Proof. Let U_1 and U_2 be two components of $\mathbb{S}^2 \setminus K$ and select $p \in U_1$ and $q \in U_2$. Then K cuts $\mathbb{S}^2$ between p and q. By Proposition 3.4 this implies the existence of a simple closed curve $C \subset K$ which also cuts $\mathbb{S}^2$ between p and q. From this follows by [16, §59, IV, Theorem 4] that every set obtained from C by deformation in $\mathbb{S}^2 \setminus \{p,q\}$ can not be a single point. Since $K \subset \mathbb{S}^2 \setminus \{p,q\}$ this holds *a fortiori* for each deformation of C in K. Thus C is a non-trivial loop in K. This proves the result. □

Proposition 4.2. *Let $K \subset \mathbb{S}^2$ be a locally arcwise connected continuum. Suppose that $\mathbb{S}^2 \setminus K$ has infinitely many components. Then the following assertions hold.*

(i) *$\pi_1(K)$ is not free.*
(ii) *$\pi_1(K)$ is uncountable.*
(iii) *K is not locally simply connected.*
(iv) *K has no universal cover.*

Proof. We show assertion (iii). Then Proposition 3.6 yields (i), (ii) and (iv). In what follows, $B_r(y)$ will denote the open ball with radius r centered at y.

Let $\{U_i\}_{i\ge1}$ be the components of $\mathbb{S}^2 \setminus K$ and select $x_i \in U_i$. Let x be an accumulation point of the sequence x_i. Note that $x \in K$ because K is closed and $\operatorname{diam}(U_i) \to 0$ for $i \to \infty$ by Proposition 3.2. Let $\varepsilon > 0$ be an arbitrarily given number and set $K_\varepsilon := K \cap B_\varepsilon(x)$ (observe that K_ε is locally arcwise connected, as it is an open subset of the locally arcwise connected space K). We need to show that the neighborhood K_ε of x is not simply connected.

Since K is a locally connected continuum there exists an $\eta \in (0,\varepsilon)$ such that every pair of points with distance less than η can be connected by an arc of diameter ε (see [16, §50, II, Theorem 4]).

Since $\lim_{i\to\infty} \operatorname{diam}(U_i) = 0$, the definition of x implies that there is an $m \in \mathbb{N}$ such that $U_m \subset B_{\eta/2}(x)$. Thus K_η is a locally arcwise connected set with disconnected complement. Applying Proposition 4.1 yields a loop $C_\eta \subset K_\eta$ which is nontrivial in K. This loop can be connected with

x by an arc contained in K_ε. By [11, Proposition 1.5] this yields a nontrivial loop in K_ε which is based in x. Thus K is not locally simply connected in x. This proves (iii). $\square$

Lemma 4.3. *Let $\mathcal{T} = \mathcal{T}(A, \mathcal{D}) \subset \mathbb{S}^2$ be a connected $\mathbb{Z}^2$-tile with disconnected complement $\mathbb{S}^2 \setminus \mathcal{T}$. Then $\mathbb{S}^2 \setminus \mathcal{T}$ has infinitely many components.*

Proof. Since $\mathcal{T}$ is compact the disconnectivity of $\mathbb{S}^2 \setminus \mathcal{T}$ implies that $\mathbb{R}^2 \setminus \mathcal{T}$ has at least one bounded component T_0. As $\mathbb{R}^2 \setminus \mathcal{T}$ is open, every component of it is open in $\mathbb{R}^2$. Thus T_0 contains points which lie in the interior of a translate $\mathcal{T} + v$ for a certain $v \in \mathbb{Z}^2 \setminus \{0\}$. Of course, each of the sets

$$\mathbb{R}^2 \setminus A^{-1}(\mathcal{T} + d) \quad (d \in \mathcal{D})$$

contains a bounded component $A^{-1}(T_0 + d)$, which contains interior points of $A^{-1}(\mathcal{T} + d + v)$. Since $v \neq 0$, by the the equation $A\mathcal{T} = \mathcal{T} + \mathcal{D}$ there exists a $d_0 \in \mathcal{D}$ for which

$$A^{-1}(\mathcal{T} + d_0 + v) \cap \mathrm{int}(\mathcal{T}) = \emptyset.$$

Set $T_1 := A^{-1}(T_0 + d_0)$. Again by the equation $A\mathcal{T} = \mathcal{T} + \mathcal{D}$ we conclude that T_1 is a component of $\mathbb{R}^2 \setminus \mathcal{T}$. Iterating this construction we can construct countably many components T_i $(i \geq 0)$ of $\mathbb{R}^2 \setminus \mathcal{T}$ which are all contained in a certain fixed disk $B_r(0)$ around the origin. $\square$

Theorem 4.4. *Let $\mathcal{T} \subset \mathbb{S}^2$ be a connected $\mathbb{Z}^2$-tile.*

- *If $\mathbb{S}^2 \setminus \mathcal{T}$ is connected then $\pi_1(\mathcal{T})$ is trivial. If, moreover, $\mathcal{T}$ contains no cut point, it is homeomorphic to a closed disk.*
- *If $\mathbb{S}^2 \setminus \mathcal{T}$ is disconnected then $\pi_1(\mathcal{T})$ is uncountable and not free. Furthermore, $\mathcal{T}$ is not locally simply connected and has no universal cover.*

In particular, the fundamental group of a $\mathbb{Z}^2$-tile is either trivial or uncountable.

Proof. Since $\mathcal{T}$ is a locally arcwise connected continuum by Hata [10, Theorem 4.6], the first part follows from Proposition 3.3. Just note that for a set M which is an absolute retract we can extend each continuous function $f : \mathbb{S}^1 \to M$ to a continuous function $\tilde{f} : \overline{D} \to M$ on the closed unit disk $\overline{D}$. This implies that each loop in M is trivial.

The second part follows from Lemma 4.3 together with Proposition 4.2. $\square$

5. On the components of the interior of a tile

In this section we prove results on the topological structure of the components $\{U_i\}_{i \geq 1}$ of a connected plane IFS attractor T. The corresponding results for connected $\mathbb{Z}^2$-tiles follow as corollaries. First we prove that $\overline{U_i}$ is a locally connected continuum for each i. This result has already been obtained by Ngai and Tang [24]. However, our proof is much shorter and easier than theirs. We need the following lemma.

Lemma 5.1. *Suppose that T is the attractor of the IFS $\{f_j\}_{j=1}^q$ of injective contractions in the plane that satisfies the open set condition. Suppose further that T is connected and that $\{U_i\}$ is the collection of the components of* $\mathrm{int}(T)$*. Then U_i is a complementary component of a locally connected continuum for each i.*

Proof. It is known from Hata [10, Theorem 4.6] that T is a locally connected continuum. There exist indices $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, q\}$, such that the small homeomorphic copy

$$H(T) := f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}(T)$$

of T is entirely contained in U_1. Then, $H^{-1}(U_1)$ is a component of the interior of $H^{-1}(T)$, which is also a locally connected continuum. Moreover, we have $T \subset H^{-1}(U_1)$. For every word $\alpha = j_1 j_2 \ldots j_m \in \{1, 2, \ldots, q\}^m$, let

$$f_\alpha = f_{j_1} \circ f_{j_2} \circ \cdots \circ f_{j_m}.$$

Then $H^{-1}(T) = \bigcup_\alpha H^{-1} \circ f_\alpha(T)$. Let

$$M := \bigcup_{\substack{\alpha \in \{1,\ldots,q\}^m \\ \alpha \neq i_1 i_2 \ldots i_m}} H^{-1} \circ f_\alpha(T).$$

We claim that M is a locally connected continuum. To prove this define a sequence M_k $(k \geq 0)$ inductively as follows.

$$\begin{aligned} M_0 &:= \partial T, \\ M_{k+1} &:= M_k \cup \bigcup_{\substack{\alpha \neq i_1 i_2 \cdots i_m \\ H^{-1}\circ f_\alpha(T)\cap M_k \neq \emptyset}} H^{-1}\circ f_\alpha(T) \qquad (k \geq 0). \end{aligned}$$

Since M_0 is a continuum by [19], we conclude that M_k is a continuum for each $k \in \mathbb{N}$ and $M_0, M_1, \ldots$ is an increasing sequence of continua. Because $\{1,\ldots,q\}^m$ is finite, this sequence becomes eventually constant, equal to a set M', say. It is clear that $M' \subseteq M$. We will now show that $M' = M$. Suppose to the contrary that $M' \neq M$. Then there is a maximal non-empty $\mathcal{I} \subset \{1,\ldots,q\}^m \setminus \{i_1 \ldots i_m\}$ such that $H^{-1}\circ f_\alpha(T) \cap M' = \emptyset$ for each $\alpha \in \mathcal{I}$. Hence, for $\alpha \in \mathcal{I}$ we have $H^{-1}\circ f_\alpha(T)\cap \partial T = \emptyset$ and thus, by the open set condition, $H^{-1}\circ f_\alpha(T)\cap T = \emptyset$. Since $H^{-1}(T) = T \cup M' \cup M''$ with $M'' := \bigcup_{\alpha\in\mathcal{I}} H^{-1}\circ f_\alpha(T)$ this yields the separation $(T\cup M')\cup M''$ of $H^{-1}(T)$. This contradicts the connectivity of T. Thus $M' = M$ and since M' is a continuum so is M. Moreover, since every $H^{-1}\circ f_\alpha(T)$ is a locally connected continuum, M is the union of finitely many locally connected continua. Thus M is locally connected by [16, §49, II, Theorem 1] which says that the union of finitely many locally connected compact sets is locally connected.
Now, we will show that each U_i is a component of the complement of M. Obviously, U_i is connected. Since $\mathrm{int}(T) \subset \mathbb{R}^2 \setminus M$ we conclude that $U_i \subset \mathbb{R}^2 \setminus M$.

We will prove that U_i is open and closed in $\mathbb{R}^2\setminus M$. Firstly, U_i is obviously open in $\mathbb{R}^2 \setminus M$ because it is open in $\mathbb{R}^2$. Secondly, by construction, $\overline{U_i} \subset T \subset H^{-1}(U_1)$ is contained in the interior of $H^{-1}(T) = M \cup \mathrm{int}(T)$. Thus the only points in $\mathbb{R}^2\setminus M$ having distance 0 from U_i are contained in $\mathrm{int}(T)$. Hence,

$$\overline{U_i}\ \text{ (w.r.t. } \mathbb{R}^2\setminus M) = \overline{U_i}\ \text{ (w.r.t. } \mathrm{int}(T)) = U_i$$

and we conclude that U_i is closed in $\mathbb{R}^2\setminus M$. Being a connected, open and closed subset of $\mathbb{R}^2\setminus M$, the set U_i is a component of $\mathbb{R}^2\setminus M$ by [16, §46, III, Theorem 4] and we are done. □

Now it is easy to prove the following proposition.

Proposition 5.2. *Suppose that T is the attractor of an IFS $\{f_j\}_{j=1}^q$ of injective contractions in the plane which satisfies the open set condition. Suppose further that T is connected and that $\{U_i\}$ is the collection of the components of* $\mathrm{int}(T)$. *Then $\overline{U_i}$ is a locally connected continuum for each i.*

Proof. Each component U_i is a complementary component of a locally connected continuum by Lemma 5.1. Thus Proposition 3.1 implies that the boundary of every U_i is a locally connected continuum. Because every U_i is a simply connected region, there is a conformal homeomorphism h_i from the open unit disk D onto U_i by the Riemann Mapping Theorem (cf. [3]). Since ∂U_i is a locally connected continuum, this homeomorphism has a continuous extension to $\overline{D}$ by Proposition 3.7 (take $g = h_i$). Therefore, the closure $\overline{U_i}$ is a continuous image of the closed disk $\overline{D}$. Thus it is a locally connected continuum by the Hahn-Mazurkiewicz-Sierpiński Theorem (cf. [16, §50, II, Theorem 2]). □

The main objective of the present section is the proof of the following theorem.

Theorem 5.3. *Suppose that $\mathcal{T}$ is a connected $\mathbb{Z}^2$-tile and that $\{U_i\}$ is the collection of the components of* $\mathrm{int}(\mathcal{T})$. *Then, for every fixed i the fundamental group $\pi_1(\overline{U_i})$ is trivial if and only if $\overline{U_i}$ is homeomorphic to a closed disk. Particularly, triviality of $\pi_1(\mathcal{T})$ implies that $\overline{U_i}$ is homeomorphic to a closed disk for every i.*

We will even prove the following more general result.

Theorem 5.4. *Suppose that T is the attractor of an IFS $\{f_j\}_{j=1}^q$ of injective contractions on the plane which satisfies the open set condition. Suppose that T is connected and that $\{U_i\}$ is the collection of the components of* $\mathrm{int}(T)$. *Then, for every fixed i the fundamental group $\pi_1(\overline{U_i})$ is trivial if and only if $\overline{U_i}$ is homeomorphic to a closed disk. Particularly, triviality of $\pi_1(T)$ implies that $\overline{U_i}$ is homeomorphic to a closed disk for every i.*

Proof. $\overline{U_i}$ is a locally connected continuum by Proposition 5.2. Thus the triviality of the fundamental group $\pi_1(\overline{U_i})$ implies that $\mathbb{S}^2 \setminus \overline{U_i}$ is connected (Proposition 4.1). Since it is easy to see that $\overline{U_i}$ has no cut point, we may apply the last assertion of Proposition 3.1 with $C = \overline{U_i}$ to derive that the boundary of (the unique component of) $\mathbb{S}^2 \setminus \overline{U_i}$ is a simple closed curve. But this boundary is exactly ∂U_i. Consequently, $\overline{U_i}$ is homeomorphic to a closed disk by a theorem of Schönflies (cf. [22, Chapter 9, Theorem 6]). The converse part is trivial.

If the fundamental group of T is trivial, $\pi_1(\overline{U_i})$ is clearly trivial for every i. □

6. Tiles with nontrivial fundamental group

In this section we want to state criteria which assure the non-triviality of the fundamental group of a $\mathbb{Z}^2$-tile. These criteria can be checked easily with help of the neighbor graph and the adding machine.

We need the following auxiliary result which will be proved using methods from plane topology. During its proof we will use Lemma 3.8 frequently.

Lemma 6.1. *Let $B_0, B_1, B_2 \subset \mathbb{R}^2$ be locally connected continua with the following properties.*
- (i) $\mathrm{int}(B_i) \cap \mathrm{int}(B_j) = \emptyset$ *for* $i \neq j$.
- (ii) B_i *is the closure of its interior* ($0 \le i \le 2$).
- (iii) $\mathbb{S}^2 \setminus \mathrm{int}(B_i)$ *is a locally connected continuum* ($0 \le i \le 2$).
- (iv) *There exist* $x_1, x_2 \in B_0 \cap B_1 \cap B_2$ *with* $x_1 \in \mathrm{int}(B_0 \cup B_1 \cup B_2)$.

Then there is an $i \in \{0,1,2\}$ such that $B_i \cup B_{i+1}$ has a bounded complementary component U with $U \cap \mathrm{int}(B_{i+2}) \neq \emptyset$ (the indices are to be taken modulo 3).

Remark 1. Every lattice tiling of $\mathbb{R}^2$ by connected and locally connected tiles has the first three properties (see the proof of Proposition 6.2), so these are natural conditions for our discussion on $\mathbb{Z}^2$-tiles.

Proof. For $i \in \{0,1,2\}$ let $\{U_j^{(i)}\}_{j\ge 0}$ be the components of $\mathrm{int}(B_i)$. Furthermore, let $\varepsilon > 0$ be small enough such that
$$\overline{B_{2\varepsilon}(x_1)} \subset B_0 \cup B_1 \cup B_2.$$
We want to distinguish two cases.

Case (i): There are infinitely many sets in the collection $\{U_j^{(i)}\}_{j\ge 0, 0\le i\le 2}$ having nonempty intersection with $B_\varepsilon(x_1)$.

There exists a fixed $i_0 \in \{0,1,2\}$ such that infinitely many sets of the shape $U_j^{(i_0)}$ intersecting $B_\varepsilon(x_1)$. W.l.o.g. we may assume that $i_0 = 0$. Since $\mathbb{S}^2 \setminus \mathrm{int}(B_0)$ is a locally connected continuum Proposition 3.2 yields
$$\lim_{j\to\infty} \mathrm{diam}(U_j^{(0)}) = 0.$$
Thus there exists j_0 such that $U_{j_0}^{(0)} \subset B_{2\varepsilon}(x_1)$. Set $U = U_{j_0}^{(0)}$. Then $\partial U \subset B_1 \cup B_2$ and U is a bounded complementary component of $B_1 \cup B_2$ with $U \cap \mathrm{int}(B_0) \neq \emptyset$.

Case (ii): There are only finitely many sets in the collection $\{U_j^{(i)}\}_{j\ge 0, 0\le i\le 2}$ intersecting $B_\varepsilon(x_1)$.

Select $r \in (0,\varepsilon)$ so small that for any $i \in \{0,1,2\}$ and any j
$$\overline{U_j^{(i)}} \cap \overline{B_r(x_1)} \neq \emptyset \Longrightarrow \begin{cases} x_1 \in \overline{U_j^{(i)}} & \text{and} \\ U_j^{(i)} \cap \partial B_r(x_1) \neq \emptyset. \end{cases}$$

For $i \in \{0,1,2\}$ denote by $\{V_j^{(i)}\}$ the family of all the $U_j^{(i)}$ whose closure intersects the closed disk $\overline{B_r(x_1)}$.

For each $i \in \{0,1,2\}$ select an arc $J_i \subset B_i$ from x_1 to x_2. Let b_i be the last point at which J_i intersects $\overline{B_r(x_1)}$. Then there exists j_i such that $b_i \in \overline{V_{j_i}^{(i)}}$. Thus $x_1, b_i \in \overline{V_{j_i}^{(i)}}$. By local connectivity of $\mathbb{R}^2 \setminus \mathrm{int}(B_i)$ the points x_1 and b_i can be connected by an arc J_i' whose interior is contained in $V_{j_i}^{(i)}$. Let a_i be the first point at which J_i' meets $\partial B_r(x_1)$. Denote by p_i the subarc in J_i' from x_1 to a_i. Let p_i' be the union of the subarc in J_i' from a_i to b_i with the subarc of J_i from b_i to x_2. Let $I_i = p_i \cup p_i'$. It is easy to see that $I_i \subset B_i$ is an arc connecting x_1 and x_2.

Now, the open disk $B_r(x_1)$ is divided by $p_0 \cup p_1 \cup p_2$ in three regions. Let Y_i be the region whose boundary does not intersect the interior of p_i. Note that $\overline{Y_i}$ is homeomorphic to a closed disk.

Subcase (ii.1): The point x_1 can be connected to infinity by an arc $\mathcal{R}$ in

$$\mathbb{S}^2 \setminus (p_0' \cup p_1' \cup p_2').$$

Let x_1' be the last point at which $\mathcal{R}$ intersects $\partial B_r(x_1)$. Then x_1' must lie on the boundary of some Y_i. Assume with no loss of generality that $i = 0$. Then x_1 and x_1' can be joined by an arc P whose interior is contained in Y_0. Let $\mathcal{R}'$ be the union of P and the subarc of $\mathcal{R}$ from x_1' to infinity. Then $\mathcal{R}'$ is an arc connecting x_1 and infinity which does not intersect $I_0 \cup I_1 \cup I_2$ except at x_1. Let $a \neq x_1$ be the first point at which I_1 meets I_2.

For $i = 1, 2$, denote by I_i' the subarc of I_i from x_1 to a. Then $I_1' \cup I_2'$ is a simple closed curve. Let R_1, R_2 its residual components and assume that $\infty \in R_1$. Then R_1 contains $\mathcal{R} \setminus \{x_1\}$. Thus, R_2 must contain $p_0 \setminus \{x_1\}$. Because B_0 is the closure of its interior this indicates that some interior point of B_0 belongs to a bounded complementary component of $(I_1 \cup I_2) \subset (B_1 \cup B_2)$.

Subcase (ii.2): The point x_1 cannot be connected to infinity by an arc $\mathcal{R}$ in

$$\mathbb{S}^2 \setminus (p_0' \cup p_1' \cup p_2').$$

Since $(p_0' \cap p_1' \cap p_2')$ contains x_2, by Proposition 3.5, the point x_1 cannot be connected to infinity by an arc $\mathcal{R}$ in

$$\mathbb{S}^2 \setminus \left(p_i' \cup p_{i+1}'\right)$$

for some $i \in \{0, 1, 2\}$. (Here, the indices are to be taken modulo 3).

We may assume without loss of generality that $i = 0$. Then x_1, and thus an interior point of B_0 near x_1, must belong to a bounded complementary component of $(p_1' \cup p_2') \subset (B_1 \cup B_2)$. This indicates that $\mathrm{int}(B_0)$ intersects a bounded complementary component of $B_1 \cup B_2$. $\square$

Proposition 6.2. *Let $\mathcal{T}$ be a connected $\mathbb{Z}^2$-tile and let $s_1, s_2 \in S \setminus \{0\}$ be disjoint. If $\#V_2(s_1, s_2) \geq 2$ and $V_2(s_1, s_2) \setminus V_3 \neq \emptyset$ then one of the following statements is true.*

- *$(\mathcal{T} + s_1) \cup (\mathcal{T} + s_2)$ has a bounded complementary component U with $U \cap \mathrm{int}(\mathcal{T}) \neq \emptyset$.*
- *$\mathcal{T} \cup (\mathcal{T} + s_1)$ has a bounded complementary component U with $U \cap \mathrm{int}(\mathcal{T} + s_2) \neq \emptyset$.*
- *$\mathcal{T} \cup (\mathcal{T} + s_2)$ has a bounded complementary component U with $U \cap \mathrm{int}(\mathcal{T} + s_1) \neq \emptyset$.*

Proof. The assertion follows if $B_0 := \mathcal{T}$, $B_1 := \mathcal{T} + s_1$ and $B_2 := \mathcal{T} + s_2$ satisfy the conditions of Lemma 6.1. First of all, $\mathcal{T}$ and its translates are locally connected continua by Hata's result [10, Theorem 4.6]. Condition (i) of Lemma 6.1 is true because of (1.1) and Condition (ii) follows because the $\mathbb{Z}^2$-tile $\mathcal{T}$ (as well as its translates) is the closure of its interior (cf. Vince [33, Theorem 3.6]). Condition (iii) is true because the connected closed set $\mathbb{R}^2 \setminus \mathrm{int}(\mathcal{T})$ can be written as a locally finite union of translates of $\mathcal{T}$. Thus it is also locally connected and its closure in $\mathbb{S}^2$, the set $\mathbb{S}^2 \setminus \mathrm{int}(\mathcal{T})$, is a locally connected continuum. The same holds for all translates of $\mathcal{T}$. To see that Condition (iv) holds just take $x_1, x_2 \in V(s_1, s_2)$ such that $x_1 \notin V_3$. Then clearly $x_1, x_2 \in B_0 \cap B_1 \cap B_2$ holds. Suppose that $x_1 \notin \mathrm{int}(B_0 \cup B_1 \cup B_2)$. Then by the tiling property there exists $s_3 \in \mathbb{Z}^2 \setminus \{0, s_1, s_2\}$ with $x_1 \in \mathcal{T} + s_3$, hence, $x_1 \in V_3$, a contradiction.

Thus the conditions of Lemma 6.1 are fulfilled by the sets $\mathcal{T}$, $\mathcal{T} + s_1$ and $\mathcal{T} + s_2$. The proposition now follows as a consequence of this lemma. $\square$

Theorem 6.3. *Let $\mathcal{T}$ be a connected $\mathbb{Z}^2$-tile. Suppose that there exist $s_1, s_2 \in S \setminus \{0\}$ with the following properties.*

- *$\#V_2(s_1, s_2) \geq 2$ and $V_2(s_1, s_2) \setminus V_3 \neq \emptyset$.*
- *For each $i \in \{0, 1, 2\}$ there exists a digit string w_i with the following property. Using w_i as input string for $G^T(S)$ starting at 0, s_1, and s_2 yields output strings c_0, c_1, c_2 satisfying*

$$\max(\mathcal{L}(c_i), \mathcal{L}(c_{i+1})) < \mathcal{L}(c_{i+2})$$

(indices are to be taken modulo 3).

Then $\pi_1(\mathcal{T})$ is uncountable and not free. Furthermore, $\mathcal{T}$ is not locally simply connected and has no universal cover.

Remark 2. Both conditions can be checked rather easily by inspecting the graph $G^T(S)$ (see Section 8, where examples are discussed).

Proof. The first condition of the theorem ensures that Proposition 6.2 holds. Suppose that the first alternative of this proposition is true (the other alternatives are treated likewise), *i.e.*, we have that

$$\mathbb{R}^2 \setminus ((\mathcal{T}+s_1) \cup (\mathcal{T}+s_2)) \text{ has a bounded component } U \text{ with } U \cap \mathrm{int}(\mathcal{T}) \neq \emptyset. \tag{6.1}$$

By the second condition we may apply Lemma 2.2 with $a = (w_1)_A$, $r_1 = s_1$, $r_2 = 0$, $o_1 = c_1$, $o_2 = c_0$ and with $a = (w_1)_A$, $r_1 = s_2$, $r_2 = 0$, $o_1 = c_2$, $o_2 = c_0$. Indeed, since

$$L = \max(\mathcal{L}(c_1), \mathcal{L}(c_2)) < \mathcal{L}(c_0)$$

Lemma 2.2 yields

$$A^{-L}(\mathcal{T} + s_i + (w_1)_A) \quad \subset \quad \mathcal{T} \quad (i = 1, 2), \tag{6.2}$$

$$\mathrm{int}(A^{-L}(\mathcal{T} + (w_1)_A)) \cap \mathcal{T} \quad = \quad \emptyset. \tag{6.3}$$

In view of the tiling property of $\mathcal{T}$ (6.3) yields the existence of an $s \in \mathbb{Z}^2 \setminus \{0\}$ with

$$\mathrm{int}(A^{-L}(\mathcal{T} + (w_1)_A)) \quad \subset \quad \mathrm{int}(\mathcal{T} + s). \tag{6.4}$$

If we translate (6.1) by $(w_1)_A$ and multiply by A^{-L} then (6.2) and (6.4) imply that $\mathcal{T}$ has a bounded complementary component U' containing $A^{-L}(U + (w_1)_A)$ with $U' \cap \mathrm{int}(\mathcal{T} + s) \neq \emptyset$. Thus $\mathbb{S}^2 \setminus \mathcal{T}$ is disconnected. Theorem 4.4 now yields the result. □

Theorem 6.4. *Let $\mathcal{T} = \mathcal{T}(A, \mathcal{D})$ be a $\mathbb{Z}^2$-tile. Suppose that there is an $a \notin \mathcal{D}$ such that $\partial(\mathcal{T}+a) \subset (\mathcal{T}+\mathcal{D})$. Then $\pi_1(\mathcal{T})$ is uncountable and not free. Furthermore, $\mathcal{T}$ is not locally simply connected and has no universal cover.*

Proof. Since $\partial(\mathcal{T}+a) \subset (\mathcal{T}+\mathcal{D})$, the interior of $\mathcal{T}+a$ is the union of bounded components of $\mathbb{S}^2 \setminus (\mathcal{T}+\mathcal{D})$. Since $\mathcal{L}(a) \geq 2$ multiplication by A^{-1} implies that the complement of $\mathcal{T}$ itself must have some bounded components. Thus $\mathbb{S}^2 \setminus \mathcal{T}$ is disconnected and the result follows from Theorem 4.4. □

7. Tiles with trivial fundamental group

In this section we give a criterion for the triviality of the fundamental group of a $\mathbb{Z}^2$-tile. The main theorem of this section is Theorem 7.2. It contains three conditions which are sufficient for the triviality of $\pi_1(\mathcal{T})$. As we will show after the proof of this theorem, these conditions can be checked with help of $G(S)$ and is transpose. Before we state the result we need a preparatory lemma.

Lemma 7.1. *Let $\mathcal{T}(A, \mathcal{D})$ be a $\mathbb{Z}^2$-tile with $\pi_1(\mathcal{T}) \neq 0$. Then at least one of the following assertions holds.*

- *There exists a non-zero $a \in \mathbb{Z}^2$ and a loop $\ell \subset \mathcal{T} \cap (\mathcal{T}+a)$ which separates a point $x_0 \in \mathcal{T}+a$ from ∞.*
- *There exists $\mathcal{D}' \subset \mathcal{D}$ with $\#\mathcal{D}' \geq 3$ and $\#V_{\mathcal{D}'} \geq \#\mathcal{D}' - 1$.*

Proof. Because $\pi_1(\mathcal{T}) \neq 0$ Theorem 4.4 implies that $\mathcal{T}$ has a bounded complementary component V. Thus there is a set $S_0 \subset S$ such that

$$V \subset \bigcup_{s \in S_0} (\mathcal{T} + s). \tag{7.1}$$

Let $w = (a_{L-1} \dots a_0)$ be a string of input digits leading from s to s' in $G^T(S)$ and define a (continuous, open) mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x) = A^{-L} \left(x + \sum_{i=0}^{L-1} A^i a_i \right).$$

Because S has property (C) we see as in the proof of Lemma 2.3 that the existence of such a labelling implies that

$$s = \sum_{i=0}^{L-1} A^i(a_i' - a_i) + A^L s'.$$

Thus

$$f(\mathcal{T}+s) = A^{-L}\left(\mathcal{T}+s+\sum_{i=0}^{L-1} A^i a_i\right) = \sum_{i=0}^{L-1} A^{i-L}a_i' + A^{-L}\mathcal{T} + s' \subset (\mathcal{T}+s'). \tag{7.2}$$

If the string of input digits w leads from S_0 to S_1 (7.1) implies together with (7.2) that

$$f(V) \subset \bigcup_{s\in S_1} (\mathcal{T}+s).$$

As S has property (C) Lemma 2.4 enables us to select $w = (a_{L-1}\dots a_0)$ in a way that for a fixed $d \in \mathcal{D}$ each element of S_1 is of the shape $d' - d$ with some $d' \in \mathcal{D}$ and S_1 contains at least one non-zero element. Choose $b_1 \in S_1 \setminus \{0\}$. Because V is open, (7.2) implies that $f(V)$ contains interior points of $\mathcal{T} + b_1$. Moreover, $f(\mathcal{T}) \subset \mathcal{T}$ implies that $\partial f(V) \subset \mathcal{T}$. Thus $f(V)$ contains a complementary component U of $\mathcal{T}$ for which

$$U \subset \bigcup_{b\in S_1\setminus\{0\}} (\mathcal{T}+b)$$

holds. Obviously, $\partial U \subset \mathcal{T}$. Because $\mathcal{T}$ is a locally connected continuum, Proposition 3.1 implies that ∂U is also a locally connected continuum. Moreover, ∂U cuts between a point $x_0 \in U \cap (\mathcal{T}+b_1)$ and ∞. Thus by Proposition 3.4 there exists a simple closed curve $\ell : \mathbb{S}^1 \to \partial U$ which also cuts between x_0 and ∞. Note that $\ell \subset \mathcal{T}$.

There are two cases to distinguish. The first one is $\ell \subset (\mathcal{T}+b)$ for some $b \in S_1 \setminus \{0\}$. Since $\ell \subset \mathcal{T}$ in this case the first assertion holds with $a := b$.

Otherwise let $S_2 \subset S_1 \setminus \{0\}$ be minimal with the property $\ell \subset \bigcup_{b\in S_2}(\mathcal{T}+b)$ (note that in this case we have $\#S_2 \ge 2$). Then for each $b \in S_2$ there exists a parameter $t_b \in \mathbb{S}^1$ with

$$\ell(t_b) \in (\mathcal{T}+b) \setminus \bigcup_{b\ \in S_2\setminus\{b\}} (\mathcal{T}+b').$$

Let I_b be an interval which is maximal with the property $\ell(\mathrm{int}(I_b)) \subset (\mathcal{T}+b)\setminus \bigcup_{b\ \in S_2\setminus\{b\}}(\mathcal{T}+b')$. Let e_b be the end point of I_b. Then there is some $b' \in S_2 \setminus \{b\}$ with

$$\ell(e_b) \in \mathcal{T} \cap (\mathcal{T}+b) \cap (\mathcal{T}+b'). \tag{7.3}$$

Since all the points $\ell(e_b)$ $(b \in S_2)$ are disjoint we see by adding d to the relation in (7.3) that $\#V_{\mathcal{D}'} \ge \#S_2 = \#\mathcal{D}' - 1$ holds with $\mathcal{D}' := \{b + d \mid b \in S_2\} \cup \{0\}$ and the second assertion is true. ☐

Theorem 7.2. *Let $\mathcal{T} = \mathcal{T}(A,\mathcal{D})$ be a connected $\mathbb{Z}^2$-tile. $\pi_1(\mathcal{T})$ is trivial provided that the following conditions hold.*

(i) *Let $z_1, z_2 \in \mathbb{Z}^2$ such that their A-adic expansion differs only in the least significant digit. Then*

$$\# \bigcup_{z\in\mathbb{Z}^2\colon \mathcal{L}(z)<\mathcal{L}(z_1)} ((\mathcal{T}+z_1)\cap(\mathcal{T}+z_2)\cap(\mathcal{T}+z)) \le 1. \tag{7.4}$$

(ii) *In each set $D_0 \subset \mathcal{D}$ with $\#D_0 \ge 3$ there exists $d \in D_0$ such that for all $d_1, d_2 \in D_0$ with d, d_1, d_2 pairwise disjoint we have*

$$(\mathcal{T}+d)\cap(\mathcal{T}+d_1) \subset (\mathcal{T}+d_2) \quad \text{or} \quad (\mathcal{T}+d)\cap(\mathcal{T}+d_2) \subset (\mathcal{T}+d_1).$$

(iii) *For all $\mathcal{D}' \subset \mathcal{D}$ with $\#\mathcal{D}' \ge 3$ we have $\#V_{\mathcal{D}'} < \#\mathcal{D}' - 1$.*

Remark 3. These conditions can be checked easily by inspecting the graph $G(S)$ and its transpose. We will give details after the proof of the theorem.

Proof. Assume that $\pi_1(\mathcal{T}) \neq 0$. We will prove that this is impossible if (i), (ii) and (iii) hold. Since we assume the truth of condition (iii), Lemma 7.1 implies that there exists $a \in \mathbb{Z}^2$ and a simple closed curve $\ell : \mathbb{S}^1 \to \mathcal{T} \cap (\mathcal{T} + a)$ which cuts between a point $x_0 \in \mathrm{int}(\mathcal{T} + a)$ and ∞. Let $m \in \mathbb{N}$ be the least integer with the property that there exists no $a_m = \sum_{i=0}^{m-1} A^i b_i$ such that $\ell \subset A^{-m}(\mathcal{T}+a_m)+a$. Thus there exists an $a_{m-1} = \sum_{i=0}^{m-2} A^i b_i$ such that $\ell \subset A^{1-m}(\mathcal{T}+a_{m-1})+a$. For $d \in \mathcal{D}$ set

$$S_d := A^{-m}(\mathcal{T} + Aa_{m-1} + d) + a \subset \mathcal{T} + a.$$

By the choice of m there is a set $D_0 \subset \mathcal{D}$ of minimal cardinality such that

$$C := \bigcup_{d \in D_0} S_d \supset \ell.$$

Note that $\# D_0 \geq 2$. By the minimality of D_0 for each $d \in D_0$ there exists a parameter $t_d \in \mathbb{S}^1$ with

$$\ell(t_d) \in S_d \setminus \bigcup_{d' \in D_0 \setminus \{d\}} S_{d'}.$$

Let I_d be the maximal closed interval containing t_d with $\ell(\mathrm{int}(I_d)) \subset S_d \setminus \bigcup_{d' \in D_0 \setminus \{d\}} S_{d'}$. Let e be an end point of I_d. Then there is a $d' \in D_0 \setminus \{d\}$ and (because $\ell \subset \mathcal{T}$) a translate $s \in \mathbb{Z}^2$ with $A^{-m}(\mathcal{T} + s) \subset \mathcal{T}$ such that $\ell(e) \in S_d \cap S_{d'} \cap A^{-m}(\mathcal{T} + s)$.

Suppose first that $\# D_0 = 2$, *i.e.*, $D_0 = \{d_1, d_2\}$. In this case we have at least two different intervals I_{d_1} and I_{d_2} whose end points are mapped by ℓ to elements of sets of the shape

$$S_{d_1} \cap S_{d_2} \cap A^{-m}(\mathcal{T} + s)$$

where $\mathcal{L}(s) \leq m$. Multiplying by A^m yields that

$$B(d_1, d_2) := \bigcup_{s:\, \mathcal{L}(s) \leq m} (\mathcal{T} + A^m a + Aa_{m-1} + d_1) \cap (\mathcal{T} + A^m a + Aa_{m-1} + d_2) \cap (\mathcal{T} + s)$$

contains all points

$$\{A^m \ell(e) |\ e \text{ endpoint of } I_d \text{ for some } d \in D_0\}.$$

Thus $\# B(d_1, d_2) \geq 2$ and condition (i) is violated.

Suppose now that $\# D_0 > 2$. If there is a pair $d_1, d_2 \in D_0$ with $\# B(d_1, d_2) \geq 2$ then we can argue as in the previous case. If such a pair does not exist then for each $d \in D_0$ there must be an interval I_d whose endpoints e_1, e_2 satisfy $\ell(e_1) \in (S_d \cap S_{d_1}) \setminus S_{d_2}$ and $\ell(e_2) \in (S_d \cap S_{d_2}) \setminus S_{d_1}$ ($d, d_1, d_2 \in \mathcal{D}$ pairwise disjoint). This violates condition (ii) and we are done. □

We will now show how the conditions of Theorem 7.2 can be checked with help of certain subgraphs of $G(\mathbb{Z}^2)$. Condition (i) is treated in the following lemma.

Lemma 7.3. *Let*

$$\tilde{S}(d_1, d_2) := \{s \in \mathbb{Z}^2 \,|\, (\mathcal{T} + d_1) \cap (\mathcal{T} + d_2) \cap (\mathcal{T} + s) \neq \emptyset\}$$

and let $c_w(q)$ be the output string obtained by the input string w starting at q in $G^T(\mathbb{Z}^2)$. Suppose that for all $d_1, d_2 \in \mathcal{D}$ and all walks $w = (\ldots a_L \ldots a_1 0)$ the number of elements $s \in \tilde{S}(d_1, d_2)$ with

$$\mathcal{L}(c_w(s)) < \mathcal{L}(c_w(d_1))$$

is at most one. Then condition (i) of Theorem 7.2 is true.

Proof. Each pair z_1, z_2 in condition (i) of Theorem 7.2 is of the shape $d_1 + (w)_A, d_2 + (w)_A$ for certain $d_1, d_2 \in \mathcal{D}$ and $w = (\ldots a_2 a_1 0)$. Thus we get a contribution to the union in (7.4) for each $s \in \tilde{S}(d_1, d_2)$ which satisfies

$$\mathcal{L}(s + (w)_A) < \mathcal{L}(d_1 + (w)_A).$$

Since this can be rewritten as $\mathcal{L}(c_w(s)) < \mathcal{L}(c_w(d_1))$ the equivalence of the two conditions is shown. □

Conditions (ii) and (iii) can be read off from the graph $G(S)$ easily (see Proposition 2.5 and the remark after it). Note that Condition (ii) is trivially true if in each set $D_0 \subset \mathcal{D}$ with $\# D_0 \geq 3$ there exists a $d \in D_0$ such that $(\mathcal{T} + d) \cap (\mathcal{T} + d') \neq \emptyset$ holds for at most one $d' \in D_0 \setminus \{d\}$.

8. Examples

In this section we want to apply our results to special fractals. We start with the $\mathbb{Z}^2$-tile $\mathcal{T}_1 := \mathcal{T}_1(A_1, \mathcal{D}_1)$ where

$$A_1 := \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, \qquad \mathcal{D}_1 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \right\}. \tag{8.1}$$

The tile $\mathcal{T}_1$ is shown in Figure 1. This tile has been studied for instance by Bandt and Wang [4].

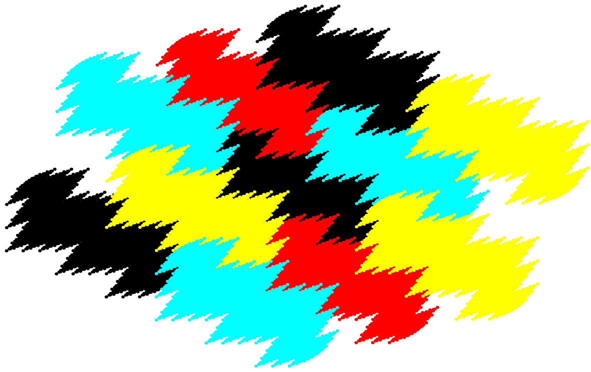

Figure 1. The tile $\mathcal{T}_1$ and its neighbors.

They showed that $\mathcal{T}_1$ is not homeomorphic to a closed disk. With a neighbor finding algorithm (cf. for instance [29, 31]) it is not hard to compute the set of neighbors S_1 of $\mathcal{T}_1$. We get

$$S_1 = \{(0,0), (\pm 1, 0), (\pm 1, \mp 1), (\pm 2, \mp 1), (\pm 3, \mp 1), (\pm 4, \mp 2)\}.$$

The graph $G^T(S_1)$ is drawn in Figure 2. Using this graph we will be able to prove the following theorem.

Theorem 8.1. *Let $\mathcal{T}_1 = \mathcal{T}_1(A_1, \mathcal{D}_1)$ be the $\mathbb{Z}^2$-tile with A_1 and $\mathcal{D}_1$ as in (8.1). Let $\{U_i\}_{i\in\mathbb{N}}$ be the set of its interior components. Then $\mathcal{T}_1$ has trivial fundamental group. Furthermore, $\mathcal{T}_1$ has a cut point and $\overline{U_i}$ is homeomorphic to a closed disk for each $i \in \mathbb{N}$.*

Proof. The main part of the proof consists in verifying the conditions of Theorem 7.2. We start with condition (i). To this matter we will use Lemma 7.3.

First by inspecting the neighbor graph in Figure 2 we observe using Proposition 2.5 that

$$\begin{aligned} \tilde{S}((0,0),(1,0)) &\subseteq \{(-1,1),(2,-1)\}, \\ \tilde{S}((0,0),(-1,0)) &\subseteq \{(1,-1),(-2,1)\}, \\ \tilde{S}((1,0),(-1,0)) &= \emptyset. \end{aligned}$$

Let $w = (\dots a_3 a_2 a_1 0)$. In what follows we write $\bar{1}$ instead of -1. If we start at a state $q \in S_1$ with the input string w this will produce a certain output string. This output string will be called $c(q)$ in the following.

We will show now that for each $w = (\dots a_3 a_2 a_1 0)$ there is at most one $s \in \tilde{S}((0,0),(1,0))$ with $\mathcal{L}(c(s)) < \mathcal{L}(c((0,0)))$. Since $\#\tilde{S}((0,0),(1,0)) \le 2$ this is tantamount to proving that there exists an $s \in \{(-1,1),(2,-1)\}$ with $\mathcal{L}(c(s)) \ge \mathcal{L}(c((0,0)))$.

Let $a_1 = 0$. Then the input string w starting at $(-1,1)$ leads to $(0,0)$ after two steps. We see form Figure 2 that $c((-1,1))$ is of the shape $(\dots b_3 b_2 1 \bar{1})$ where the b_i coincide with the corresponding digits of $c((0,0))$. Thus $\mathcal{L}(c((-1,1))) \ge \mathcal{L}(c((0,0)))$ and we are done in this case.

Let $a_1 = \bar{1}$. Again the input string of w starting at $(-1,1)$ leads to $(0,0)$ after two steps. here $c((-1,1))$ is of the shape $(\dots b_3 b_2 0 \bar{1})$ where the b_i coincide with the corresponding digits of $c((0,0))$. Thus $\mathcal{L}(c((-1,1))) = \mathcal{L}(c((0,0)))$ unless $b_i = 0$ for each $i \ge 2$. However, in order to get $b_i = 0$ for $i \ge 2$ we need $a_i = 0$ for each $i \ge 2$, *i.e.*, $w = (\dots 00\bar{1}0)$. In this case $c((2,-1)) = ((\bar{1}1)^\infty \bar{1}0\bar{1})$ which yields $\mathcal{L}(c((2,-1))) \ge \mathcal{L}(c((0,0)))$ and we are done.

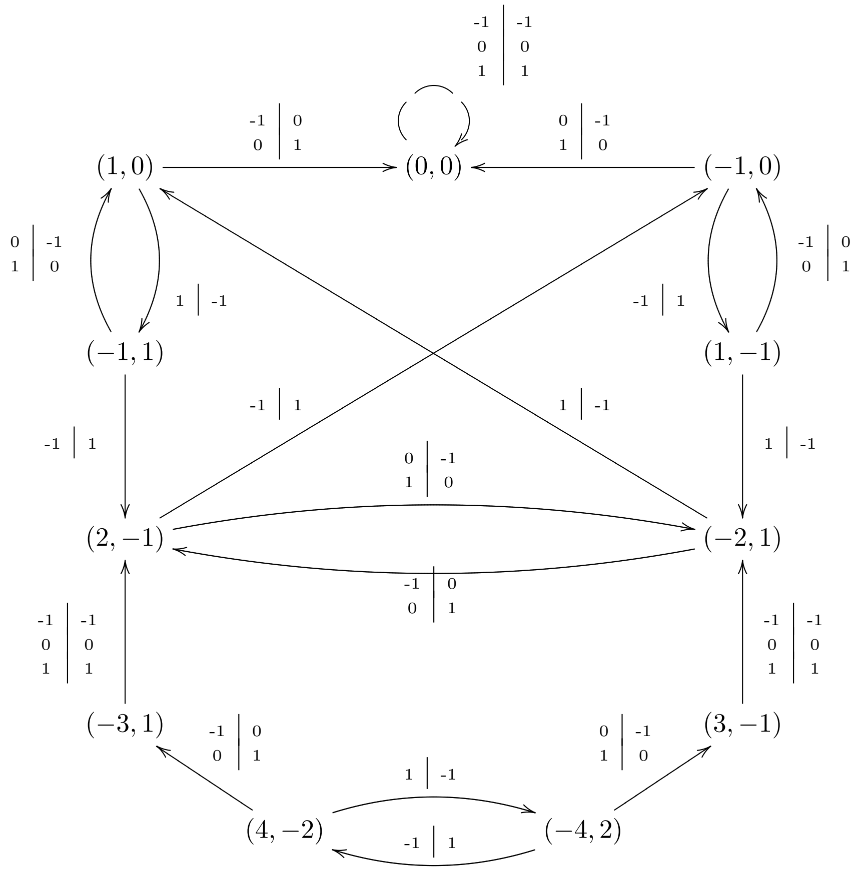

FIGURE 2. The graph $G^T(S_1)$ for $\mathcal{T}_1$.

Thus we must have $a_1 = 1$, *i.e.*, $w = (\dots a_3 a_2 10)$. Now we put our attention to a_2. If $a_2 \neq 1$ then we see from Figure 2 that starting from $(2,-1)$ with the input string w we arrive at $(0,0)$ after three steps. By a similar reasoning as before this leads either to $\mathcal{L}(c((-1,1))) \geq \mathcal{L}(c((0,0)))$ or to $\mathcal{L}(c((2,-1))) \geq \mathcal{L}(c((0,0)))$. Thus we only have to consider the case $a_2 = 1$.

By induction we see that if $a_1 = \cdots = a_{2k} = 1$ and $a_{2k+1} \neq 1$ then $(-1,1)$ goes to $(0,0)$ after $2k+2$ steps with input string w. Furthermore, $a_1 = \cdots = a_{2k-1} = 1$ and $a_{2k} \neq 1$ then $(2,-1)$ goes to $(0,0)$ after $2k+1$ steps. By the same reasoning as above this leads to $\max(\mathcal{L}(c((-1,1))), \mathcal{L}(c((-2,1)))) \geq \mathcal{L}(c((0,0)))$.

So $w = (1^\infty 0)$ is the only input string which remains to be checked. However, it is easy to see that for this w we have $\mathcal{L}(c((-1,1))) = \infty$ and we are done.

Just by changing signs we obtain an analogous result for the set $\tilde{S}((0,0),(-1,0))$. Since $\tilde{S}((-1,0),(1,0))$ is empty we proved the truth of the first assertion in Lemma 7.3. Thus condition (i) of Theorem 7.2 holds.

Condition (ii) is checked as follows. First observe that it is trivial for $\#D_0 \leq 2$. Thus the only possible choice is $D_0 = \mathcal{D}$. It is clear that $\mathcal{T} + (-1,0)$ and $\mathcal{T} + (1,0)$ do not touch each other (because otherwise we would have $(2,0) \in S$). Thus

$$(\mathcal{T}_1 + (1,0)) \cap (\mathcal{T}_1 + (-1,0)) = \emptyset \subset \mathcal{T}$$

and we are done. Condition (iii) is easily checked. Indeed, we have to check $V_{\mathcal{D}'}$ only for $\mathcal{D}' = \mathcal{D}$. However, $V_{\mathcal{D}} = \emptyset$ because $\mathcal{T} + (-1,0)$ and $\mathcal{T} + (1,0)$ do not touch each other.

Thus $\pi_1(\mathcal{T}_1)$ is trivial. In [4] it is proved that $\mathcal{T}_1$ is not homeomorphic to a closed disk. Since $\pi_1(\mathcal{T}_1) = 0$ the first part of Theorem 4.4 ensures the existence of a cut point of $\mathcal{T}_1$. The assertions on U_i follow from Theorem 5.3. $\square$

The next example is devoted to a tile which has been studied in [9]. Let $\mathcal{T}_2 = \mathcal{T}_2(A_2, \mathcal{D}_2)$ be the $\mathbb{Z}^2$-tile with

$$A_2 := \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \mathcal{D}_2 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \tag{8.2}$$

The tile $\mathcal{T}_2$ together with its 10 neighbors is depicted in Figure 3.

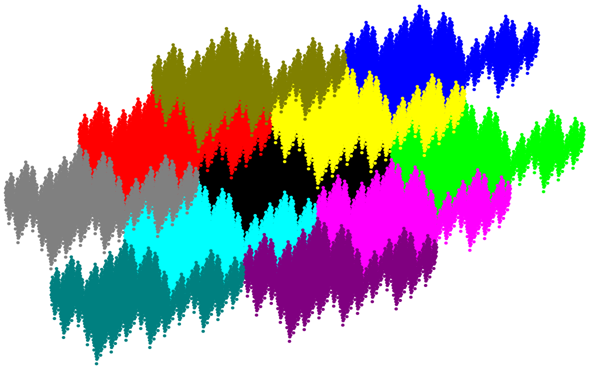

FIGURE 3. The tile $\mathcal{T}_2$ and its neighbors.

In a similar way as for the tile $\mathcal{T}_1$ we get the following analogous result for $\mathcal{T}_2$.

Theorem 8.2. *Let $\mathcal{T}_2 = \mathcal{T}_2(A_2, \mathcal{D}_2)$ be the $\mathbb{Z}^2$-tile with A_2 and $\mathcal{D}_2$ as in (8.1). Let $\{U_i\}_{i\in\mathbb{N}}$ be the set of its interior components. Then $\mathcal{T}_2$ has trivial fundamental group. Furthermore, $\mathcal{T}_2$ has a cut point and $\overline{U_i}$ is homeomorphic to a closed disk for each $i \in \mathbb{N}$.*

Rough sketch of the proof. First we need to check conditions (i),(ii) and (iii) of Theorem 7.2. To this matter we need to construct $G^T(S_2)$. This is easily done with help of one of the neighbor finding algorithms (cf. [29, 31]).

Condition (i) is checked in a very similar way as in Theorem 8.1. To check condition (ii) we have to observe that the easier variant of (ii) indicated after Lemma 7.3 is not fulfilled in this case. We have to use Proposition 2.5 here. However, because one of the intersections $(\mathcal{T}_2+d)\cap(\mathcal{T}_2+d_1)$ and $(\mathcal{T}+d)\cap(\mathcal{T}_2+d_2)$ always contains at most one point the truth of (ii) is easy to see. Condition (iii) can easily be derived with help of Proposition 2.5. Thus $\pi_1(\mathcal{T})$ is trivial. However, since $\mathcal{T}_2$ has 10 neighbors, it can not be homeomorphic to a closed disk in view of [4, Proposition 1.1]. Thus Theorem 4.4 yields the existence of a cut point of $\mathcal{T}_2$. The assertions on U_i again follow from Theorem 5.3. □

The next example is devoted to a tile with non-trivial fundamental group. Let $\mathcal{T}_3 = \mathcal{T}_3(A_3, \mathcal{D}_3)$ be the $\mathbb{Z}^2$-tile with

$$A_3 := \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}, \qquad \mathcal{D}_3 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right\}. \tag{8.3}$$

The tile $\mathcal{T}_3$ is shown in Figure 4. It has been studied for instance in [1, 25]. Again by using a neighbor finding algorithm we can construct the set of neighbors S_3 and with this set we easily get the graph $G^T(S_3)$ depicted in Figure 5 (note that

$$\begin{array}{c|c} a & a' \\ \cdot & \cdot \\ b & b' \end{array}$$

is a shortcut for $a|a'$, $a+1|a'+1, \dots, b|b'$). We will show the following result.

Theorem 8.3. *Let $\mathcal{T}_3 = \mathcal{T}_3(A_3, \mathcal{D}_3)$ be the $\mathbb{Z}^2$ tile with A_3 and $\mathcal{D}_3$ as in (8.3). Then $\pi_1(\mathcal{T}_3)$ is uncountable and not free. Furthermore, $\mathcal{T}_3$ is not locally simply connected and has no universal cover.*

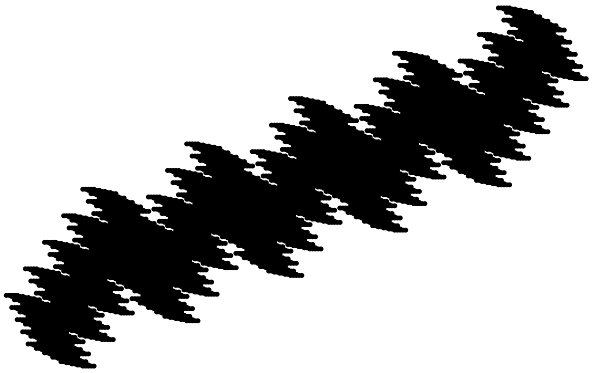

FIGURE 4. The tile $\mathcal{T}_3$.

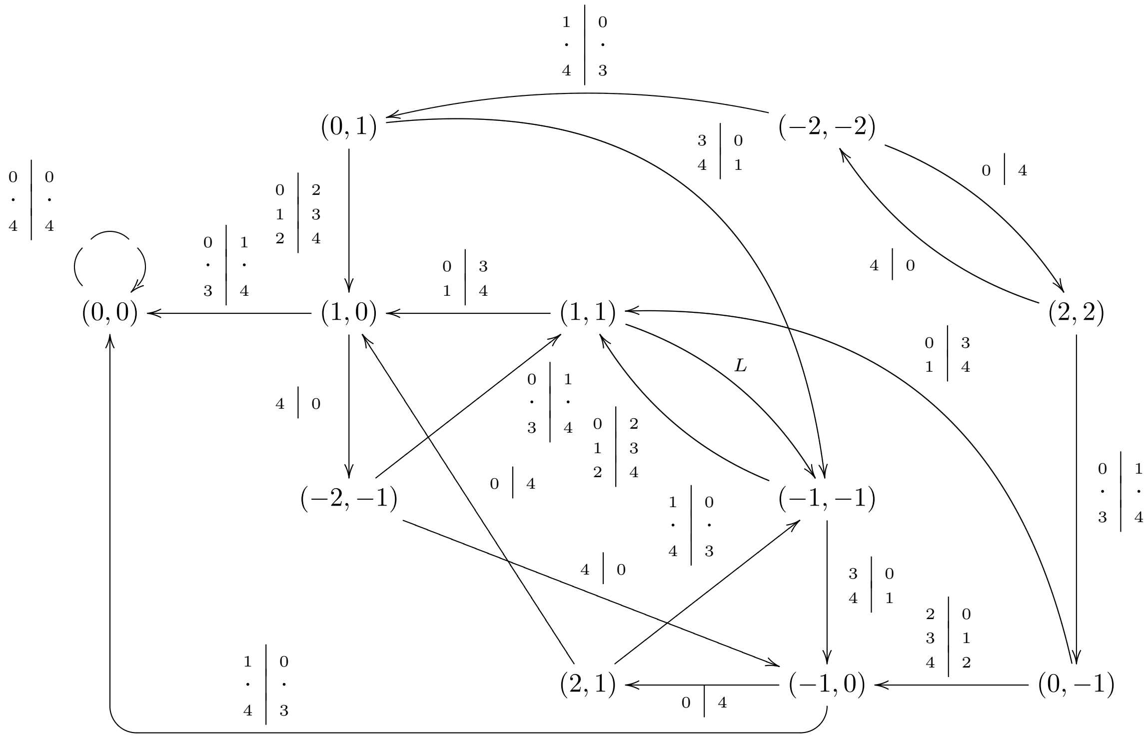

FIGURE 5. The graph $G^T(S_3)$ (L is an abbreviation for $d|d-2$, $2 \le d \le 4$)

Proof. We have to check the conditions of Theorem 6.3. Let $s_1 = (1,0)$ and $s_2 = (-1,-1)$. Then the first condition of Theorem 6.3 follows from [1, Theorem 11.1] for the special choice $A = 4, B = 5$. In order to check the second condition let the same notation as in Theorem 6.3 be in force. First we take $w_0 = 0^\infty$. With help of Figure 5 we see (using the notation of Theorem 6.3) that this choice leads to

$$c_0 = 0^\infty, \quad c_1 = 0^\infty 1, \quad c_2 = 0^\infty 132.$$

Thus $\max(\mathcal{L}(c_0), \mathcal{L}(c_1)) < \mathcal{L}(c_2)$. Now let $w_1 = 0^\infty 14404$. Then by looking at Figure 5 we see that

$$c_0 = 0^\infty 14404, \quad c_1 = 0^\infty 1210, \quad c_2 = 0^\infty 1341.$$

Thus $\max(\mathcal{L}(c_1), \mathcal{L}(c_2)) < \mathcal{L}(c_0)$. For w_2 we take $w_2 = 0^\infty 144$. Again we easily check that we get for the output strings c_0, c_1, c_2 that $\max(\mathcal{L}(c_2), \mathcal{L}(c_0)) < \mathcal{L}(c_1)$. Thus the conditions of Theorem 6.3 are satisfied and the result follows from the conclusion of that theorem. □

Next we want to give an easy example to which Theorem 6.4 can be applied. In [4, Section 2] the $\mathbb{Z}^2$-tile $\mathcal{T}_4 = \mathcal{T}_4(A, \mathcal{D})$ with

$$A_4 := \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \mathcal{D}_4 := \left\{ \begin{pmatrix} i \\ j \end{pmatrix} \;\middle|\; 0 \le i, j \le 3 \right\} \setminus \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\} \tag{8.4}$$

has been considered. It is depicted in Figure 6. From Theorem 6.4 one easily derives the following

FIGURE 6. The tile $\mathcal{T}_4$.

result on $\mathcal{T}_4$.

Theorem 8.4. *Let $\mathcal{T}_4 = \mathcal{T}_4(A_4, \mathcal{D}_4)$ be the $\mathbb{Z}^2$ tile with A_4 and $\mathcal{D}_4$ as in* (8.4)*. Then $\pi_1(\mathcal{T}_4)$ is uncountable and not free. Furthermore, $\mathcal{T}_4$ is not locally simply connected and has no universal cover.*

We conclude this paper with an example of a plane reptile which does not fit in our framework. For $\mathbb{Z}^2$-tiles the fundamental group can not be non-trivial and countable. However, this can happen for a plane tile, that is, a compact subset of $\mathbb{R}^2$ with interior points satisfying $T = \bigcup_i f_i(T)$ for a family of contractive similarities $\{f_1, \dots, f_m\}$ with the same contraction ratio $r = \sqrt{\frac{1}{m}}$. Particularly, if we divide the square $[0,1] \times [0,1]$ into 9 small squares of side length $\frac{1}{3}$ and remove the two small squares $D_1 = [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$ and $D_2 = [\frac{2}{3}, 1] \times [0, \frac{1}{3}]$, then the union T of the remaining 7 small squares and the two small squares $D_1' = [1, \frac{4}{3}] \times [\frac{2}{3}, 1]$ and $D_2' = [\frac{4}{3}, \frac{5}{3}] \times [\frac{1}{3}, \frac{2}{3}]$ is Grünbaum's 36-reptile [5]. In fact, this tile has $\pi_1(T) = \mathbb{Z}$ as can be easily seen from Figure 7. Furthermore, this tile has two interior components the closure of one of them being not not homeomorphic to a closed disc.

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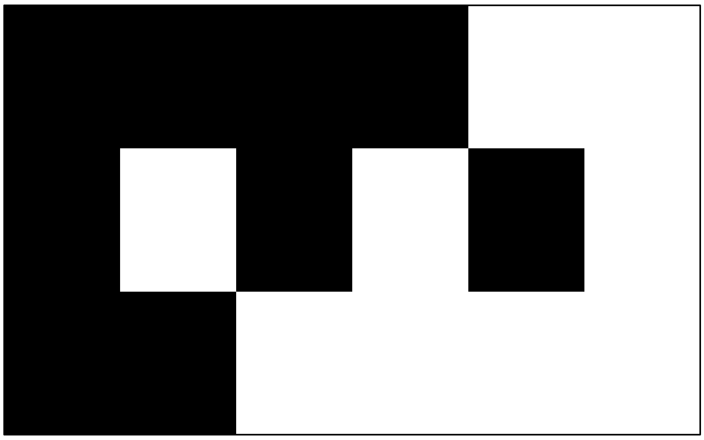

FIGURE 7. A tile T with $\pi_1(T) = \mathbb{Z}$.

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