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# On Waring's and Tarry's problem with digital restrictions

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**Abstract.** *In the present paper we prove Hardy-Littlewood type asymptotic formulæ for several variants of Waring's problem subject to restrictions defined in terms of digital sums. In particular, we provide "digital" versions of Waring's problem, Waring's problem for polynomials and Tarry's problem.*

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## 1. Introduction

A set $A \subseteq \mathbb{N}$ is called a basis of $\mathbb{N}$ if there exists an integer $s$ such that each $N \in \mathbb{N}$ can be written as a sum of $s$ elements of $A$. If this is true only for each sufficiently large integer $N \in \mathbb{N}$, $A$ is called an asymptotic basis. The least $s$ with this property is called the order of the basis (or asymptotic basis, respectively).

The choice $A = \mathcal{A}_d = \{n^d \mid n \in \mathbb{N}\}$ corresponds to Waring's problem. We use the common notations $g(d)$ and $G(d)$ for the smallest possible number $s$ such that $\mathcal{A}_d$ is a basis or asymptotic basis of order $s$, respectively.

Let $r_{s,d}(N)$ denote the number of solutions of the equation

$$n_1^d + \cdots + n_s^d = N.$$

Hardy and Littlewood [**7**] were the first to give an asymptotic formula of the type

$$r_{s,d}(N) = \mathfrak{S}(N)\,\Gamma\Big(1+\frac{1}{d}\Big)^s \Gamma\Big(\frac{s}{d}\Big)^{-1} N^{s/d-1} + \mathcal{O}\big(N^{s/d-1-\varepsilon}\big) \quad (s > 2^d). \tag{1}$$

Here, $\mathfrak{S}$ denotes a positive bounded function, $\Gamma$ is the gamma function and the implied constant depends only on $s, d$ and $\varepsilon$. Obviously, this shows that $\mathcal{A}_d$ forms an asymptotic basis of order $2^d + 1$.

While the problem of determining $g(d)$ is "nearly completely" solved (cf. for instance [**16, 17**]), the upper bound for $G(d)$ has been continuously improved throughout the last century. The best currently known bound is due to Wooley [**19, 20**] and reads

$$G(d) \leq d\left(\log d + \log\log d + 2 + \mathcal{O}\left(\frac{\log\log d}{\log d}\right)\right).$$

For a survey on previous bounds for $G(d)$ as well as other results related to Waring's problem we refer to [**16, 17**] where also an extensive list of references is given.

The least $s$ which admits an asymptotic formula of the shape (1) has also been investigated. Call this number $\tilde{G}(d)$. Ford [**4**] proved that

$$\tilde{G}(d) \leq d^2\big(\log d + \log\log d + \mathcal{O}(1)\big). \tag{2}$$

Closely connected to Waring's problem is Tarry's problem, which asks for the least number $s$ that permits a simultaneous solution of the system

$$\begin{aligned} n_1 + \cdots + n_s &= N_1,\\ n_1^2 + \cdots + n_s^2 &= N_2,\\ &\vdots\\ n_1^d + \cdots + n_s^d &= N_d. \end{aligned} \tag{3}$$

Obviously, this system can have solutions only if there exist some relations between the integers $N_i$ $(1 \leq i \leq d)$ (cf. Hua [**11**] and Section 4). This problem is also referred to as Prouhet-Tarry-Escott problem in the literature. Solutions of the corresponding symmetric system

$$n_1^i + \cdots + n_s^i = n_{s+1}^i + \cdots + n_{2s}^i \quad (1 \leq i \leq d)$$

are often called *ideal solutions* (cf. Borwein and Ingalls [**2**], Chen [**3**], Weisstein [**18**], Wooley [**21**]).

In the present paper we are interested in solutions of Waring's and Tarry's problem subject to certain restrictions defined via congruence properties of sum of digits functions.

For $q \in \mathbb{N}$, $q \geq 2$ let

$$n = c_0 + c_1 q + \cdots + c_r q^r$$

be the $q$-ary representation of an integer $n \in \mathbb{N}$. The $q$-ary *sum of digits function* $S_q$ of $n$ is defined by

$$S_q(n) = c_0 + \cdots + c_r.$$

Gel'fond [**6**] showed that $S_q$ is uniformly distributed in residue classes, i.e. that

$$\big|\{1 \leq n \leq N \mid S_q(n) \equiv a \bmod m\}\big| = \frac{N}{m} + \mathcal{O}(N^{1-\delta})$$

for any $a$, $q$, $m \in \mathbb{N}$ with $q \geq 2$, $m \geq 2$ and $\gcd(q-1, m) = 1$. The implied constant as well as $\delta$ only depend on $q$ and $m$. Gel'fond conjectured that an analogous result would also hold for systems of $L \geq 1$ simultaneous congruences

$$\begin{aligned} S_{q_1}(n) &\equiv a_1 \bmod m_1, \\ S_{q_2}(n) &\equiv a_2 \bmod m_2, \\ &\vdots \\ S_{q_L}(n) &\equiv a_L \bmod m_L. \end{aligned} \tag{4}$$

We introduce the vector notation

$$\underline{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_L \end{pmatrix}, \qquad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{pmatrix}, \qquad \underline{m} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_L \end{pmatrix},$$

to write (4) shortly as

$$S_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}.$$

In the following, we will speak of *sum of digits congruence restrictions* or *digital restrictions* whenever conditions of this type appear. Besineau [**1**] could prove Gel'fond's conjecture without obtaining an explicit error term, and recently Kim [**13**] was able to establish the result with the expected error term. Namely, he proved that

$$\left|\{1 \leq n \leq N \mid S_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}\}\right| = \frac{N}{M} + \mathcal{O}(N^{1-\delta}),$$

where $M = \prod_{l=1}^{L} m_l$. The implied constant as well as $\delta$ only depend on $\underline{q}$ and $\underline{m}$.

A version of Waring's problem with digital restrictions has been considered by Pfeiffer, Thuswaldner and Tichy [**14, 15**]. They could establish a Hardy-Littlewood type asymptotic formula for the number of solutions of the equation

$$n_1^d + \cdots + n_s^d = N$$

where the numbers $n_i \in \mathbb{N}$ $(1 \leq i \leq s)$ satisfy the system of congruences (4).

The present paper has the following aims.

- We present an improvement of the bound for $s$ from [**15**] (compare the alternative approach in [**14**]). In fact, our asymptotic formula for Waring's problem with digital restrictions is valid for all $s > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$. Note that this bound is the same as the currently best known bound for $\tilde{G}(d)$ given in (2).
- We want to prove Hardy-Littlewood type asymptotic formulæ for certain variants of Waring's problem. In particular, we will provide results for "digital" versions of Waring's problem with polynomials as well as Tarry's problem.

The paper is organized as follows. In Section 2 we will present a fairly general way which permits to reduce a "Waring-type" problem with digital restrictions to the corresponding problem without restrictions (see Proposition 1). In Sections 3 and 4 we apply these results to improve the bounds for $s$ obtained in [**15**]. Furthermore, we give the asymptotic formula for the "digital versions" of Waring's problem with polynomials as well as Tarry's problem.

## 2. The general setting

Let $D \geq 1$, and let $f_1(x), \ldots, f_D(x) \in \mathbb{Q}[x]$ denote arbitrary integer-valued polynomials. We may assume without loss of generality that $\deg f_1 \leq \deg f_2 \leq \ldots \leq \deg f_D$. Consider the system of Diophantine equations

$$\begin{aligned} f_1(n_1) + \cdots + f_1(n_s) &= N_1, \\ f_2(n_1) + \cdots + f_2(n_s) &= N_2, \\ &\vdots \\ f_D(n_1) + \cdots + f_D(n_s) &= N_D. \end{aligned}$$

This is a natural generalization of Waring's as well as Tarry's problem. Using the vector notation

$$\underline{f}(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \\ \vdots \\ f_D(n) \end{pmatrix}, \qquad \underline{N} = \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_D \end{pmatrix}, \qquad \underline{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_D \end{pmatrix}$$

(where the vector $\underline{\alpha}$ is going to be used later), this system of equations can be written as a single vector equation

$$\underline{f}(n_1) + \cdots + \underline{f}(n_s) = \underline{N}. \tag{5}$$

As outlined in the introduction, we are interested in those solutions of (5), which satisfy digital restrictions of the form

$$S_{\underline{q}}(n_i) \equiv \underline{a} \bmod \underline{m} \quad \text{for} \quad 1 \leq i \leq s. \tag{6}$$

We adopt the following notations:

- $r_{s,\underline{f}}(\underline{N})$ denotes the number of solutions of (5).
- $r_{s,\underline{f},\underline{a},\underline{m}}(\underline{N})$ denotes the number of solutions of (5) satisfying (6).
- $r_{s,\underline{f}}(\underline{N}, P)$ denotes the number of solutions of (5) satisfying $1 \leq n_i \leq P$ for all $1 \leq i \leq s$.
- $r_{s,\underline{f},\underline{a},\underline{m}}(\underline{N}, P)$ denotes the number of solutions of (5) satisfying $1 \leq n_i \leq P$ for all $1 \leq i \leq s$ and (6).

Let

$$F_{\underline{a},\underline{m}}(\underline{\alpha}) = \sum_{\substack{n=1 \\ S_{\underline{q}}(n) \equiv \underline{a} \bmod \underline{m}}}^{P} e\big(\underline{\alpha} \cdot \underline{f}(n)\big), \tag{7}$$

where "$\cdot$" denotes the usual scalar product. Using the orthogonality relation

$$\int_0^1 e(n\alpha) d\alpha = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

of the function $e(z)=\exp(2\pi iz)$, standard methods of additive number theory lead to the formula

$$r_{s,\underline{f},\underline{a},\underline{m}}(\underline{N},P)=\int_0^1\cdots\int_0^1 F_{\underline{a},\underline{m}}(\underline{\alpha})^s e(-\underline{\alpha}\cdot\underline{N})d\underline{\alpha}.$$

In the next step we want to get rid of the system of simultaneous congruence relations occurring in the range of the sum in (7). To this matter we use the discrete orthogonality relation

$$\frac{1}{m}\sum_{h=0}^{m-1}e\Big(\frac{n}{m}h\Big)=\begin{cases}1 & \text{if } n\equiv 0 \bmod m,\\ 0 & \text{otherwise.}\end{cases}$$

Indeed, this relation implies that

$$\prod_{l=1}^{L}\frac{1}{m_l}\sum_{h_l=0}^{m_l-1}e\Big(\frac{h_l}{m_l}\big(S_{q_l}(n)-a_l\big)\Big)=\begin{cases}1 & \text{if } S_{\underline{q}}(n)\equiv \underline{a} \bmod \underline{m},\\ 0 & \text{otherwise.}\end{cases}$$

Using this in (7) we obtain

$$F_{\underline{a},\underline{m}}(\underline{\alpha})=\frac{1}{M}\sum_{h_1=0}^{m_1-1}\cdots\sum_{h_L=0}^{m_L-1}\sum_{n=1}^{P}e\Big(\underline{\alpha}\cdot\underline{f}(n)+\sum_{l=1}^{L}\frac{h_l}{m_l}\big(S_{q_l}(n)-a_l\big)\Big).$$

With the help of the $(L\times s)$-matrices

$$\mathcal{H}=\begin{pmatrix}h_{11} & h_{12} & \dots & h_{1s}\\ h_{21} & h_{22} & \dots & h_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ h_{L1} & h_{L2} & \dots & h_{Ls}\end{pmatrix}\quad\text{and}\quad \mathcal{M}=\begin{pmatrix}m_1 & m_1 & \dots & m_1\\ m_2 & m_2 & \dots & m_2\\ \vdots & \vdots & \ddots & \vdots\\ m_L & m_L & \dots & m_L\end{pmatrix}$$

we can express $F_{\underline{a},\underline{m}}(\underline{\alpha})^s$ by

$$F_{\underline{a},\underline{m}}(\underline{\alpha})^s=\frac{1}{M^s}\sum_{0\le\mathcal{H}<\mathcal{M}}\prod_{i=1}^{s}\sum_{n_i=1}^{P}e\Big(\underline{\alpha}\cdot\underline{f}(n_i)+\sum_{l=1}^{L}\frac{h_{li}}{m_l}\big(S_{q_l}(n_i)-a_l\big)\Big),$$

where the matrix inequality $0\le\mathcal{H}<\mathcal{M}$ means that the sum is to be taken over the range

$$\{\mathcal{H}\in\mathbb{Z}^{(L\times s)}\mid 0\le h_{li}<m_l \text{ for } 1\le i\le s \text{ and } 1\le l\le L\}.$$

Splitting the sum over $\mathcal{H}$ into the summand corresponding to $\mathcal{H}=0$ and the remaining summands corresponding to $\mathcal{H}\neq 0$ we obtain

$$r_{s,\underline{f},\underline{a},\underline{m}}(\underline{N},P)=\frac{1}{M^s}\Big(\mathcal{I}_0(\underline{N},P)+\sum_{\substack{0\le\mathcal{H}<\mathcal{M}\\ \mathcal{H}\neq 0}}\mathcal{I}_{\mathcal{H}}(\underline{N},P)\Big),\tag{8}$$

where

$$\mathcal{I}_0(\underline{N},P)=\int_0^1\cdots\int_0^1\prod_{i=1}^{s}\sum_{n_i=1}^{P}e\big(\underline{\alpha}\cdot\underline{f}(n_i)\big)e(-\underline{\alpha}\cdot\underline{N})d\underline{\alpha}\tag{9}$$

and

(10)
$$\mathcal{I}_{\mathcal{H}}(\underline{N},P)=\int_0^1\cdots\int_0^1\prod_{i=1}^{s}\sum_{n_i=1}^{P}e\Big(\underline{\alpha}\cdot\underline{f}(n_i)+\sum_{l=1}^{L}\frac{h_{li}}{m_l}\big(S_{q_l}(n_i)-a_l\big)\Big)e(-\underline{\alpha}\cdot\underline{N})d\underline{\alpha}$$

for $\mathcal{H}\neq 0$. As we will see, only the term $\mathcal{I}_0(\underline{N},P)$ contributes to the main term, while the sum over the other terms $\mathcal{I}_{\mathcal{H}}(\underline{N},P)$ is of smaller order.

Interpreting the integral in (9) properly we immediately get

$$\mathcal{I}_0(\underline{N},P)=r_{s,\underline{f}}(\underline{N},P). \tag{11}$$

Let us therefore turn our attention to $\mathcal{I}_{\mathcal{H}}(\underline{N},P)$. Shifting the modulus to the integrand yields

$$\big|\mathcal{I}_{\mathcal{H}}(\underline{N},P)\big|\leq\int_0^1\cdots\int_0^1\prod_{i=1}^{s}\Big|\sum_{n_i=1}^{P}e\Big(\underline{\alpha}\cdot\underline{f}(n_i)+\sum_{l=1}^{L}\frac{h_{li}}{m_l}S_{q_l}(n_i)\Big)\Big|d\underline{\alpha}.$$

Using the abbreviation

$$T_i(\underline{\alpha}):=\sum_{n=1}^{P}e\Big(\underline{\alpha}\cdot\underline{f}(n)+\sum_{l=1}^{L}\frac{h_{li}}{m_l}S_{q_l}(n)\Big),$$

we can further bound $\mathcal{I}_{\mathcal{H}}(\underline{N},P)$ by

$$\big|\mathcal{I}_{\mathcal{H}}(\underline{N},P)\big|\leq\max_i\sup_{\underline{\alpha}}\big|T_i(\underline{\alpha})\big|^{s-2t}\cdot\max_j\int_0^1\cdots\int_0^1\big|T_j(\underline{\alpha})\big|^{2t}d\underline{\alpha} \tag{12}$$

for any $t$ with $s>2t$. The estimate of $\mathcal{I}_{\mathcal{H}}(\underline{N},P)$ can therefore be divided into the tasks of evaluating the supremum and the mean value of $T_i(\underline{\alpha})$ in the domain $\underline{\alpha}\in[0,1)^d$.

Concerning the first task, we refer to Pfeiffer and Thuswaldner [**14**], Theorem 2.2. In this theorem they showed that if $d$, $m_l$, $h_l$, $q_l$ and $N$ are positive integers with $m_l\geq 2$, $q_l\geq 2$ and $m_l\nmid h_l(q_l-1)$, then the estimate

$$\sum_{n=1}^{P}e\Big(\theta n^d+\sum_{l=1}^{L}\frac{h_l}{m_l}S_{q_l}(n)\Big)\ll P^{1-\delta}\quad\text{for } P\to\infty$$

holds uniformly in $\theta\in[0,1)$. Here $\delta>0$ can be calculated explicitly. It depends, like the implied constant, only on $m_l,q_l$ and $d$. A closer examination of the proof of this result reveals that it even holds in a more general form. The key step of the proof depends on the application of difference operators of order $d$. These operators cancel out $\theta n^d$. If $\theta n^d$ is replaced by $p(n)$ where $p$ denotes an arbitrary polynomial of degree $d$ over $\mathbb{R}$, the proof works in the same way. Setting $p(n):=\underline{\alpha}\cdot\underline{f}(n)$ the result is therefore applicable to our situation and we obtain the Weyl-type estimate

$$\sup_{\underline{\alpha}\in[0,1)^d}\big|T_i(\underline{\alpha})\big|\ll P^{1-\delta}. \tag{13}$$

Concerning the second task, we expand the integral in (12) in the form

$$\int_0^1 \cdots \int_0^1 \left|T_j(\underline{\alpha})\right|^{2t} d\underline{\alpha} = \underbrace{\sum_{n_1=1}^{P} \cdots \sum_{n_{2t}=1}^{P}}_{\Sigma} e\Bigg(\sum_{l=1}^{L} \frac{h_{lj}}{m_l} \sum_{i=1}^{t} \big(S_{q_l}(n_i) - S_{q_l}(n_{t+i})\big)\Bigg), \tag{14}$$

where the sum $\sum$ is extended over the solutions $n_i \in \{1, \dots, P\}$ $(1 \le i \le 2t)$ of the symmetric system

$$\underline{f}(n_1) + \cdots + \underline{f}(n_t) = \underline{f}(n_{t+1}) + \cdots + \underline{f}(n_{2t}), \tag{15}$$

again without digital restrictions. Replacing the summands of $\sum$ in (14) by 1 obviously yields the upper bound

$$\int_0^1 \cdots \int_0^1 \left|T_j(\underline{\alpha})\right|^{2t} d\underline{\alpha} \le I_{t,\underline{f}}(P), \tag{16}$$

where $I_{t,\underline{f}}(P)$ denotes the number of solutions of (15) with $n_i \in \{1, \dots, P\}$ $(1 \le i \le 2t)$.

Assume that there is a constant $s_0 \in \mathbb{N}$ such that the estimate

$$I_{t,\underline{f}}(P) \ll P^{2t - \sum_{i=1}^{d} \deg f_i + \varepsilon_1} \qquad (P \to \infty) \tag{17}$$

holds for all $t \ge s_0$ with a constant $\varepsilon_1 \in [0, \delta)$ where $\delta$ is the constant occurring in (13).

The estimate in (17) implies together with (16) that

$$\int_0^1 \cdots \int_0^1 \left|T_j(\underline{\alpha})\right|^{2t} d\underline{\alpha} \le I_{t,\underline{f}}(P) \ll P^{2t - \sum_{i=1}^{d} \deg f_i + \varepsilon_1}. \tag{18}$$

Applying the estimates (13) and (18) in (12), we conclude that

$$|\mathcal{I}_{\mathcal{H}}(\underline{N}, P)| \ll (P^{1-\delta})^{s-2t} \cdot P^{2t - \sum_{i=1}^{d} \deg f_i + \varepsilon_1} = P^{s - \sum_{i=1}^{d} \deg f_i - \varepsilon_2}, \tag{19}$$

where $\varepsilon_2 = \delta \cdot (s - 2t) - \varepsilon_1$. This estimate is valid for $s > 2t \ge 2s_0$. Inserting (11) and the previous estimate for $\mathcal{I}_{\mathcal{H}}(\underline{N}, P)$ in (8) yields the following result.

**Proposition 1.** *Let $s$, $d \in \mathbb{N}$, $a_l$, $m_l$, $q_l \in \mathbb{N}$ with $m_l \ge 2$, $q_l \ge 2$, $\gcd(q_l - 1, m_l) = 1$ for $1 \le l \le L$ and $\gcd(q_l, q_k) = 1$ for $1 \le l < k \le L$.*

*Assume that there exists a constant $s_0 \in \mathbb{N}$ such that the estimate*

$$I_{t,\underline{f}}(P) \ll P^{2t - \sum_{i=1}^{d} \deg f_i + \varepsilon_1}$$

*holds for all $t \ge s_0$ with a constant $\varepsilon_1 \in [0, \delta)$ where $\delta$ is the constant occurring in (13). Then there exists $\varepsilon_2 > 0$ such that*

$$r_{s,\underline{f},\underline{a},\underline{m}}(\underline{N}, P) = \frac{1}{M^s} r_{s,\underline{f}}(\underline{N}, P) + \mathcal{O}\left(P^{s - \sum_{i=1}^{d} \deg f_i - \varepsilon_2}\right)$$

*holds uniformly for all $\underline{N} \in \mathbb{N}^d$ and $s \ge 2s_0 + 1$.*

As one can see, the term responsible for saving the quantity $P^{-\varepsilon_2}$ in the error term originates from the "digital" version (13) of Weyl's Lemma.

**Remark 1.** Note that all of the above considerations would also hold for the more general system

$$\begin{aligned} f_{11}(n_1) + \cdots + f_{1s}(n_s) &= N_1, \\ f_{21}(n_1) + \cdots + f_{2s}(n_s) &= N_2, \\ &\vdots \\ f_{D1}(n_1) + \cdots + f_{Ds}(n_s) &= N_D \end{aligned}$$

provided that $\deg f_{i1} = \cdots = \deg f_{is}$ holds for all $i \in \{1, \dots, D\}$. The only difference is that we have to keep track of the additional indices.

In the next section we will use Proposition 1 in order to derive Hardy-Littlewood type asymptotic formulas for various versions of Waring's problem with digital restrictions. To this matter we just need to know mean value estimates for the corresponding "unrestricted" problem and have to relate $\underline{N}$ to the parameter $P$ in a proper way. This is possible because of the uniformity in $\underline{N}$ of the estimates in Proposition 1.

## 3. Waring's Problem for powers and polynomials

The classical Waring's problem with digital restrictions has been studied in [**14, 15**]. Due to a result of Ford [**4**] we are able to improve the results contained in these papers as follows.

**Theorem 1.** *Let $s, d \in \mathbb{N}$, $a_l, m_l, q_l \in \mathbb{N}$ with $m_l \geq 2$, $q_l \geq 2$, $\gcd(q_l - 1, m_l) = 1$ for $1 \leq l \leq L$ and $\gcd(q_l, q_k) = 1$ for $1 \leq l < k \leq L$ and set $M := \prod_{l=1}^{L} m_l$. Let $r_{s,d,\underline{a},\underline{m}}(N)$ denote the number of representations of $N$ in the form*

$$n_1^d + \cdots + n_s^d = N \quad \textit{with} \quad S_{\underline{q}}(n_i) \equiv \underline{a} \bmod \underline{m} \quad \textit{for} \quad 1 \leq i \leq s. \tag{20}$$

*Then asymptotically*

$$r_{s,d,\underline{a},\underline{m}}(N) = \frac{1}{M^s} \mathfrak{S}(N) \Gamma\left(1 + \frac{1}{d}\right)^s \Gamma\left(\frac{s}{d}\right)^{-1} N^{s/d-1} + \mathcal{O}(N^{s/d-1-\varepsilon})$$

*for $s > d^2\big(\log d + \log\log d + \mathcal{O}(1)\big)$. Here $\varepsilon$ depends only on $s$ and $d$, while the implied constant depends on $s$, $d$ and $\varepsilon$.*

**Proof.** In this case we have $D = 1$ and $f(x) = x^d$ in (5). For this choice [**4**], Equation (5.4), implies

$$I_{t,f}(P) \ll P^{2t-d}$$

for $t \geq \frac{1}{2} d^2(\log d + \log\log d + \mathcal{O}(1))$. Applying Proposition 1 yields

$$r_{s,f,\underline{a},\underline{m}}(N, P) = \frac{1}{M^s} r_{s,f}(N, P) + \mathcal{O}\big(P^{s-d-\varepsilon_2}\big) \qquad (P \to \infty)$$

for $s > d^2(\log d + \log\log d + \mathcal{O}(1))$. Since this equation holds uniformly in $N$ we may set $P = \lfloor N^{1/d} \rfloor$. Observing that $r_{s,f,\underline{a},\underline{m}}(N) = r_{s,f,\underline{a},\underline{m}}(N, P)$ as well as $r_{s,f}(N) = r_{s,f}(N, P)$ holds for this choice of $P$ we arrive at

$$r_{s,f,\underline{a},\underline{m}}(N) = \frac{1}{M^s} r_{s,f}(N) + \mathcal{O}\big(N^{s/d-1-\varepsilon_3}\big) \qquad (N \to \infty),$$

which holds for $s > d^2(\log d + \log\log d + \mathcal{O}(1))$. The result now follows from the asymptotic formula (1) for $r_{f,d}(N)$, which by (2) holds for $s > d^2(\log d + \log\log d + \mathcal{O}(1))$.

Next we turn to Waring's problem with polynomial summands. Let $f$ be an integer valued polynomial of degree $d$ with the property that

(21) there exists no $k \in \mathbb{N}$ such that $k | f(n)$ holds for all $n \in \mathbb{N}$.

Then we consider the equation

$$f(n_1) + \cdots + f(n_s) = N. \tag{22}$$

Let $G(f)$ denote the least number $s$ for which (22) has a solution for each $N$ sufficiently large. It is known that for this problem the local solvability, i.e. the solvability of

$$f(n_1) + \cdots + f(n_s) \equiv N \bmod Q \quad (Q \in \mathbb{N}) \tag{23}$$

is much harder to decide than in the case of the ordinary Waring's problem. Indeed, Hua [**10**] found a polynomial $f$ of degree $d$ which admits no solution for (23) if $s = 2^d - 2$, $Q = 2^d$ and $N \equiv (-1)^d \bmod Q$. Obviously this implies that $G(f) \geq 2^d - 1$ for this choice of $f$. On the other hand, Hua [**9**] proved that

$$G(f) \leq 2^d + 1$$

holds for any polynomial $f$ having property (21). Using a slight modification of Proposition 1 we can prove the same bound for Waring's problem with polynomial summands with digital restrictions. However, as many other authors (see for instance [**5, 22**]), we ask for an upper bound for the solvability of (22) under the assumption that (23) can be solved for each $Q$. It is known that this is equivalent to the fact that the singular series

$$\mathfrak{S}_{s,f}(N) := \sum_{Q=1}^{\infty} \sum_{\substack{k=1 \\ (k,Q)=1}}^{Q} \left( \frac{1}{Q} \sum_{j=1}^{Q} e(kf(j)/Q) \right)^s e(-kN/Q)$$

is strictly positive. Following [**5**] let $\bar{G}(f)$ be the least number $s$ so that for every $\delta > 0$ and every $N > N_0(\delta)$ with $\mathfrak{S}_{s,f}(N) \geq \delta$, (22) is solvable. The best known bounds for $\bar{G}(f)$ are contained in [**5**].

In the present paper we are interested in $\bar{G}_{\underline{a},\underline{m}}(f)$ which is the least number $s$ so that for every $\delta > 0$ and every $N > N_0(\delta)$ with $\mathfrak{S}_{s,f}(N) \geq \delta$, (22) with digital restrictions is solvable. The following result implies that

$$\bar{G}_{\underline{a},\underline{m}}(f) \leq d^2(\log d + \log\log d + \mathcal{O}(1)).$$

**Theorem 2.** *Let $s$, $d \in \mathbb{N}$, $a_l$, $m_l$, $q_l \in \mathbb{N}$ with $m_l \geq 2$, $q_l \geq 2$, $\gcd(q_l - 1, m_l) = 1$ for $1 \leq l \leq L$ and $\gcd(q_l, q_k) = 1$ for $1 \leq l < k \leq L$ and set $M := \prod_{l=1}^{L} m_l$. Let $f(x) \in \mathbb{Q}[x]$ be an integer-valued polynomial of degree $d$ with property (21) and positive leading coefficient $c$. Let $r_{s,f,\underline{a},\underline{m}}(N)$ denote the number of solutions of*

$$f(n_1) + \cdots + f(n_s) = N \quad \textit{with} \quad S_{\underline{q}}(n_i) \equiv \underline{a} \bmod \underline{m} \quad \textit{for} \quad 1 \leq i \leq s.$$

*Then asymptotically*

$$r_{s,f,\underline{a},\underline{m}}(N) = \frac{1}{M^s} c^{-s/d} \mathfrak{S}_{s,f}(N) \Gamma\left(1+\frac{1}{d}\right)^s \Gamma\left(\frac{s}{d}\right)^{-1} N^{s/d-1} + \mathcal{O}(N^{s/d-1-\varepsilon})$$

*for* $s > d^2(\log d + \log\log d + \mathcal{O}(1))$. *Here* $\varepsilon$ *depends only on* $s$ *and* $d$, *while the implied constant depends on* $s$, $d$ *and* $\varepsilon$.

**Proof.** This theorem corresponds to the choice $D = 1$ in (5). The vector $\underline{f}(x)$ becomes a scalar $f(x)$, where $f$ is a polynomial with property (21). By [4,Equation (5.4)] we have

$$I_{t,f}(P) \ll P^{2t-d}$$

for $t \geq \frac{1}{2} d^2(\log d + \log\log d + \mathcal{O}(1))$. Using this result instead of [**8**], Lemma 6.1, in the proof of Theorem 1 of the same paper, this theorem yields a Hardy-Littlewood asymptotic formula for $r_{s,f}(N)$ which is valid for $s > d^2(\log d + \log\log d + \mathcal{O}(1))$.

The theorem now follows from Proposition 1.

**Remark 2.** By Remark 1 it is possible to prove the same theorem for the problem

$$f_1(n_1) + \cdots + f_s(n_s) = N \quad \text{with} \quad S_{\underline{q}}(n_i) \equiv \underline{a} \bmod \underline{m} \quad \text{for} \quad 1 \leq i \leq s.$$

Here $f_1, \ldots, f_s$ are integer valued polynomials of degree $d$ with property (21). We just have to note that [8,Theorem 1] also holds in this more general setting.

## 4. Tarry's problem

For the moment consider the unrestricted version of Tarry's problem. Contrary to Waring's problem it is not clear whether (3) has a solution for $n_1, \ldots, n_s \in \mathbb{R}^+$. We call (3) positively solvable, if such a solution exists. It can be deduced that positive solvability is equivalent to $b_0 > 0$, where

$$\begin{aligned} b_0 \quad := \quad & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \int_0^1 e(\alpha_d x^d + \cdots + \alpha_1 x) dx \right)^s \times \\ & \times e\left( -\frac{N_d}{P^d}\alpha_d - \ldots - \frac{N_1}{P}\alpha_1 \right) d\underline{\alpha}. \end{aligned}$$

A necessary and sufficient criterion for the positivity of $b_0$ is given in [**12**], Chapter XI.

The "congruent solvability" of (3) is controlled by the singular series $\mathfrak{S}(N_1, \ldots, N_d)$ (cf. [**11**], Section XIX). Criteria for $\mathfrak{S}(N_1, \ldots, N_d) > 0$ are studied in [**11**] (cf. also [**12**], Chapter XI, where positivity properties of a similar function are studied).

$b_0$ and $\mathfrak{S}$ also play a rôle in the following version of Tarry's problem with digital restrictions.

**Theorem 3.** *Let* $s$, $d \in \mathbb{N}$, $a_l$, $m_l$, $q_l \in \mathbb{N}$ *with* $m_l \geq 2$, $q_l \geq 2$, $\gcd(q_l - 1, m_l) = 1$ *for* $1 \leq l \leq L$ *and* $\gcd(q_l, q_k) = 1$ *for* $1 \leq l < k \leq L$, *and* $N_1, \ldots, N_d \in \mathbb{N}$. *Set*

*$M := \prod_{l=1}^{L} m_l$, let $P = \lfloor N_d^{1/d} \rfloor$, and let $r_{s,d,\underline{a},\underline{m}}(N_1,\dots,N_d,P)$ denote the number of solutions of the simultaneous equations*

$$\left.\begin{aligned} n_1 + \cdots + n_s &= N_1, \\ n_1^2 + \cdots + n_s^2 &= N_2, \\ &\vdots \\ n_1^d + \cdots + n_s^d &= N_d, \end{aligned}\right. \quad \text{with} \quad S_{\underline{q}}(n_i) \equiv \underline{a} \bmod \underline{m} \quad \text{for} \quad 1 \le i \le s.$$

*Then asymptotically*

$$r_{s,d,\underline{a},\underline{m}}(N_1,\dots,N_d,P) = \frac{1}{M^s} b_0 \mathfrak{S}(N_1,\dots,N_d) P^{s-d(d+1)/2} + \mathcal{O}(P^{s-d(d+1)/2-\varepsilon})$$

*for $s > 2d^2\big(\log d + 2\log\log d + \mathcal{O}(1)\big)$. Here $\varepsilon$ depends only on $s$ and $d$, while the implied constant depends on $s$, $d$ and $\varepsilon$.*

**Proof.** In order to obtain the setting of this theorem we have to choose $D = d$ and $\underline{f}(x) = (x, x^2, \dots, x^d)$ in (5). Again we want to apply Proposition 1. The mean value estimate needed in order to apply this proposition in the present setting is contained in Wooley [**21**], Theorem 3. This theorem implies that

$$I_{t,\underline{f}}(P) \ll P^{2t-\frac{d(d+1)}{2}}$$

holds for $t > d^2\big(\log d + 2\log\log d + \mathcal{O}(1)\big)$. After the application of Proposition 1 we need an asymptotic formula for Tarry's problem without digital restriction. For $s \ge d^2(6\log d + 2\log\log d + 8)$ such an asymptotic formula is shown in [**11**], Section XIX. With help of [**21**], Theorem 3, this formula can be shown even for $s > 2d^2\big(\log d + 2\log\log d + \mathcal{O}(1)\big)$. This proves the result.

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