WARING'S PROBLEM RESTRICTED BY A SYSTEM OF SUM OF DIGITS CONGRUENCES

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Abstract. The aim of the present paper is to generalize earlier work by Thuswaldner and Tichy on Waring's Problem with digital restrictions to systems of digital restrictions. Let $s_q(n)$ be the q-adic sum of digits function and let $d,\ s,\ a_l,\ m_l,\ q_l \in \mathbb{N}$. Then for $s > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$ there exists $N_0 \in \mathbb{N}$ such that each integer $N \geq N_0$ has a representation of the form

$$N = x_1^d + \cdots + x_s^d \quad \text{where} \quad s_{q_l}(x_i) \equiv a_l \bmod m_l \qquad (1 \leq i \leq s \text{ and } 1 \leq l \leq L).$$

The result, together with an asymptotic formula of the number of this representations, will be shown with the help of the circle method together with exponential sum estimates.

1. Notation

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of positive integers, integers and real numbers, respectively. A set of the shape $\{n \in \mathbb{Z} \mid a \leq n \leq b\}$ will be called *interval of integers*. The notations $e(z)$ for $\exp(2\pi i z)$, $\lfloor x \rfloor$ for the greatest integer less than or equal to $x \in \mathbb{R}$, and $\lceil x \rceil$ for the smallest integer greater than or equal to x will be used frequently. For the sake of shortness, we are going to make extensive use of vector and matrix notation throughout this paper. For example, if $v_1, \ldots, v_d$ is a finite collection of indexed numbers, then $\underline{v} = (v_1, \ldots, v_d)$ will denote the corresponding vector.

Furthermore we will use the notations $f(x) = \mathcal{O}\big(g(x)\big)$ as well as $f(x) \ll g(x)$ to express that $|f(x)| \leq c|g(x)|$ for some positive constant c and all sufficiently large $x \in \mathbb{R}$.

A function f is said to be *completely q-additive*, if for any $p,\ r,\ t \in \mathbb{N}$ with $0 \leq r < q^t$ the property $f(p \cdot q^t + r) = f(p) + f(r)$ holds. The classical example of a completely q-additive function is the the q-adic *sum of digits* function s_q which assigns to each positive integer n the sum

$$s_q(n) = c_0 + \cdots + c_r$$

of digits in its (unique) q-adic representation

$$n = c_0 + c_1 q + \cdots + c_r q^r.$$

This function will play a prominent role throughout the paper.

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2. Main results and preliminaries

A fundamental problem in additive number theory is to decide whether a given set $A \subseteq \mathbb{N}$ is a *basis* of $\mathbb{N}$, that is, if each $N \in \mathbb{N}$ admits a representation of the form

$$N = x_1 + \cdots + x_s \quad \text{with} \quad x_1, \ldots, x_s \in A.$$

We call $s \in \mathbb{N}$ the *order* of the basis. If a representation of this shape only exists if N is sufficiently large we call A an *asymptotic basis* of $\mathbb{N}$. In Waring's Problem the set A is to be taken

$$A = A_d = \{n^d \mid n \in \mathbb{N}\} \quad (d \in \mathbb{N} \text{ fixed}).$$

This problem and variants of it have been studied extensively. For details and references we refer for example to Hua [5], Nathanson [8], Vaughan [10] or Vaughan and Wooley [11].

In [9], Thuswaldner and Tichy investigated the number of representations

$$N = x_1^d + \ldots + x_s^d, \tag{2.1}$$

where the integers x_i have been additionally restricted by sum of digits congruences of the type $s_{q_i}(x_i) \equiv a_i \bmod m_i$ for given a_i, q_i, and m_i, $1 \leq i \leq s$ (cf. [9, Theorem 3.1]). As consequence, they deduced that the set $\{n^d \mid s_q(n) \equiv a \bmod m\}$ forms an asymptotic basis of order $2^d + 1$ (cf. [9, Theorem 3.2]). In the present paper we go one step further and generalize this work to systems of digital restrictions. In particular, for given positive integer $N \in \mathbb{N}$ we consider the number $r_{d,s,\underline{a},\underline{m}}(N)$ of representations (2.1) where each x_i, $1 \leq i \leq s$, simultaneously obeys a system of $L \geq 1$ sum of digits congruences

$$\begin{gathered} s_{q_1}(x_i) \equiv a_1 \bmod m_1, \\ \vdots \\ s_{q_L}(x_i) \equiv a_L \bmod m_L. \end{gathered}$$

We are going to provide an asymptotic formula for $r_{d,s,\underline{a},\underline{m}}(N)$ from which the fact that the corresponding restricted set forms an asymptotic basis will follow. We will use the abbreviation

$$s_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}$$

if $s_{q_l}(n) \equiv a_l \bmod m_l$ holds for $1 \leq l \leq L$, and denote the set of all integers that fulfill this condition by

$$U_{\underline{a},\underline{m}} = \{n \in \mathbb{N} \mid s_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}\}.$$

Sets of that kind have also been studied with different setups at first by Gelfond [3] and subsequent authors as for example by Besineau [1], Mauduit and Sárközy [7] and Kim [6].

Our main result can be summarized as follows.

Theorem 2.1. *Let d, $s \in \mathbb{N}$ and a_l, m_l, $q_l \in \mathbb{N}$ with $m_l \geq 2$, $q_l \geq 2$, $\gcd(q_l - 1, m_l) = 1$ for $1 \leq l \leq L$ and $\gcd(q_l, q_k) = 1$ for $1 \leq l < k \leq L$. If $r_{d,s,\underline{a},\underline{m}}(N)$ denotes the number of representations of N in the form*

$$N = x_1^d + \cdots + x_s^d \qquad (x_1, \ldots, x_s \in U_{\underline{a},\underline{m}}),$$

then for $s > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$ (the implied constant is absolute) there exists a positive constant δ such that

$$r_{d,s,\underline{a},\underline{m}}(N) = \frac{1}{M^s}\mathfrak{S}(N)\Gamma\left(1+\frac{1}{d}\right)^s \Gamma\left(\frac{s}{d}\right)^{-1} N^{\frac{s}{d}-1} + \mathcal{O}(N^{\frac{s}{d}-1-\delta}),$$

where $M = \prod_{l=1}^{L} m_l$. The implied constant depends only on d, s, L and $\underline{m}$. $\mathfrak{S}$ is an arithmetic function for which there exist positive constants $0 < c_1 < c_2$ depending only on d and s such that

$$c_1 < \mathfrak{S}(N) < c_2.$$

This implies that $A_{d,\underline{a},\underline{m}} = \{n^d \mid s_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}\}$ forms an asymptotic basis of order $d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$ of $\mathbb{N}$.

In order to establish this theorem, we will need the following higher correlation result for $s_q(n)$ proved by Thuswaldner and Tichy in the aforementioned paper [9]. To formulate it, let $\Delta_d\big(f(n), \underline{k}\big)$ denote the d-th iterated difference operator applied to an arithmetic function f with differences $k_1, \dots, k_d$, i.e.,

$$\Delta_1\big(f(n), k_1\big) = f(n+k_1) - f(n),$$
$$\Delta_d\big(f(n), k_1, \dots, k_d\big) = \Delta_1\Big(\Delta_{d-1}\big(f(n), k_1, \dots, k_{d-1}\big), k_d\Big) \qquad (d \geq 2).$$

Proposition 2.1 ([9], Theorem 3.3)**.** *Let d, m, h, q and N be positive integers with $m \geq 2$, $q \geq 2$ and $m \nmid h(q-1)$, and let*

$$p(d,q) = \left\lceil 2\frac{d(d+2)}{q-1} + 2d + 5 \right\rceil.$$

Let $I_1, \dots, I_d$, J be intervals of integers with $\sqrt{N} \leq |I_j|, |J| \leq N$ for $1 \leq j \leq d$. Then the estimate

$$\sum_{k_1 \in I_1} \cdots \sum_{k_d \in I_d} \left| \sum_{n \in J} e\Big(\frac{h}{m}\Delta_d\big(s_q(n), \underline{k}\big)\Big) \right|^2 \ll |I_1| \cdots |I_d| |J|^2 N^{-\eta}$$

holds with $\eta = 1/m^2 q^{p(d,q)} > 0$.

With the help of this result we will derive the following estimate which is crucial in the proof of Theorem 2.1.

Theorem 2.2. *Let d, m_l, h_l, q_l and N be positive integers with $m_l \geq 2$, $q_l \geq 2$ for $1 \leq l \leq L$, $\gcd(q_l, q_k) = 1$ for $1 \leq l < k \leq L$ and $m_l \nmid h_l(q_l - 1)$ for at least one $1 \leq l \leq L$. Then the estimate*

$$\sum_{n=0}^{N-1} e\Big(\theta n^d + \sum_{l=1}^{L} \frac{h_l}{m_l} s_{q_l}(n)\Big) \ll N^{1-\gamma}$$

holds uniformly in $\theta \in [0,1)$ with $\gamma = \eta/(6DL^2)$, where $D = 2^d$, $\eta = \min \eta_l > 0$ and $\eta_l = 1/m_l^2 q_l^{p(d,q_l)} > 0$ with $p(d,q)$ as in Proposition 2.1.

Kim [6, Proposition 2] proved a version of this result with $\theta = 0$, i.e., where the term θn^d is missing. More precisely, he showed that under the same conditions as in Theorem 2.2 for all positive integers N

$$\sum_{n=0}^{N-1} e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} f_l(n)\Big) \ll N^{1-\delta},$$

where $\delta = 1/(120m^2q^3L^2)$ with $m = \max m_l$, $q = \max q_l$, and the implied constant depends only on q and L. In fact, Kim's result is even more general since it admits arbitrary completely q_l-additive functions f_l instead of the s_{q_l}.

On the other hand, Thuswaldner and Tichy [9, Theorem 3.4] provided the case $L = 1$ of our result. They showed that if d, m, h, q and N are positive integers with $m \geq 2$, $q \geq 2$ and $m \nmid h(q-1)$, then the estimate

$$\sum_{n=0}^{N-1} e\left(\theta n^d + \frac{h}{m} s_q(n)\right) = N^{1-\varepsilon}$$

holds uniformly in $\theta \in [0, 1)$ with $\varepsilon = \eta 2^{-(d+1)}$ and η as in Proposition 2.1. Comparing this with the special case $L = 1$ of Theorem 2.2, their saving ε is obviously better than our γ, which is inherently due to the different method (following Kim [6]) applied.

Theorem 2.2 constitutes a generalization of both of these results. Note that it even remains valid if the term θn^d is replaced by an arbitrary polynomial in n of degree d. In order to establish Theorem 2.2 we have to adapt the proof of [6, Proposition 2] to our more sophisticated situation. This will lead to exponential sums which can be estimated with help of Proposition 2.1. Theorem 2.1 will then follow from Theorem 2.2 by an application of the circle method.

Next we are going to provide two preliminary lemmata. The following is a generalization of [6, Lemma 6].

Lemma 2.1. *Let f be a completely q-additive function. Then*

$$\Delta_d\big(f(n), \underline{k}\big) = \Delta_d\big(f(r), \underline{k}\big)$$

for all positive $k_1, \ldots, k_d$ and n with $n \equiv r \bmod q^t$, where $0 \leq r < q^t - k_1 - \ldots - k_d$.

Proof. It is easy to see that the d-th iterated difference operator can be written explicitly as

$$\Delta_d\big(f(n), \underline{k}\big) = \sum_{I \subseteq \{1,\ldots,d\}} (-1)^{d-|I|} f\Big(n + \sum_{i \in I} k_i\Big).$$

Let $n = p \cdot q^t + r$. Since $r + \sum_{i\in I} k_i < q^t$ for any selection of the subset of indices I, we can exploit the q-additivity of f and obtain

$$\Delta_d\big(f(n), \underline{k}\big) = \sum_{I\subseteq\{1,\dots,d\}} (-1)^{d-|I|} f\Big(pq^t + r + \sum_{i\in I} k_i\Big) =$$

$$= f(p) \underbrace{\sum_{I\subseteq\{1,\dots,d\}} (-1)^{d-|I|}}_{=0} + \sum_{I\subseteq\{1,\dots,d\}} (-1)^{d-|I|} f\Big(r + \sum_{i\in I} k_i\Big) = \Delta_d\big(f(r), \underline{k}\big).$$

□

The following inequality is a variant of [4, Lemma 2.7] which is itself an iteration of the ordinary Weyl-van der Corput inequality.

Lemma 2.2. *Let $D = 2^d$ and $K \geq 1$. Then the inequality*

$$\left|\sum_{n=0}^{N-1} e\big(\varphi(n)\big)\right|^D \leq 32^{D-1}\left(\frac{N^D}{K} + \frac{N^{D-1}}{K^d}\sum_{k_1=1}^{K}\dots\sum_{k_d=1}^{K}\left|\sum_{n=0}^{N-k_1-\dots-k_d-1} e\Big(\Delta_d\big(\varphi(n), \underline{k}\big)\Big)\right|\right) \tag{2.2}$$

holds for any arithmetic function $\varphi(n)$.

Proof. We only give a sketch of the easy proof. Let A_j be defined recursively by $A_1 = 1$ and $A_j = A_{j-1}^2 \cdot j$ for $j \geq 2$. Starting from the ordinary Weyl-van der Corput inequality (cf. [6, Lemma 4]), we obtain

$$\left|\sum_{n=0}^{N-1} e\big(\varphi(n)\big)\right|^D \leq$$

$$\leq A_d\left(N^D \sum_{j=1}^{d} \frac{2^{2D-3\cdot 2^{d-j}}}{K^{2^{d-j}}} + N^{D-1}\frac{2^{2D-2}}{K^d}\sum_{k_1=1}^{K}\dots\sum_{k_d=1}^{K}\left|\sum_{n=0}^{N-k_1-\dots-k_d-1} e\Big(\Delta_d\big(\varphi(n), \underline{k}\big)\Big)\right|\right)$$

by induction and iterated application of Cauchy-Schwarz's inequality. Again by induction one can show that $A_d \leq 2^{2D-d-1}$ which is in turn $\leq 2^{2D-2}$. Now

$$\sum_{j=1}^{d} \frac{2^{2D-3\cdot 2^{d-j}}}{K^{2^{d-j}}} \leq \frac{\max_j 2^{2D-3\cdot 2^{d-j}}}{\min_j K^{2^{d-j}}} \sum_{j=1}^{d} 1 = \frac{d\cdot 2^{2D-3}}{K}.$$

Since we are only interested in a result similar to [4, Lemma 2.7], we generously estimate the nominator by 2^{3D-3}. This yields inequality (2.2). □

3. Proof of Theorem 2.2

In this section we are going to derive Theorem 2.2 from Proposition 2.1 and we do this by following the proof of [6, Proposition 2]. Let $q = \max q_l$. We have to investigate the problem only for $N \geq q^{3L}$, because for $1 \leq N \leq q^{3L}$ the estimate holds trivially. Set $K := \lfloor N^{1/3L}\rfloor \geq q \geq 2$, and let $Q_l = q_l^{e_l}$ such that

$$2 \leq K \leq K^2 q_l^{-1} \leq Q_l \leq K^2 \tag{3.1}$$

for $1 \le l \le L$. This can be achieved by choosing $e_l = \lfloor 2\log K/\log q_l \rfloor$.

We start from the iterated Weyl-van der Corput inequality (2.2) with

$$\varphi(n) = \theta n^d + \sum_{l=1}^{L} \frac{h_l}{m_l} s_{q_l}(n),$$

so that the left hand side of inequality (2.2) is the D-th power of the exponential sum we want to estimate. Since $\Delta_d(\theta n^d, \underline{k}) = \theta d!\ k_1 \dots k_d$ is constant with respect to n, and by the linearity of Δ_d,

$$\Delta_d\bigl(\varphi(n), \underline{k}\bigr) = \theta d! \cdot k_1 \dots k_d + \sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(n), \underline{k}\bigr),$$

we have for fixed $k_1, \dots, k_d$

(3.2)

$$\Bigl|\sum_{n=0}^{N-k_1-\dots-k_d-1} e\Bigl(\Delta_d\bigl(\varphi(n), \underline{k}\bigr)\Bigr)\Bigr| = \underbrace{|e(\theta d!\ k_1 \dots k_d)|}_{=1} \cdot \Bigl|\sum_{n=0}^{N-k_1-\dots-k_d-1} e\Bigl(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(n), \underline{k}\bigr)\Bigr)\Bigr|.$$

At this point we distinguish in which residue class $n \bmod \underline{Q}$ lies. To accomplish this, let the sets $\mathcal{R}$ and $\mathcal{R}_0$ be defined by

$$\mathcal{R} = \{\underline{r} \in \mathbb{Z}^L \mid 0 \le r_l \le Q_l - 1 \text{ for } 1 \le l \le L\},$$
$$\mathcal{R}_0 = \{\underline{r} \in \mathbb{Z}^L \mid 0 \le r_l \le Q_l - d \cdot K - 1 \text{ for } 1 \le l \le L\}.$$

Furthermore, for $\underline{r} \in \mathcal{R}$ let

$$P_{\underline{r}} = \{n \in \mathbb{Z} \mid n \equiv \underline{r} \bmod \underline{Q}\},$$

where $n \equiv \underline{r} \bmod \underline{Q}$ means that $n \equiv r_l \bmod Q_l$ for $1 \le l \le L$. With help of these sets we rewrite the sum under the rightmost modulus of equation (3.2) in the following way:

$$\sum_{n=0}^{N-k_1-\dots-k_d-1} e\Bigl(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(n), \underline{k}\bigr)\Bigr) = \sum_{\underline{r}\in\mathcal{R}} \sum_{\substack{n=0\\ n\in P_{\underline{r}}}}^{N-k_1-\dots-k_d-1} e\Bigl(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(n), \underline{k}\bigr)\Bigr) = \tag{3.3}$$
$$= \sum_{\underline{r}\in\mathcal{R}_0} + \sum_{\underline{r}\in\mathcal{R}\setminus\mathcal{R}_0}.$$

To the first sum we can now apply Lemma 2.1 to obtain

$$\sum_{\underline{r}\in\mathcal{R}_0} \sum_{\substack{n=0\\ n\in P_{\underline{r}}}}^{N-k_1-\dots-k_d-1} e\Bigl(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(n), \underline{k}\bigr)\Bigr) =$$
$$= \sum_{\underline{r}\in\mathcal{R}_0} e\Bigl(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\bigl(s_{q_l}(r_l), \underline{k}\bigr)\Bigr) \sum_{\substack{n=0\\ n\in P_{\underline{r}}}}^{N-k_1-\dots-k_d-1} 1 = \sum_{\underline{r}\in\mathcal{R}} - \sum_{\underline{r}\in\mathcal{R}\setminus\mathcal{R}_0}.$$

Substituting this back into equation (3.3) leads to

(3.4)
$$\sum_{n=0}^{N-k_1-\ldots-k_d-1} e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\big(s_{q_l}(n), \underline{k}\big)\Big) = \sum_{\underline{r}\in\mathcal{R}} e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\big(s_{q_l}(r_l), \underline{k}\big)\Big) \sum_{\substack{n=0\\ n\in P_{\underline{r}}}}^{N-k_1-\ldots-k_d-1} 1 +$$
$$+ \sum_{\underline{r}\in\mathcal{R}\setminus\mathcal{R}_0} \sum_{\substack{n=0\\ n\in P_{\underline{r}}}}^{N-k_1-\ldots-k_d-1} \bigg(e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\big(s_{q_l}(n), \underline{k}\big)\Big) - e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\big(s_{q_l}(r_l), \underline{k}\big)\Big)\bigg) =$$
$$= \sum\nolimits_1 + \sum\nolimits_2 .$$

Now we argue along the same lines as Kim [6, p. 330]. Since the Q_l are pairwise coprime (as powers of the q_l), the system of congruences that defines the set $P_{\underline{r}}$ is equivalent to a single congruence mod $\prod_{l=1}^{L} Q_l$ by the Chinese Remainder Theorem, and therefore

$$\big|\{n < N \mid n \equiv \underline{r} \bmod \underline{Q}\}\big| = \frac{N}{\prod_{l=1}^{L} Q_l} + \mathcal{O}(1).$$

Hence for the first sum we get

$$\sum\nolimits_1 = \sum_{r_1=0}^{Q_1-1} \cdots \sum_{r_L=1}^{Q_L-1} \prod_{l=1}^{L} e\Big(\frac{h_l}{m_l} \Delta_d\big(s_{q_l}(r_l), \underline{k}\big)\Big)\bigg(\frac{N}{\prod_{l=1}^{L} Q_l} + \mathcal{O}(1)\bigg) =$$
$$= N \cdot \prod_{l=1}^{L} \frac{1}{Q_l} \sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l} \Delta_d\big(s_{q_l}(r_l), \underline{k}\big)\Big) + \mathcal{O}\Big(\prod_{l=1}^{L} Q_l\Big).$$

Since $\prod_{l=1}^{L} Q_l \leq K^{2L} \leq N/K$ by (3.1), the error term in the last equation is of order $\mathcal{O}(N/K)$. The second sum is bounded trivially by

$$\sum\nolimits_2 \leq 2 \cdot |\mathcal{R}\setminus\mathcal{R}_0| \cdot \bigg(\frac{N}{\prod_{l=1}^{L} Q_l} + \mathcal{O}(1)\bigg) \leq 2dqL\frac{N}{K} + \mathcal{O}\Big(\frac{1}{K} \prod_{l=1}^{L} Q_l\Big),$$

where we used the estimate from [6, p. 331] for $\mathcal{R}\setminus\mathcal{R}_0$. By the same argument as before the final error term is of order $\mathcal{O}(N/K)$. Substituting these two estimates again back into equation (3.4) we obtain

$$\sum_{n=0}^{N-k_1-\ldots-k_d-1} e\Big(\sum_{l=1}^{L} \frac{h_l}{m_l} \Delta_d\big(s_{q_l}(n), \underline{k}\big)\Big) = N \cdot \prod_{l=1}^{L} \frac{1}{Q_l} \sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l} \Delta_d\big(s_{q_l}(r_l), \underline{k}\big)\Big) + \mathcal{O}\Big(\frac{N}{K}\Big).$$

Now from the iterated Weyl-van der Corput inequality (2.2) we get

$$(3.5)\quad \left|\sum_{n=0}^{N-1} e\Big(\theta n^d + \sum_{l=1}^{L} \frac{h_l}{m_l} s_{q_l}(n)\Big)\right|^D \leq$$
$$\leq 32^{D-1}\left(\frac{N^D}{K} + \frac{N^D}{K^d}\sum_{k_1=1}^{K}\cdots\sum_{k_d=1}^{K}\prod_{l=1}^{L}\left|\frac{1}{Q_l}\sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l}\Delta_d\big(s_{q_l}(r_l),\underline{k}\big)\Big)\right|\right) + \mathcal{O}\Big(\frac{N^D}{K^{d+1}}\Big).$$

By setting

$$\sum\nolimits_3 = \frac{1}{K^d}\sum_{k_1=1}^{K}\cdots\sum_{k_d=1}^{K}\prod_{l=1}^{L}\left|\frac{1}{Q_l}\sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l}\Delta_d\big(s_{q_l}(r_l),\underline{k}\big)\Big)\right|$$

and applying Hölder's inequality to this term we obtain

$$\sum\nolimits_3 \leq \frac{\big(K^d\big)^{\frac{1}{L+1}}}{K^d}\cdot\prod_{l=1}^{L}\left(\sum_{k_1=1}^{K}\cdots\sum_{k_d=1}^{K}\left|\frac{1}{Q_l}\sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l}\Delta_d\big(s_{q_l}(r_l),\underline{k}\big)\Big)\right|^{L+1}\right)^{\frac{1}{L+1}} \leq$$
$$\leq \prod_{l=1}^{L}\left(\frac{1}{K}\sum_{k_1=1}^{K}\cdots\frac{1}{K}\sum_{k_d=1}^{K}\left|\frac{1}{Q_l}\sum_{r_l=0}^{Q_l-1} e\Big(\frac{h_l}{m_l}\Delta_d\big(s_{q_l}(r_l),\underline{k}\big)\Big)\right|^{2}\right)^{\frac{1}{L+1}},$$

where the last inequality is valid since the value under the modulus is at most 1. Now the last term is a product of sums of the type being estimated in Proposition 2.1. By assumption, for at least one $1 \leq l \leq L$ we have $m_l \nmid h_l(q_l-1)$, say ℓ. Since $K \leq Q_l \leq K^2$ for any $1 \leq l \leq L$ by (3.1), or in other terms $\sqrt{Q_l} \leq K \leq Q_l$, this means that we can in fact apply Proposition 2.1 to get

$$\frac{1}{K}\sum_{k_1=1}^{K}\cdots\frac{1}{K}\sum_{k_d=1}^{K}\left|\frac{1}{Q_\ell}\sum_{r_\ell=0}^{Q_\ell-1} e\Big(\frac{h_\ell}{m_\ell}\Delta_d\big(s_{q_\ell}(r_\ell),\underline{k}\big)\Big)\right|^{2} \ll Q_\ell^{-\eta_\ell} \ll K^{-\eta_\ell}$$

with $\eta_l > 0$ as in Theorem 2.2. Setting $\eta = \min \eta_l > 0$ yields $K^{-\eta_\ell} \ll K^{-\eta}$, and this is $\ll N^{-\eta/3L}$ because $K = \lfloor N^{1/3L}\rfloor \geq (1/2)N^{1/3L}$. Estimating the remaining factors corresponding to $l \neq \ell$ trivially by 1, we obtain

$$\sum\nolimits_3 \ll N^{-\eta/(3L(L+1))} \ll N^{-\eta/(6L^2)}$$

and, observing inequality (3.5), finally

$$\left|\sum_{n=0}^{N-1} e\Big(\theta n^d + \sum_{l=1}^{L} \frac{h_l}{m_l} s_{q_l}(n)\Big)\right|^D \leq 32^{D-1}N^D\sum\nolimits_3 + \mathcal{O}\big(\frac{N^D}{K}\big) + \mathcal{O}\big(\frac{N^D}{K^{d+1}}\big) =$$
$$= \mathcal{O}\big(N^{D-\eta/(6L^2)}\big) + \mathcal{O}\big(N^{D-1/3L}\big) + \mathcal{O}\big(N^{D-(d+1)/(3L)}\big).$$

Setting $\gamma = \eta/(6DL^2)$, the three summands on the right hand side are all $\mathcal{O}\big(N^{D(1-\gamma)}\big)$, and taking the D-th root yields Theorem 2.2.

4. Application of the circle method

In this section we are going to prove Theorem 2.1. We will follow the lines of Thuswaldner and Tichy [9, Section 9] to a great extent. Let $P = \lfloor N^{1/d} \rfloor$. Using the identity

$$\prod_{l=1}^{L} \frac{1}{m_l} \sum_{h_l=0}^{m_l-1} e\Big(\frac{h_l}{m_l}\big(s_{q_l}(n) - a_l\big)\Big) = \begin{cases} 1 & \text{if } s_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}, \\ 0 & \text{otherwise,} \end{cases}$$

elementary methods of additive number theory yield

$$r_{d,s,\underline{a},\underline{m}}(N) = \int_0^1 F(\theta)^s e(-\theta N) d\theta \tag{4.1}$$

for the number $r_{d,s,\underline{a},\underline{m}}(N)$ in question, where

$$\begin{aligned} F(\theta) &= \sum_{\substack{n=0 \\ n \in U_{\underline{a},\underline{m}}}}^{P-1} e(\theta n^d) = \sum_{n=0}^{P-1} e(\theta n^d) \prod_{l=1}^{L} \frac{1}{m_l} \sum_{h_l=0}^{m_l-1} e\Big(\frac{h_l}{m_l}\big(s_{q_l}(n) - a_l\big)\Big) = \\ &= \frac{1}{M} \sum_{n=0}^{P-1} \sum_{h_1=0}^{m_1-1} \dots \sum_{h_L=0}^{m_L-1} e\Big(\theta n^d + \sum_{l=1}^{L} \frac{h_l}{m_l}\big(s_{q_l}(n) - a_l\big)\Big) \end{aligned}$$

and $M = \prod_{l=1}^{L} m_l$. With the help of the $(L \times s)$-matrices $\mathcal{H} = (h_{li})_{1 \le l \le L, 1 \le i \le s}$ and $\mathcal{M} = (m_l)_{1 \le l \le L, 1 \le i \le s}$ we can express $F(\theta)^s$ as

$$\begin{aligned} F(\theta)^s &= \frac{1}{M^s} \prod_{i=1}^{s} \sum_{n_i=0}^{P-1} \sum_{h_{1i}=0}^{m_1-1} \dots \sum_{h_{Li}=0}^{m_L-1} e\Big(\theta n_i^d + \sum_{l=1}^{L} \frac{h_{li}}{m_l}\big(s_{q_l}(n_i) - a_l\big)\Big) = \\ &= \frac{1}{M^s} \sum_{0 \le \mathcal{H} < \mathcal{M}} \prod_{i=1}^{s} \sum_{n_i=0}^{P-1} e\Big(\theta n_i^d + \sum_{l=1}^{L} \frac{h_{li}}{m_l}\big(s_{q_l}(n_i) - a_l\big)\Big), \end{aligned}$$

where the matrix inequality is to be understood as follows:

$$0 \le \mathcal{H} < \mathcal{M} = \{\mathcal{H} \in \mathbb{Z}^{(L \times s)} \mid 0 \le h_{li} < m_l \text{ for } 1 \le i \le s \text{ and } 1 \le l \le L\}.$$

Inserting this expression into (4.1) leads to

$$r_{d,s,\underline{a},\underline{m}}(N) = \frac{1}{M^s} \sum_{0 \le \mathcal{H} < \mathcal{M}} \underbrace{\int_0^1 \prod_{i=1}^{s} \sum_{n_i=0}^{P-1} e\Big(\theta n_i^d + \sum_{l=1}^{L} \frac{h_{li}}{m_l}\big(s_{q_l}(n_i) - a_l\big)\Big) e(-\theta N) d\theta}_{\mathcal{I}_{\mathcal{H}}}.$$

Concerning the integrals $\mathcal{I}_{\mathcal{H}}$ with $\mathcal{H} \ne 0$ we observe that

$$|\mathcal{I}_{\mathcal{H}}| \le \int_0^1 \prod_{i=1}^{s} |S_i(\theta)| d\theta \le \sup_{\theta, i} \big(|S_i(\theta)|^{s-2t}\big) \cdot \max_j \Big(\int_0^1 |S_j(\theta)|^{2t} d\theta\Big) \tag{4.2}$$

for any t with $s > 2t$, where

$$S_i(\theta) = \sum_{n=0}^{P-1} e\Big(\theta n^d + \sum_{l=1}^{L} \frac{h_{li}}{m_l} s_{q_l}(n)\Big)$$

are the essential sums to which we can apply Theorem 2.2. Writing integral (4.2) as in the proof of the classical Lemma of Hua (cf. [10, Lemma 2.5])

$$\int_0^1 |S_j(\theta)|^{2t} d\theta = \sum_{\substack{n_1,\dots,n_{2t}<P \\ n_1^d+\dots+n_t^d = n_{t+1}^d+\dots+n_{2t}^d}} e\Big(\sum_{l=1}^{L} \frac{h_{lj}}{m_l} \sum_{k=1}^{t} \big(s_{q_l}(n_k) - s_{q_l}(n_{t+k})\big)\Big)$$

and applying Ford [2, Equation 5.4], this integral can obviously be further bounded from above by

$$\int_0^1 |S_j(\theta)|^{2t} d\theta \le \Big|\big\{n_1,\dots,n_{2t} < P \mid n_1^d + \dots + n_t^d = n_{t+1}^d + \dots + n_{2t}^d\big\}\Big| \ll P^{2t-d}$$

for the currently best known lower bound $t > (1/2)d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$ (where the implied constant is absolute). Substituting the last estimate together with the one from Theorem 2.2 back into equation (4.2) we obtain

$$\mathcal{I}_{\mathcal{H}} \ll \big(P^{1-\gamma}\big)^{s-2t} \cdot P^{2t-d} = P^{s-d-\delta}$$

with $\delta = \gamma(s-2t) > 0$, which holds for $s > 2t > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$.

The integral $\mathcal{I}_0$ is well-known from the ordinary Waring's Problem and can be evaluated using the Hardy-Littlewood asymptotic formula, which also holds for $s > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$ (cf. Nathanson [8, Theorem 5.7], Vaughan and Wooley [11, p. 11] and Ford [2]). Putting $P = \lfloor N^{1/d} \rfloor$ back into the estimates completes the proof.

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